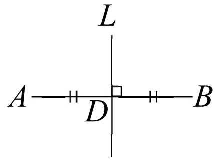
中垂線與角平分線的性質
Properties of the perpendicular bisector and angle bisector

Class:__________ Name: __________

1. Perpendicular bisector

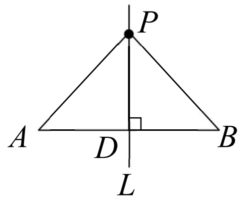
We know the perpendicular bisector must be perpendicular to the segment and divides the segment into two halves. That is, if L is the perpendicular bisector of $\overline{AB}$, and intersects $\overline{AB}$ at D, then (1) L is perpendicular to $\overline{AB}$ (2) $\overline{DA}=\overline{DB}$.



Next, we use the congruency of triangles to learn other properties of the perpendicular bisector.

Example 1

Line L is the perpendicular bisector of $\overline{AB}$. P is an arbitrary point on L. Prove $\overline{PA}=\overline{PB}$.



[Solution]

In $\triangle PAD$ and $\triangle PBD$,

(1) $\overline{DA}=\overline{DB}$ (L is the perpendicular bisector of $\overline{AB}$)

(2) $\angle PDA = \angle PDB = 90°$($L$ is the perpendicular bisector of $\overline{AB}$)

(3) $\overline{PD}=\overline{PD}$ (Common side)

Therefore, $\triangle PAD \cong \triangle PBD$ (SAS congruence postulate), and then $\overline{PA}=\overline{PB}$.

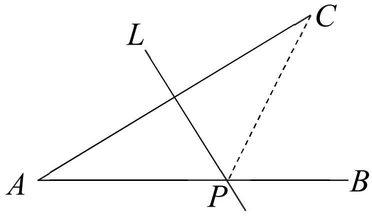
From the result above, we know that if P is on the perpendicular bisector of $\overline{AB}$, then $\overline{PA}=\overline{PB}$.

★ The property of perpendicular bisector-1:

Any point on the perpendicular bisector of a line segment is equidistant from both endpoints of the line segment.

Exercise 1

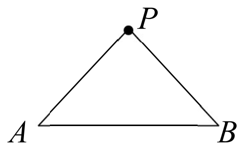
Line L is the perpendicular bisector of $\overline{AC}$, and intersects $\overline{AB}$ at P. $\overline{AB}=10$. $\overline{PB}=4$. Find $\overline{PC}=$?



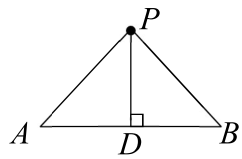
From the proof above, we know that if P is an arbitrary point on the perpendicular bisector of $\overline{AB}$, then P is equidistant from both A and B. Conversely, if P is equidistant from both A and B, does this mean that P is on the perpendicular bisector of $\overline{AB}$?

Example 2

Prove that if $\overline{PA}=\overline{PB}$, then P is on the perpendicular bisector of $\overline{AB}$.



[Solution]



Draw a line segment through P, which is perpendicular to $\overline{AB}$ and intersects $\overline{AB}$ at D.

In $\triangle APD$ and $\triangle BPD$,

(1) $\overline{PA}=\overline{PB}$ (Given)

(2) $\angle PDA=\angle PDB=90°$ ($\overline{PD}\perp\overline{AB}$)

(3) $\overline{PD}=\overline{PD}$ (Common side)

Therefore, $\triangle APD\cong\triangle BPD$ (RHS congruence postulate), and then $\overline{AD}=\overline{BD}$.

Because $\overline{PD}\perp\overline{AB}$ and $\overline{AD}=\overline{BD}$, $\overleftrightarrow{PD}$ is the perpendicular bisector of $\overline{AB}$.

Then, P is on the perpendicular bisector of $\overline{AB}$.

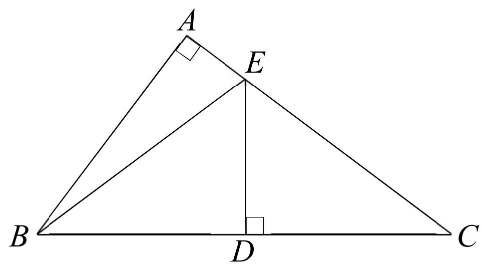
★ The property of perpendicular bisector-2:

Any point which is equidistant from both endpoints of the line segment must be on the perpendicular bisector of the line segment.

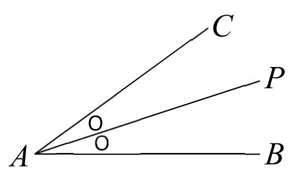
Exercise 2

In $\triangle ABC$, $\angle A=90°$. $\overline{DE}\perp\overline{BC}$ and $\overline{BE}=\overline{CE}$.

$\overline{AB}=6$. $\overline{BD}=5$. Find $\overline{AE}=$?



2. Angle bisector

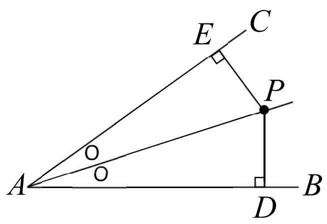


We know the angle bisector must divide an angle into two halves.

That is, if $\overrightarrow{AP}$ is an angle bisector of $\angle CAB$, then $\angle CAP=\angle PAB$.

Next, we use the congruency of triangles again to learn other properties of the angle bisector.

Example 3



$\overrightarrow{AP}$ is an angle bisector of $\angle CAB$. $\overline{PD} \perp \overline{AB}$ and $\overline{PE} \perp \overline{AC}$.

Prove that $\overline{PD} = \overline{PE}$.

[Solution]

In $\triangle PAD$ and $\triangle PAE$,

(1) $\angle PAD = \angle EAP$ ($\overrightarrow{AP}$ is an angle bisector of $\angle CAB$)

(2) $\angle PDA = \angle PEA = 90°$(Given)

(3) $\overline{PA} = \overline{PA}$ (Common side)

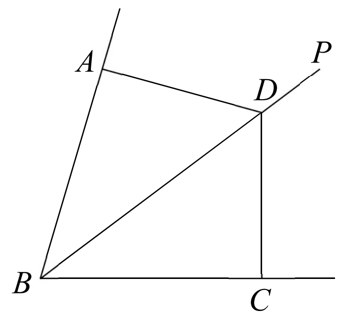
Therefore, $\triangle PAD \cong \triangle PAE$ (AAS congruence postulate), and then $\overline{PD} = \overline{PE}$.

From the result above, we know that if P is on the angle bisector of $\angle CAB$, then P is equidistant from two sides of $\angle CAB$.

★ The property of angle bisector-1:

Any point on the angle bisector of an angle is equidistant from both sides of the angle.

Exercise 3

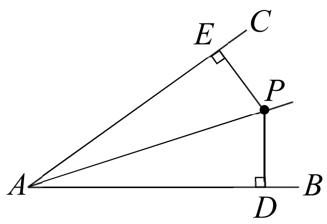


$\overline{BP}$ is an angle bisector of $\angle ABC$. D is on $\overline{BP}$. $\overline{DA} \perp \overline{AB}$ and $\overline{DC} \perp \overline{BC}$. $\overline{AB} = 16$. $\overline{DC} = 12$. Find $\overline{BD} =$?

From the proof above, we know that if P is an arbitrary point on the angle bisector of $\angle CAB$, then P is equidistant from both $\overline{AB}$ and $\overline{AC}$. Conversely, if P is equidistant from both $\overline{AB}$ and $\overline{AC}$, whether P is on the angle bisector of $\angle CAB$?

Example 4

Given that $\overline{PD} \perp \overline{AB}$ and $\overline{PE} \perp \overline{AC}$. Prove that if $\overline{PD} = \overline{PE}$, then P is on the angle bisector of $\angle CAB$.



[Solution]

In $\triangle PAD$ and $\triangle PAE$,

(1) $\overline{PD} = \overline{PE}$ (Given)

(2) $\angle PDA = \angle PEA = 90°$(Given)

(3) $\overline{PA} = \overline{PA}$(Common side)

Therefore, $\triangle PAD \cong \triangle PAE$ (RHS congruence postulate), and then $\angle PAD = \angle EAP$.

Because $\angle PAD = \angle EAP$, $\overrightarrow{AP}$ is an angle bisector of $\angle CAB$ and P is on the angle bisector of $\angle CAB$.

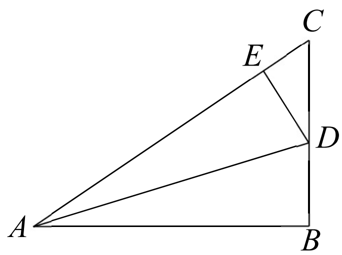
★ The property of angle bisector-2:

Any point which is equidistant from both sides of the angle must be on the angle bisector of the angle.

Exercise 4

In $\triangle ABC$, $\angle B = \angle AED = 90°$. $\overline{DB} = \overline{DE}$. $\angle C = 66°$.

Find $\angle ADB = ?$



一、設計理念：

1. 中垂線與角平分線性質在各版的所放置的章節不同，但其主要目標皆是讓學生應用全等性質。
2. 此章節的兩個性質，後者在教科書中稱為中垂線/角平分線的判斷性質，但為方便起見，本教案直接以 1 和 2 區別中垂線/角平分線的兩個性質。
3. 三角形全等性質的英文全名為「triangle congrence postulate」，但因為在本節課之前已經學過三角型的全等性質，故使用性質時，口語上可直接說「SSS」，而不需要完整說出「SSS triangle congrence postulate」。
4. 三角形全等性質中的「性質」一詞有多種單字，常見的包括 postulate 及 theorem，亦有人使用 criterion 或 rule，本教案使用最常見的 postulate 一詞。
5. 國外通常以代表「HL(Hypotenuse-Leg) congrence postulate」，但配合臺灣課本使用「RHS 全等」一詞，故本文中將其譯為「RHS congruence postulate」。

二、英文詞彙：

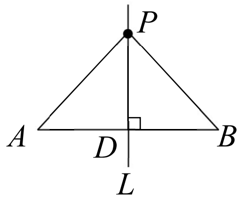
中文	英文
中垂線	perpendicular bisector
角平分線	angle bisector
線段	line segment
角	angle
相交	intersect
SSS 全等	SSS congruence postulate
SAS 全等	SAS congruence postulate
RHS 全等	RHS congruence postulate
ASA 全等	ASA congruence postulate
AAS 全等	AAS congruence postulate
對應邊	corresponding side
對應角	corresponding angle
共用邊	common side
共用角	common angle

三、數學英文用法：

數學表示法	英文
$\overline{AB}$	line segment AB
$\overrightarrow{AB}$	ray AB
$\overleftrightarrow{AB}$	line AB
$\angle A$	angle A
$90°$	90 degrees
$\overline{AB}$ 與 L 垂直	line segment AB is perpendicular to L

四、教學參考範例：

<table>
<tr>
<td rowspan="2">1
【中垂線性質】</td>
<td>Example 1

L is the perpendicular bisector of $\overline{AB}$. P is an arbitrary point on L. Prove $\overline{PA}=\overline{PB}$.
</td>
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<tr>
<td>Let's see example 1. We try to prove $\overline{PA}$ equals $\overline{PB}$. It seems that $\triangle PAD$ and $\triangle PBD$ are congruent, and we have to find conditions to prove our guess.
Before our proof, we should analyze the conditions we have. L is the perpendicular bisector of $\overline{AB}$ implies two things: (1) D is the midpoint of $\overline{AB}$, and (2) L is perpendicular to $\overline{AB}$. We will use both of them in our proof.
First, D is the midpoint of $\overline{AB}$, so $\overline{DA}$ equals $\overline{DB}$.
Second, L is perpendicular to $\overline{AB}$, so either $\angle PDA$ or $\angle PDB$ equals 90°.
Third, don't forget some information given in the figure. $\overline{PD}$ is the common side of these two triangles, so $\overline{PD}$ equals $\overline{PD}$.
These three conditions can show that $\triangle PAD$ and $\triangle PBD$ are congruent by the SAS congruence postulate. The corresponding sides of two congruent triangles should be equal, so $\overline{PA}$ equals $\overline{PB}$.
Notice that P is an arbitrary point on L, so the statement still holds if we choose another point on L, and we can prove it through the same process. Therefore, we can get the following conclusion: Any point on the perpendicular bisector of a line segment is equidistant from both endpoints of the line segment.</td>
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【中垂線判斷性質】</td>
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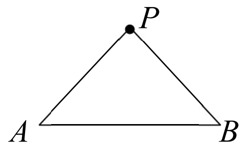
Example 2

Prove that if $\overline{PA}=\overline{PB}$, then P is on the perpendicular bisector of $\overline{AB}$.



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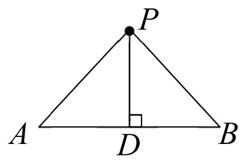

To start with, we draw a line through P, which is perpendicular to $\overline{AB}$ and intersects $\overline{AB}$ at point D. We try to prove P is on the perpendicular bisector of $\overline{AB}$. That is, we have to prove that $\overleftrightarrow{PD}$ is the perpendicular bisector of $\overline{AB}$.

If we can prove that $\triangle PAD$ is congruent to $\triangle PBD$, then $\overline{DA}$ equals $\overline{DB}$. Therefore, we get $\overleftrightarrow{PD}$ is the perpendicular bisector of $\overline{AB}$. Let's try to prove it.

First, we know that $\overline{PA}$ equals $\overline{PB}$ from the statement above.

Second, because $\overline{PD}$ is perpendicular to $\overline{AB}$, $\angle PDA$ and $\angle PDB$ are both equal to 90°.

Third, $\overline{PD}$ is the common side of $\triangle PAD$ and $\triangle PBD$, so $\overline{PD}$ equals $\overline{PD}$.

From these three conditions above, we can show that $\triangle PAD$ and $\triangle PBD$ are congruent by the RHS congruence postulate. The corresponding sides of two congruent triangles should be equal, so $\overline{DA}$ equals $\overline{DB}$.

We can see that $\overleftrightarrow{PD}$ is not only perpendicular to $\overline{AB}$, but also divides $\overline{AB}$ into two halves, so $\overleftrightarrow{PD}$ is the perpendicular bisector of $\overline{AB}$. In other words, P is on the perpendicular bisector of $\overline{AB}$.

From the proof above, we can get another conclusion: Any point which is equidistant from both endpoints of the line segment must be on the perpendicular bisector of the line segment.

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