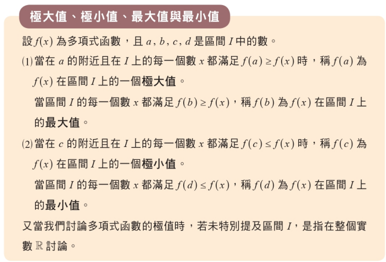
微分的應用（二）

Applications of Differentiation II

Material	Vocabulary
 **極大值、極小值、最大值與最小值** 設 $f(x)$ 為多項式函數，且 a, b, c, d 是區間 I 中的數。 (1) 當在 a 的附近且在 I 上的每一個數 x 都滿足 $f(a) \geq f(x)$ 時，稱 $f(a)$ 為 $f(x)$ 在區間 I 上的一個**極大值**。 當區間 I 的每一個數 x 都滿足 $f(b) \geq f(x)$，稱 $f(b)$ 為 $f(x)$ 在區間 I 上的**最大值**。 (2) 當在 c 的附近且在 I 上的每一個數 x 都滿足 $f(c) \leq f(x)$ 時，稱 $f(c)$ 為 $f(x)$ 在區間 I 上的一個**極小值**。 當區間 I 的每一個數 x 都滿足 $f(d) \leq f(x)$，稱 $f(d)$ 為 $f(x)$ 在區間 I 上的**最小值**。 又當我們討論多項式函數的極值時，若未特別提及區間 I，是指在整個實數 $\mathbb{R}$ 討論。	1. interval (區間), 2. relative (相對的), 3. maximum (極大值), 4. endpoint (端點), 5. y-coordinate (y 坐標), 6. absolute (絕對的), 7. minimum (極小值), 8. local (局部的), 9. global (全域), 10. extreme (極端), 11. extrema (極值), 12. occur (發生).

Illustrations I

Let function f be defined on an **interval**[1] I containing a , b , c and d .

1. See Figure 1, point B is higher than any other points nearby and it is called a **relative**[2] **maximum**[3]. Point D is an **endpoint**[4] and is also higher than any other points around point D. So, point D is called **endpoint maximum**. The **y-coordinates**[5] $f(b)$ and $f(d)$ are maxima of f on I . Moreover, point D is the highest point of f on I , so $f(d)$ is called the **absolute**[6] **maximum** of f on I .

   見圖 1，點 B 高於在其附近上的每個點，因此點 B 稱為**極大值**；點 D 是一端點且點 D 高於在其附近上的每個點，因此點 D 稱為**端點極大值**。其 y 坐標 $f(b)$ 和 $f(d)$ 皆為 $f(x)$ 在區間 I 上的極大值。

   然而，點 D 在區間 I 上為最高的點，稱 $f(d)$ 為在區間 I 上的**最大值**。

2. Point C is lower than any other points nearby and it is called a **relative minimum**[7]. Point A is an endpoint and is also lower than any other point around point A. So, point A is called **endpoint minimum**. The y-coordinates $f(a)$ and $f(c)$ are a minimum of f on I . Moreover, point C is the lowest point on I , so $f(c)$ is called the **absolute minimum** of f on I .

   點 C 低於在其附近上的每個點，因此點 C 稱為**極小值**；點 A 是一端點且點 A 低於在其附近上的每個點，因此點 A 稱為**端點極小值**。其 y 坐標 $f(a)$ 和 $f(c)$ 皆為 $f(x)$ 在區間 I 上的極小值。

   然而，點 C 在區間 I 上為最低的點，稱 $f(c)$ 為在區間 I 上的**最小值**。

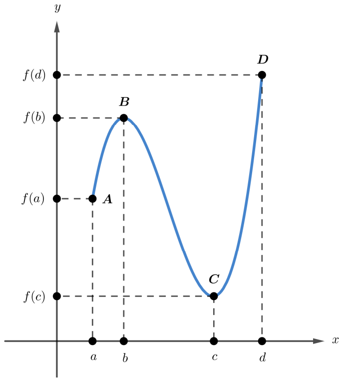


Figure 1

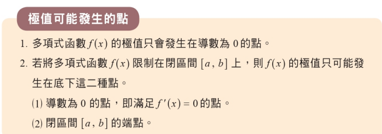
Definition of Extrema

Let function f be defined on an interval I containing a, b, c and d.

1. $f(a)$ is a **maximum** (or **local**[8] **maximum**) of f on I when $f(a) \geq f(x)$ for x around a in I.
 當在 a 的附近且在區間 I 上的每一個數 x 都滿足 $f(a) \geq f(x)$ 時，稱 $f(a)$ 為 $f(x)$ 在區間上的一個**極大值**（**局部極大值**）。
 $f(b)$ is the **absolute maximum** (or **global**[9] **maximum**) of f on I when $f(b) \geq f(x)$ for all x in I. 當區間 I 上的每一個數 x 都滿足 $f(b) \geq f(x)$ 時，稱 $f(b)$ 為 $f(x)$ 在區間 I 上的**最大值**（**絕對極大值**或**全域極大值**）。
2. $f(c)$ is a **minimum** (or **local minimum**) of f on I when $f(c) \leq f(x)$ for x around c in I.
 當在 c 的附近且在區間 I 上的每一個數 x 都滿足 $f(c) \leq f(x)$ 時，稱 $f(c)$ 為 $f(x)$ 在區間上的一個**極小值**（**局部極小值**）。
 $f(d)$ is the **absolute minimum** (or **global minimum**) of f on I when $f(d) \leq f(x)$ for all x in I. 當區間 I 上的每一個數 x 都滿足 $f(d) \leq f(x)$ 時，稱 $f(d)$ 為 $f(x)$ 在區間 I 上的**最小值**（**絕對極小值**或**全域極小值**）。

The minimum and maximum of a function on an interval are the **extreme**[10] **values**, or **extrema**[11] of the function on the interval. Extrema that **occur**[12] at the endpoints are called **endpoint extrema**.

極小值和極大值發生在區間內稱為**極值**，若極值發生在區間的端點則稱**端點極值**。

Material	Vocabulary
 極值可能發生的點 1. 多項式函數 $f(x)$ 的極值只會發生在導數為 0 的點。 2. 若將多項式函數 $f(x)$ 限制在閉區間 $[a, b]$ 上，則 $f(x)$ 的極值只可能發生在底下這二種點。 (1) 導數為 0 的點，即滿足 $f'(x) = 0$ 的點。 (2) 閉區間 $[a, b]$ 的端點。	13. continuous (連續的), 14. horizontal (水平的), 15. tangent (切線), 16. derivative (導數的), 17. domain

(定義域), 18. polynomial (多項式), 19. differentiable (可微分的).

Illustrations II

Extrema and Extreme Candidates

See Figure 2, for a **continuous**[13] function, a maximum occurs on a "hill" on the graph (like point C and F); a minimum occurs in a "valley" on the graph (like point E and G). The graph has a **horizontal**[14] **tangent**[15] line at the high point (or the low point). Hence, the common characteristic of these points is that they have horizontal tangent lines. And, the **derivative**[16] of the function at these points is equal to zero.

觀察圖 2，連續函數圖形的極大值發生在波峰（C, F 兩點）且極小值發生在波谷（E, G 兩點）；這些點的共同特徵為切線為水平切線，這四個發生極值的點其導數皆為 0。

In general, the **domain**[17] of the **polynomial**[18] function is all real numbers, which is an open interval. The extrema of a polynomial function can occur only at the numbers which the derivative of x is zero of the function. But, when it changes the domain to a closed interval $[a,b]$, the extrema can occur at the endpoints (like point A and B), where f is not **differentiable**[19] at.

一般而言，多項式函數的定義域為所有實數為一開區間，它的極值只會發生在導數為 0 的點。但若將定義域限制在閉區間 $[a,b]$ 上，則端點 A、B 也有可能是發生極值的點。

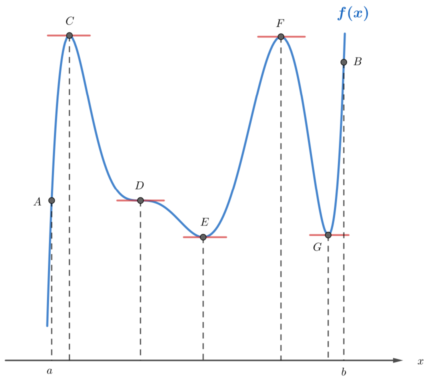


Figure 2

Points Where Extrema Might Occur

1. The extrema (maxima and minima) of a polynomial function $f(x)$ can occur at points where its derivative is zero. 多項式函數 $f(x)$ 的極值只會發生在導數為 0 的點。
2. There are two types of points where extrema might occur for a polynomial function $f(x)$ on a

closed interval $[a,b]$. 若將多項式函數限制在閉區間 $[a,b]$ 上，則 $f(x)$ 的極值只可能發生在下面兩種情形。

(1) The points where the derivative of the function is zero, denoted by $f'(x)=0$. 導數為 0 的點，則滿足 $f'(x)=0$ 的點。

(2) The endpoints of the closed interval $[a,b]$. 閉區間 $[a,b]$ 的端點。

Examples I

Find the extrema of $f(x)=x^3-3x+2$ on the interval $[-3,3]$.

Solution

Begin with differentiating the function.

$f'(x)=3x^2-3$ Differentiate.

Find all x-values for which $f'(x)=0$.

$3x^2-3=0$ Set $f'(x)$ equal to 0.

$3(x-1)(x+1)=0$ Factor.

$x=1\vee -1$ Solve the equation.

By evaluating f at these two numbers and at the endpoints of $[-3,3]$, we can determine that the maximum is $f(3)=20$ and the minimum is $f(-3)=-16$, as shown in the table. The graph of f is shown in Figure 3.

Left Endpoint $x=-3$	$f'(x)=0$ $x=-1$	$f'(x)=0$ $x=1$	Right Endpoint $x=3$
$f(-3)=-16$ Minimum	$f(-1)=4$	$f(1)=0$	$f(3)=20$ Maximum

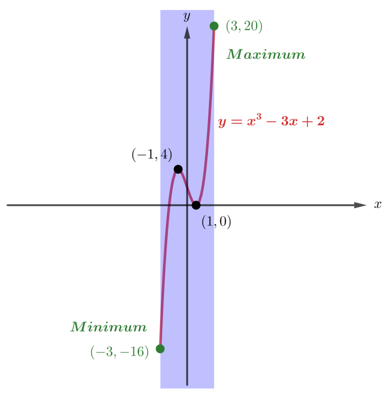


Figure 3

Material	Vocabulary
極值的一階檢定法 對於多項式函數 $f(x)$，先找出所有滿足 $f'(c)=0$ 的 c，再用下列方式針對這些 c 作檢定。 (1)若 $x=c$ 的附近滿足「當 $x<c$ 時，$f'(x)>0$；當 $x>c$ 時，$f'(x)<0$」，則 $f(c)$ 是極大值。 (2)若 $x=c$ 的附近滿足「當 $x<c$ 時，$f'(x)<0$；當 $x>c$ 時，$f'(x)>0$」，則 $f(c)$ 是極小值。	20. particular (特別的), 21. candidate (候選的), 22. identify (辨認), 23. increasing (遞增), 24. decreasing (遞減).

Illustrations III

When $f'(x)=0$ at a **particular**[20] point, we want to identify whether that point is an extreme **candidate**[21]. How can we **identify**[22] whether an extremum is a maximum or a minimum?

如何判斷滿足 $f'(x)=0$ 的「候選點」是否為發生極值的點？又如何區分一個極值是極大值或極小值？

After determining the intervals on which a function is **increasing**[23] or **decreasing**[24], it is not difficult to locate the relative extrema of the function.

在決定區間為遞增還是遞減之後，就可以容易的找到函數的極值。

1. In Figure 4(a), f has a relative maximum at $(c, f(c))$ because f is increasing immediately to the left of $x=c$ and decreasing immediately to the right of $x=c$. That is $f'(x)>0$ to the left of $x=c$, and $f'(x)<0$ to the right of $x=c$.

   圖 4(a) 中，函數在點 $(c, f(c))$ 有極大值，因為在函數 f 的左邊為嚴格遞增，右邊為嚴格遞減。即在 $x=c$ 處的左側 $f'(x)>0$，右側 $f'(x)<0$，此時 $f(c)$ 是極大值。

2. Similarly, in Figure 4(b), f has a relative minimum at $(c, f(c))$ because f is decreasing immediately to the left of $x=c$ and increasing immediately to the right of $x=c$. That is $f'(x)<0$ to the left of $x=c$, and $f'(x)>0$ to the right of $x=c$.

   圖 4(b) 中，函數在點 $(c, f(c))$ 有極小值，因為在函數 f 的左邊為嚴格遞減，右邊為嚴格遞增。即在 $x=c$ 處的左側 $f'(x)<0$，右側 $f'(x)>0$，此時 $f(c)$ 是極小值。

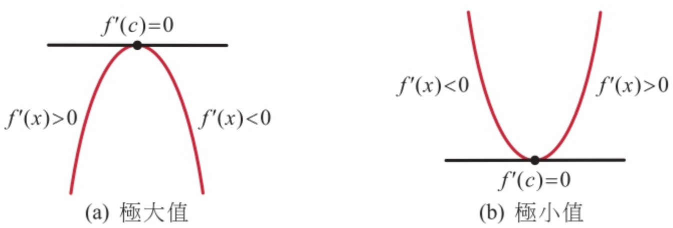


Figure 4

The First Derivative Test

Let c be the number of point C where $f'(c)=0$ for a polynomial function $f(x)$. The value $f(c)$ can be classified as follows. 我們將多項式函數 $f(x)$ 中所有滿足 $f'(c)=0$ 的點 C，其函數值 $f(c)$ 分類如下。

1. If $f'(x)$ changes from positive to negative at c, then f has a **relative maximum** at $(c,f(c))$.
   若 $f'(x)$ 在 $x=c$ 從左至右的值從正變負，則函數在點 $(c,f(c))$ 有極大值。若 $x=c$ 的附近滿足「當 $x<c$ 時，$f'(x)>0$；當 $x>c$ 時，$f'(x)<0$」，則是 $f(c)$ 極大值。
2. If $f'(x)$ changes from negative to positive at c, then f has a **relative minimum** at $(c,f(c))$.
   若 $f'(x)$ 在 $x=c$ 從左至右的值從負變正，則函數在點 $(c,f(c))$ 有極小值。若 $x=c$ 的附近滿足「當 $x<c$ 時，$f'(x)<0$；當 $x>c$ 時，$f'(x)>0$」，則是 $f(c)$ 極小值。

Note that if $f'(x)$ is positive on both side of c or negative on both side of c, then $f(c)$ is neither a relative minimum nor a relative maximum. 若 $f'(x)$ 在 $x=c$ 的左右兩側值皆正或皆負，則其值 $f(c)$ 不是極小值也不是極大值。

Examples II

Applying the First Derivative Test

Find the relative extrema of $f(x)=x^4-6x^2+5$

Solution

Note that f is continuous on the entire real number line. The derivative of f

$$f'(x)=4x^3-12x \quad \text{Differentiate.}$$

$$4x^3-12x=0 \quad \text{Set } f'(x) \text{ equal to 0.}$$

$$4x\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)=0 \quad \text{Factor.}$$

$$x=0\vee\sqrt{3}\vee-\sqrt{3} \quad \text{Solve the equation.}$$

The table summarizes the testing of the four intervals determined by these three numbers. By applying the First Derivative Test, we can conclude that f has a relative minimum at the point $\left(-\sqrt{3},-4\right)$, and another relative minimum at the point $\left(\sqrt{3},-4\right)$, a relative maximum at the point $(0,5)$, as shown in Figure 5.

Interval	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
Sign of $f'(x)$	$-$	0	$+$	0	$-$	0	$+$
Conclusion $f(x)$	Decreasing	-4	Increasing	5	Decreasing	-4	Increasing

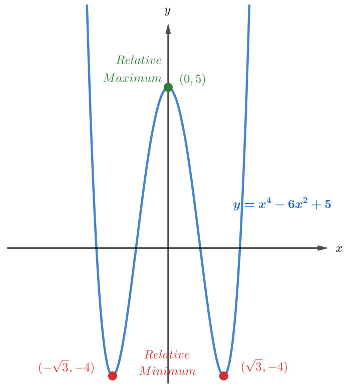
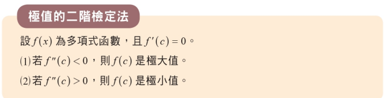


Figure 5

Material	Vocabulary
 極值的二階檢定法 設 $f(x)$ 為多項式函數，且 $f'(c)=0$。 (1)若 $f''(c)<0$，則 $f(c)$ 是極大值。 (2)若 $f''(c)>0$，則 $f(c)$ 是極小值。	25. concavity (凹向性).

Illustrations IV

The first derivative helps locate extrema by finding where a function is decreasing or increasing. Now, we introduce another test, the second derivative test, which is used to analyze the **concavity**[25] of a function and determine whether a point is a relative maximum or minimum.

極值的一階檢定法是利用函數的遞增遞減情形來判定極值。現在我們介紹二階檢定法，它是一個利用函數的凹向性來判定極值的方法。

When $f'(c)=0$,

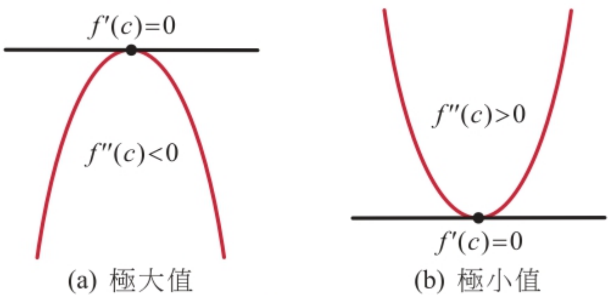


Figure 6

1. In figure 6(a), the graph of a function f is concave downward on an open interval containing c , and $f'(c)=0$, then $f(c)$ must be a relative maximum of f. 圖6(a)中，在 $x=c$ 處附近

的圖形是凹向下，此時 $f(c)$ 是極大值。

2. In figure 6(b), the graph of a function f is concave upward on an open interval containing c , and $f'(c)=0$, then $f(c)$ must be a relative minimum of f. 圖6(b)中，在 $x=c$ 處附近的圖形是凹向上，此時 $f(c)$ 是極小值。

Second Derivative Test

Let f be a polynomial function such that $f'(c)=0$ and the second derivative of f exists on an open interval containing c. 設 $f(x)$ 為多項式函數，且 $f'(c)=0$ 、函數的二階導數在包含 c 的開區間存在。

1. If $f''(c)<0$, then f has a relative maximum at $(c,f(c))$. 若 $f''(c)<0$ ，則 $f(c)$ 是極大值。
2. If $f''(c)>0$, then f has a relative minimum at $(c,f(c))$. 若 $f''(c)>0$ ，則 $f(c)$ 是極小值。

Examples III

Using the Second Derivative Test

Find the relative extrema of $f(x)=-x^4+8x^2-3$.

Solution

Differentiating twice produces the following.

$f'(x)=-4x^3+16x$ Find the first derivative.

$f''(x)=-12x^2+16$ Find the second derivative.

Set $f'(x)=0$, then we get $x=-2$,0, and 2.

We apply the Second Derivative Test as shown below.

Point	$(-2,13)$	$(0,-3)$	$(2,13)$
Sing of $f''(x)$	$f''(-2)<0$	$f''(0)>0$	$f''(2)<0$
Conclusion	Relative maximum	Relative minimum	Relative maximum

Hence, f has relative maxima at $(-2,13)$ and $(2,13)$, and a relative minimum at $(0,-3)$. The graph of f is shown in Figure 7.

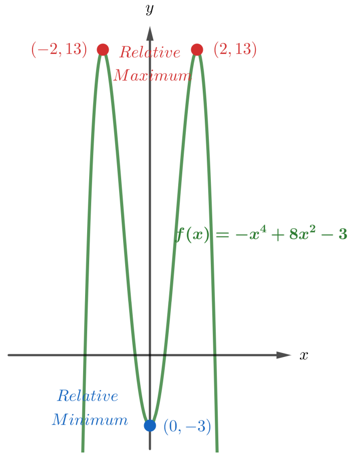


Figure 7

Examples IV

Using the Second Derivative Test

Find the relative extrema of $f(x) = x^4 + 4x^3 + 2$.

Solution

Differentiating twice produces the following.

$f'(x) = 4x^3 + 12x^2$ Find the first derivative.

$f''(x) = 12x^2 + 24x$ Find the second derivative.

Set $f'(x) = 0$, and we get $x = -3$ and 0.

We apply the Second Derivative Test as shown below.

Point	$(-3,-25)$	$(0,2)$
Sing of $f''(x)$	$f''(-3) > 0$	$f''(0) = 0$
Conclusion	Relative minimum	Test fails

Because the Second Derivative Test fails at $(0,2)$, we can use the First Derivative Test and observe that f increase to the left and right of $x = 0$.

Interval	$(-\infty,-3)$	-3	$(-3,0)$	0	$(0,\infty)$
Sign of $f'(x)$	$-$	0	$+$	0	$+$
Conclusion $f(x)$	Decreasing	-25	Increasing	2	Increasing

So, $(0,2)$ is neither a relative minimum nor a relative maximum. The graph of f is shown in Figure 8.

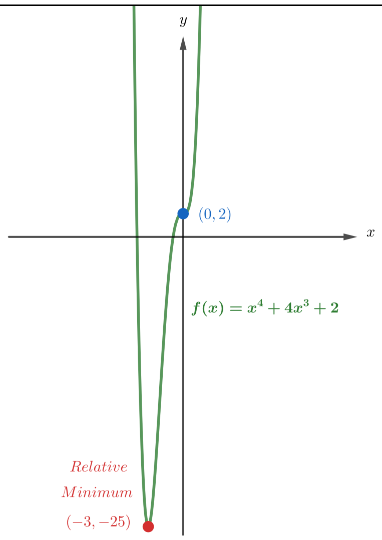


Figure 8

Thus, f has a minimum at $(-3,-25)$ and does not have a maximum.

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