Limits of sequences and series

I. Key mathematical terms

Terms	Symbol	Chinese translation
Limit		
Convergent/Convergence		
Divergent/Divergence		

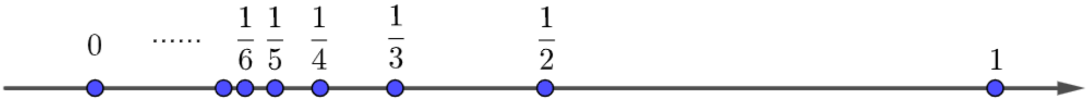
II. Limits of sequences

We learned the definition of a sequence in 1st grade. A sequence is an ordered list of numbers or some mathematical objects that follow a particular pattern. For example, we can list numbers $a_1, a_2, a_3, a_4, \ldots, a_n$ in an order to get a sequence. ***n*** represents the number of the terms of the sequence, and for $1 \le i \le n$, a_i is the i-th term of the sequence.

If we have a sequence with finite terms, we said this is a "finite sequence". (Just like the example above.) Otherwise, if a sequence has infinite terms, for example, $1, 2, 3, 4, 5, \ldots, n, \ldots$, we said this is an "infinite sequence". Whaat we are really interested in an infinite sequence is what the n-th term will be as n tends to the infinite.

Let's start with an infinite sequence $< a_n > = < \frac{1}{n} >$. We can list the terms of this sequence:

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{100}, \frac{1}{101}, \ldots, \frac{1}{1000}, \ldots,$$



We can easily find that as the number n increases, the n-th term $\frac{1}{n}$ will decrease and get closer to 0. Actually, we can make $\frac{1}{n}$ as close to 0 as we want.

For example:

(1) If we want $\left|\frac{1}{n}-0\right|<\frac{1}{10}$. Take $n>10$.

(2) If we want $\left|\frac{1}{n}-0\right|<\frac{1}{100}$. Take $n>100$.

(3) If we want $\left|\frac{1}{n}-0\right|<\frac{1}{N}$. Take $n>N$.

We can make $\frac{1}{n}$ as close to *0* as we want.

Hence, we can say "the infinite sequence $<a_n>=<\frac{1}{n}>$ tends to a definite value of *0*", and we can denote this result with the sign "lim" as follows,

$$\lim_{n\to\infty}\frac{1}{n}=0$$

The arrow "$\to$" stands for "approaches to or tends to". The sign "∞" stands for "infinity." We say "The limit as n approaches infinity of one over n equals zero."

<key>
In mathematics "infinite" is not an exact number. "Infinite" can be considered as a concept that something is boundless, endless, or larger than any number you think of.

Limits of infinite sequences

Given an infinite sequence $<a_n>$, as n increases, if a_n tends to a definite value L, then we say the limit of sequence $<a_n>$ exists, and is denoted by:

$$\lim_{n\to\infty}a_n=L$$

The sequence that has a definite limit is a "convergent sequence". The sequence that doesn't have a definite limit is a "divergent sequence".

<key> If the limit of an infinite sequence exists, then it has only one limit.

Example 1

Determine whether the following sequence is convergent or divergent. If the sequence is convergent, find the limit of the sequence.

(1) $<(-1)^n>$ (2) $<1+\frac{1}{n}>$ (3) $<5>$

Example 2

Determine whether the following sequence is convergent or divergent. If the sequence is convergent, find the limit of the sequence.

(1) $<\frac{n^2+n}{n}>$ (2) $<\sin n\pi>$ (3) $<\cos\frac{n\pi}{2}>$

Convergent and divergent of geometric sequences

We've known that the geometric sequence(progression) has form $<a_n>=<a_1r^{n-1}>$.

While a_1 stands for the first term of the sequence and r ($r \neq 0$) stands for the common ratio of the sequence. Both a_1, r are constants, and a_1 won't affect the convergent and divergent of the sequence. To understand if a sequence diverge or converge, we only need to discuss about the sequence $<r^n>$ ($r \neq 0$).

(1) $|r|=1$

In this case, the common ratio can be 1 or -1.

① $r=1$

Every term of infinite geometric sequence equals 1. $\lim_{n\to\infty} 1^n = 1$.

② $r=-1$

In this case, the sequence jumps between 1 and -1. $\lim_{n\to\infty}(-1)^n$ doesn't exist.

(2) $|r|<1$

No matter if the common ratio r is positive or negative, the sequence will approach 0 as n gets larger. (You can try some examples, such as $<(-0.1)^n>$, $<(\frac{1}{3})^n>$.)

$\lim_{n\to\infty} r^n = 0$ in this case.

(3) $|r|>1$

If the given r is positive, r^n will get bigger as n increase. If the given r is negative, r^n still get further from zero as n increase. (You can try some examples, such as $< 2^n >$, $< (-2)^n >$.)

$\lim_{n\to\infty} r^n$ doesn't exist in this case.

(4) $r = 0$ (extra)

In this case $< r^n >$ is not a geometric sequence but $\lim_{n\to\infty} 0^n = 0$.

With all the discussion above, we can have the following conclusion:

The convergent of the sequence $< r^n >$

The convergent condition of infinite sequence $< r^n >$

(1) (Converge) When $r = 1$, $\lim_{r\to\infty} r^n = 1$.

(2) (Converge) When $-1 < r < 1$, $\lim_{r\to\infty} r^n = 0$

(3) (Diverge) When $|r| > 1$ or $r = -1$, $\lim_{r\to\infty} r^n$ doesn't exist.

Example 3

Determine whether the following sequence is convergent or divergent. If the sequence is convergent, find the limit of the sequence.

(1) $< (0.01)^n >$ (2) $< (-1.01)^n >$ (3) $< (\frac{5}{7})^n >$

(4) $< (\tan 45°)^n >$ (5) $< (\sqrt{3} - 1)^n >$ (6) $< (-0.999)^n >$

III. Arithmetic of convergent sequences

Given two convergent sequences $\langle a_n \rangle$, $\langle b_n \rangle$, and we have $\lim_{n\to\infty} a_n = \alpha$, $\lim_{n\to\infty} b_n = \beta$.

We can composite these two sequences by addition, subtraction, multiplication and division to get some new sequences:

$$\langle a_n + b_n \rangle, \langle a_n - b_n \rangle, \langle a_n b_n \rangle, \left\langle \frac{a_n}{b_n} \right\rangle \ (b_n \neq 0)$$

Will the new sequence still converge? If these sequences converge, can we find the limit? You can try some examples.

<Hint>

Convergent sequences have various kinds of form. You can try to composite the convergent sequence: $<\frac{1}{n}>$, $<(0.5)^n>$, $<1^n>$, $<(-0.7)^n>$, $<\sin n\pi>$

For example:

Take $\langle a_n \rangle = <\frac{1}{n}>$, $\langle b_n \rangle = <2-\frac{1}{2n}>$ with $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 2$. We have:

$\langle a_n + b_n \rangle = \left\langle (\frac{1}{n}) + (2-\frac{1}{2n}) \right\rangle = \left\langle 2+\frac{1}{2n} \right\rangle$. $\lim_{n\to\infty} a_n + b_n = 2$.

$\langle a_n b_n \rangle = \left\langle (\frac{1}{n})(2-\frac{1}{2n}) \right\rangle = \left\langle \frac{2}{n} - \frac{1}{2n^2} \right\rangle$. $\lim_{n\to\infty} a_n b_n = 0$.

Actually, the arithmetic of convergent sequences has the following properties:

Arithmetic of convergent sequences

Sequences $\langle a_n \rangle$, $\langle b_n \rangle$ are convergent sequences, and $\lim_{n\to\infty} a_n = \alpha$, $\lim_{n\to\infty} b_n = \beta$. Then the following holds true:

(1) Sequences $\langle a_n + b_n \rangle$, $\langle a_n - b_n \rangle$, $\langle a_n b_n \rangle$ are convergent sequences, and

$$\lim_{n\to\infty} a_n + b_n = \alpha + \beta, \quad \lim_{n\to\infty} a_n - b_n = \alpha - \beta, \quad \lim_{n\to\infty} a_n b_n = \alpha\beta .$$

(2) When $b_n \neq 0$ and $\beta \neq 0$, $\left\langle \frac{a_n}{b_n} \right\rangle$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta}$.

Example 4

Determine whether the following sequence is convergent or divergent. If the sequence is convergent, find the limit of the sequence.

(1) $\langle a_n \rangle = \left\langle 5 - \frac{2}{n+3} + \frac{5}{n^2+1} \right\rangle$

(2) $\langle b_n \rangle = \left\langle (\frac{3}{4})^n + (\frac{101}{100})^n \right\rangle$

(2) $\langle c_n \rangle = \left\langle (1-\frac{1}{n^2})(-5+\frac{2}{2n}) \right\rangle$

(4) $\langle d_n \rangle = \left\langle \frac{3^n - 5^n}{7^n} \right\rangle$

Example 5

Determine whether the following sequence is convergent or divergent. If the sequence is convergent, find the limit of the sequence.

(1) $\langle a_n \rangle = \left\langle \frac{2n+4}{-7n+9} + (0.03)^n \right\rangle$ (2) $\langle b_n \rangle = \left\langle \frac{5n^2-3n+7}{(n+1)(n-2)} \right\rangle$

Example 6

Find the limit: $\lim_{n\to\infty} \frac{1^2+2^2+3^2+...+n^2}{3n^3}$ (Hint: $1^2+2^2+3^2+...+n^2 = \frac{1}{6}n(n+1)(2n+1)$)

IV. Limits of series

If $a_1, a_2, a_3, a_4, ..., a_n$ is a sequence, then the corresponding series is given by

$a_1 + a_2 + a_3 + a_4 + ... + a_n$. To represent a series in a more efficient way, we need the "sigma notation" (or "summation notation"). We can abbreviate the series above into:

$$a_1 + a_2 + a_3 + a_4 + ... + a_n = \sum_{i=1}^{n} a_i$$

"Σ" is the symbol that pronounced as "sigma".

"i" is called the index of summation.

"1" is the first value of "i".

"n" is the last value of "i".

Try the following examples:

(1) $\sum_{k=2}^{5} a_k =$ ______________________________ 。

(2) $\sum_{n=1}^{6} 5 =$ ______________________________ 。

(3) $\sum_{k=1}^{3} (2k+1) =$ ______________________________ 。

(4) $a_6 + a_7 + a_8 + ... + a_{100} = \sum$

(5) $e^1 + e^2 + e^3 + ... + e^{20} = \sum$

(6) $g(\frac{1}{n}) + g(\frac{2}{n}) + g(\frac{3}{n}) + ... + g(\frac{n}{n}) = \sum_{k=1} f(\frac{}{})$

(Hint: You can cross check your sigma notation by expanding it.)

With the examples above, we found some properties about sigma notation.

Properties about sigma notation

$\langle a_n \rangle$, $\langle b_n \rangle$ are two series and c is a constant. Then the following holds true:

(1) $\sum_{k=1}^{n} c = nc.$

(2) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k.$

(3) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Frequently used formula

Here are some frequently used formulas with sigma notation.

(1) $\sum_{k=1}^{n} k = 1+2+3+...+n = \frac{n(n+1)}{2}$

(2) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+...+n^2 = \frac{1}{6}n(n+1)(2n+1)$

(3) $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+...+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

<Key> You can use mathematical induction to prove the formulas above.

Example 7

(1) Use sigma notation to represent: $1\times 4+2\times 7+3\times 11+...+n(3n+1)$

(2) Find the sum of the series in (1)

(3) Find the sum of the series: $1\times 4+2\times 7+3\times 11+...+17\times 52$

Example 8

(1) Use sigma notation to represent: $1\times 31+2\times 28+3\times 35+...+30\times(-56)$

(2) Find the sum of the series in (1)

Infinite series

A finite series is the sum of a given number of terms that comes to an end. (The series in Example 7,8 are finite series.) An infinite series is the sum of given number that do not come to an end. The terms are infinite. It's possible for us to calculate the sum of infinite terms. We'll use **partial sums** to describe the sum of infinite series. Finite and infinite series use identical properties for addition, subtraction and multiplication. We'll also use the sigma notation to represent the infinite series, for example:

$a_1+a_2+a_3+...+a_n+... = \sum_{k=1}^{\infty} a_k$ is the sum of infinite sequence $\langle a_n \rangle, n=1,2,3,...$

The sum of infinite series $\sum_{k=1}^{\infty} a_k$

Given an infinite series $a_1+a_2+a_3+...+a_n+...$, Since we cannot find the sums of these terms by adding all of it, we can analyze it's value by the partial sums. Which is denoted by S_n, n denote the index of the last term in the sum.

$$S_n = a_1+a_2+a_3+...+a_n = \sum_{k=1}^{n} a_k$$

(1) If the infinite sequence $\langle S_n \rangle$ is a convergent sequence, and $\lim_{n\to\infty} S_n = \alpha$. We say this infinite series is a convergent series with limit α. That is:

$$\sum_{k=1}^{\infty} a_k = a_1+a_2+a_3+...+a_n+... = \lim_{n\to\infty} S_n = \alpha$$

(2) If the infinite sequence $\langle S_n \rangle$ is a divergent sequence. We say this infinite series is a divergent series with no limit exist.

Example 9

Find the sum of the infinite series: $\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+...+\frac{1}{n\times(n+2)}+...$

*Example 10

Find the sum of the infinite series: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$

Example 11

Find the sum of the infinite series: $(\frac{2}{3})^1+(\frac{2}{3})^2+(\frac{2}{3})^3+...+(\frac{2}{3})^n+...$

Example 11

Find the sum of the infinite series: $\frac{1}{7}+\frac{2}{7^2}+\frac{4}{7^3}+...+\frac{2^{n-1}}{7^n}+...$

<資料來源>

1. **Limit of Sequence**
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2. Infinite series
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2. 南一書局數學甲下冊

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