The concept of functions

I. Key mathematical terms

Terms	Symbol	Chinese translation
Independent variable/ Dependent variable		
Domain/ Codomain		
Range		

II. Definition of a function

We learned the definition of a function in 1st grade. First, we'll revisit the definition we have learned:

Definition of a function (1st grade)

Let x and y be two variables. If the value of y changes according to the values taken by x, and for each x, there exists a unique correspondence with y, we refer to y as a function of x and represent it as $y = f(x)$.

After learning the concept of sets, we are now trying to define a function from a set perspective:

Definition of a function (set version)

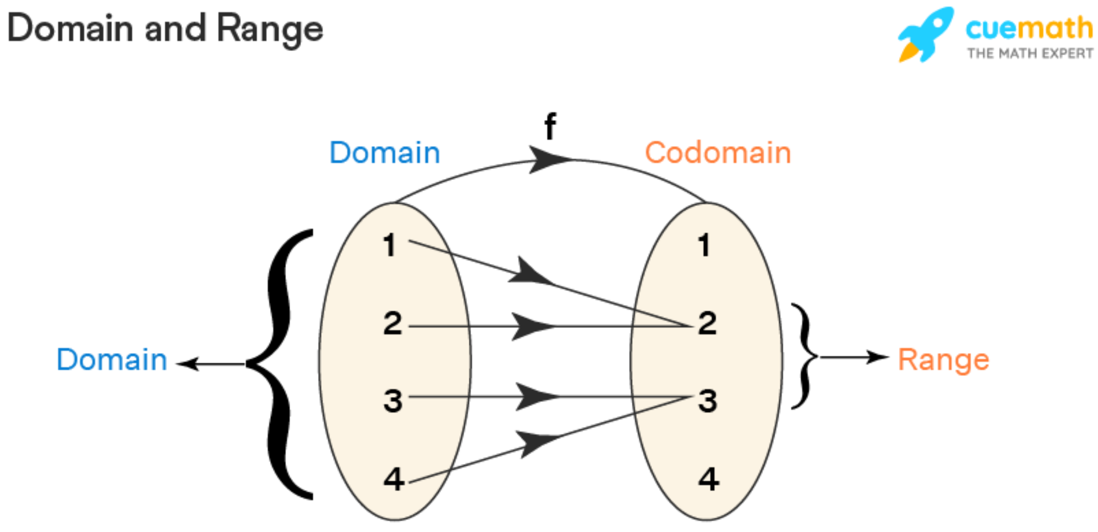
Let A and B be two nonempty sets. For any element x in set A, there is exactly one element y in set B such that x corresponds to y, this type of correspondence is called a function f from A to B, and the value corresponding to x is denoted as $f(x)$. We can represent this relation by:

$$f : A \to B$$

In such a relationship, x is the independent variable(自變數), y is the dependent variable(應變數). $f(x)$ represents the corresponding element of x in B, referred to as the function value of the function f at x. Set A is the domain(定義域) of function f, and set B is the codomain(對應域) of function f. The set formed by the values of the function $f(x)$, is the range(值域) of function f. It's denoted by:

$$\{ f(x) \mid x \in A \}$$ (It's the subset of set B)

<key> The relationship of these sets can be shown by the following graph:



https://www.cuemath.com/calculus/domain-and-range-of-a-function/

<key> Domain and Range

The **domain** of a function is the set of values that we are allowed to plug into our function. This set is the x values in function $f(x)$.

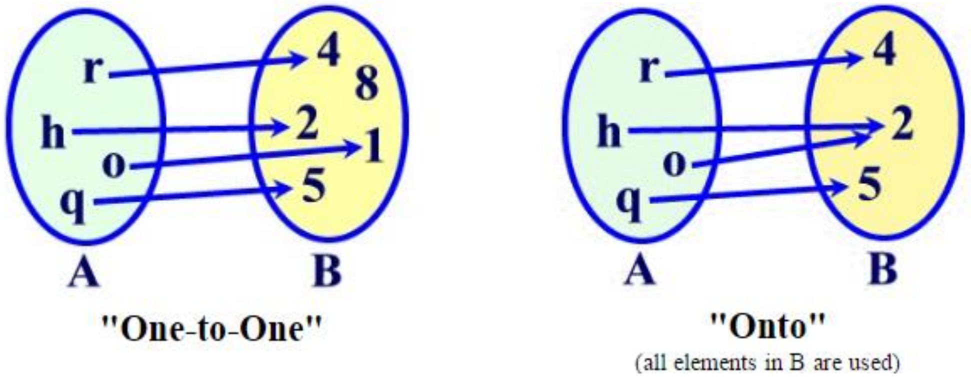
The **range** of a function is the set of values that the function assumes. This set consists of the y values corresponding to the plugged-in x values.

<key> One-to-one and onto

A function is said to be **one-to-one (injective)** if for all $f(a)=f(b)$ we have $a=b$. (Equivalently, if $a \neq b$, then $f(x) \neq f(y)$.)

A function maps elements form set A to set B. If for every element of B, there is at least one or more than one element matching with A. We said this function is an **onto(surjective)** function.

The figure below shows two functions, where the left-hand side is the one-to-one function and the right-hand side is the onto function.



https://www.aplustopper.com/one-to-one-and-onto-functions/

Example 1

Find the domain of the following functions:

(1) $f(x)=2^x$ (2) $g(x)=\log x$ (3) $h(x)=\sqrt{(2x-3)}$ (4) $k(x)=\dfrac{1}{(x-2)(x+3)}$

Example 2

Find the range of the following functions:

(1) $f(x)=\sin x$ (2) $g(x)=x^2+1$ (3) $h(x)=\sqrt{4-x^2}$ (4) $k(x)=-x^2+8x-10$

Odd and even function

Let A, B are two nonempty sets and $f: A\to B$

A function f is odd if for all $x\in A$, we have $f(-x)=-f(x)$. In this case, the function is symmetric with respect to the origin.

A function f is even if for all $x\in A$, we have $f(-x)=f(x)$. In this case the function is symmetric with respect to the y axis.

Example 3

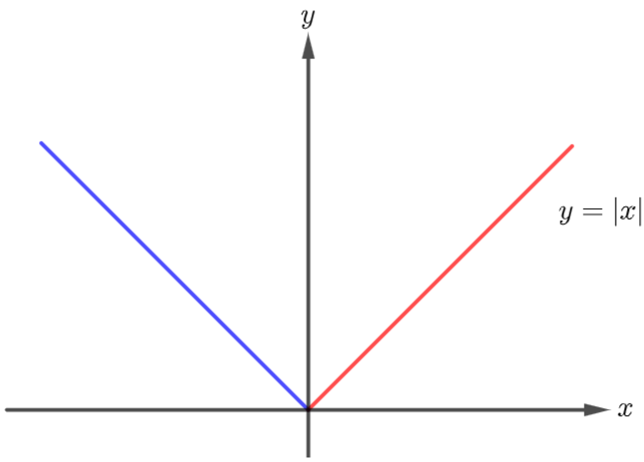
Determine whether the following function is an odd function, an even function, or neither.

(1) $f_1(x)=\sin x$ (2) $f_2(x)=\cos x$ (3) $f_3(x)=x^2$ (4) $f_4(x)=x^3-2x$

(5) $f_5(x)=\log x$ (6) $f_6(x)=x^5-3x^2+7$

III. Absolute value function and piecewise function

We know that the absolute value represents the distance of a number from zero. In other word it is the magnitude or size of a number, no negative allowed. Now we'll take about the "absolute value function". For example, an absolute value function $f(x)=|x|$ has graph shown below:



As you can see, on the right-hand side of the graph (the red line), the equation of the function is $f(x)=x$, and on the left-hand side of the graph (the blue line), the equation is $f(x)=-x$. In this function, whatever the input value is, the output is the value without regard to sing. We can define any absolute value function as a piecewise function (分段函數). In this case the absolute value function can be rewritten by:

$$f(x)=|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

A piecewise function is a function that is defined by different rules/formula for different intervals. In the previous case, when $x\geq 0$ the function will be $f(x)=x$ and when $x<0$ the function will be $f(x)=-x$.

A well-known piecewise function is the "greatest integer function" (floor function). Greatest integer function is a function that gives the greatest integer less than or equal to the given number, and we'll use the (square) bracket "[ ]" to represent this function. For example the greatest integer function $f(x)=[x]$, $f(-2)=[-2]=-2$, $f(2.5)=[2.5]=2$, $f(-1.5)=[-1.5]=-2$.

Example 4

Find the following values:

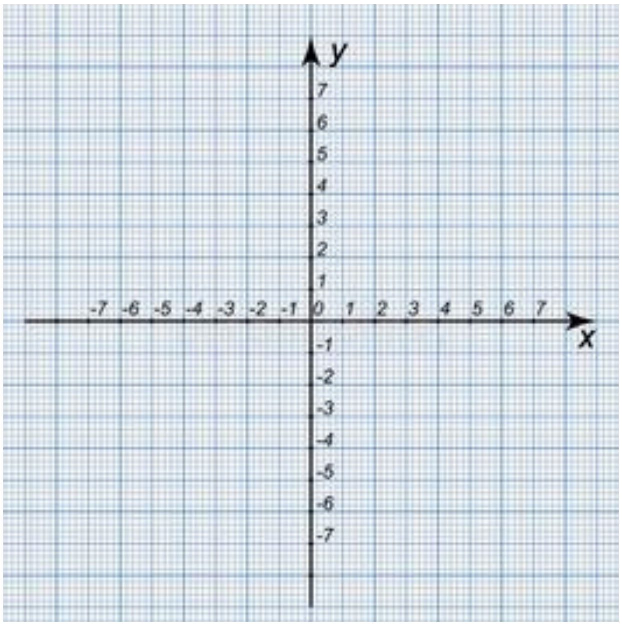
(1) $[1.3]$ (2) $[0]$ (3) $[-0.7]$ (4) $[\sqrt{8}]$ (5) $[\sqrt{7-2\sqrt{6}}]$

Example 5

Given a greatest integer function $g(x)=[2x-5]$.

(1) For $[2x-5]=3$, find the range of x.

(2) Plot the graph of $g(x)$



IV. Operations of functions and composite functions

Now we'll introduce the arithmetic of functions and composite functions. Be careful! The domain of the new function obtained through different operations may differ from the original functions' domain.

Operations of functions

Let f and g be functions defined on the sets A and B, where A and B are subsets of real numbers. The following definitions allow us to add, subtract, multiply and divide functions:

(1) Sum: $(f+g)(x)=f(x)+g(x)$.

(2) Difference: $(f-g)(x)=f(x)-g(x)$.

(3) Product: $(fg)(x)=f(x)g(x)$.

(4) Quotient: $(\frac{f}{g})(x)=\frac{f(x)}{g(x)}$. $(g(x)\neq 0)$

<key>The domain of the functions

The domain of the functions $f+g,\ f-g,\ fg,\ \dfrac{f}{g}$ is the intersection of the domain A and B. For example, function $f(x)=\sqrt{x}$ has domain $\{x\,|\,x\in\mathbb{R},\ x\geq 0\}$ and function $g(x)=x$ has domain $\{x\,|\,x\in\mathbb{R}\}$. Then the domain of function after addition operation ($f(x)+g(x)=x+\sqrt{x}$) will have domain $\{x\,|\,x\in\mathbb{R},\ x\geq 0\}$, which is the intersection of $\{x\,|\,x\in\mathbb{R},\ x\geq 0\}$ and $\{x\,|\,x\in\mathbb{R}\}$.

Example 6

Suppose function $f(x)=\sqrt{16-x^2},\quad g(x)=\sqrt{x},$ find the domain and range of the following functions:

(1) function $f(x)$ and $g(x)$.

(2) function $f(x)+g(x),\ f(x)g(x)$ and $\dfrac{f(x)}{g(x)}$.

Composition of functions

The composition of functions is the process of combining two or more functions into a single function. Many complex functions are compositions of some other simpler functions. For example, an absolute value function $h(x)=|x+5|$, can be considered as the composition of functions $f(x)=x+2$ and $g(x)=|x|$. First, we send x to $x+2$ with equation $f(x)$, and we send $x+2$ to $|x+2|$ with equation $g(x)$.

Composition of functions

Given two functions $g: A \to \mathbb{R}, f: B \to \mathbb{R},$

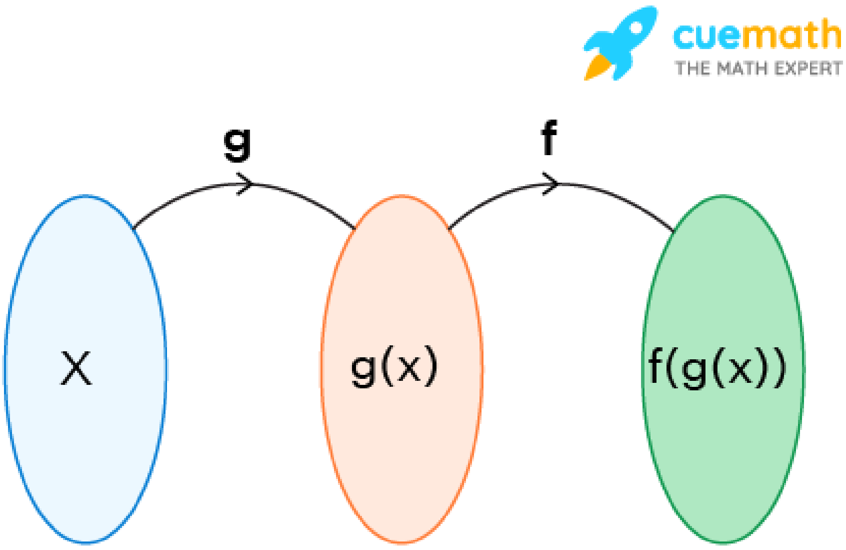
Then the composition of f and g denoted by $f \circ g$, is defined as the function:

$$(f \circ g)(x) = f(g(x)),$$

The domain of this composite function is determined by the domain of inner function $g(x)$ and the domain of the resulting composite function. $f(g(x))$.

<key>The mapping process can be represented by the following graph:

$$f \circ g : x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$



https://www.cuemath.com/calculus/composite-funtions/

Example 7

Let $f(x) = x^2 + 5x + 6,\ g(x) = \sqrt{x}\,.(f \circ g)(x),$

(1) Find the composite function $(f \circ g)(x),$ and the domain of function $(f \circ g)(x).$

(2) Find the composite function $(g \circ f)(x),$ and the domain of function $(g \circ f)(x).$

V. Inverse function

We've learned the exponential and logarithm functions. For example, an exponential function $f(x)=3^x$ and a logarithm function $g(x)=\log_3 x$ have opposite independent and dependent variables. Besides, the graphs of these two functions are symmetric about the line $x=y$. Actually, functions with these special relations are called "inverse function". Inverse functions in most general sense, are functions that "reverse" each other. For example, if a function $f(x)$ takes a to b, then the inverse function must take b to a.

Inverse function

Let A, B are two nonempty sets. If $f: A \to B,\ g: B \to A$ satisfy:

(1) For all $x \in B,\ f(g(x)) = x.$

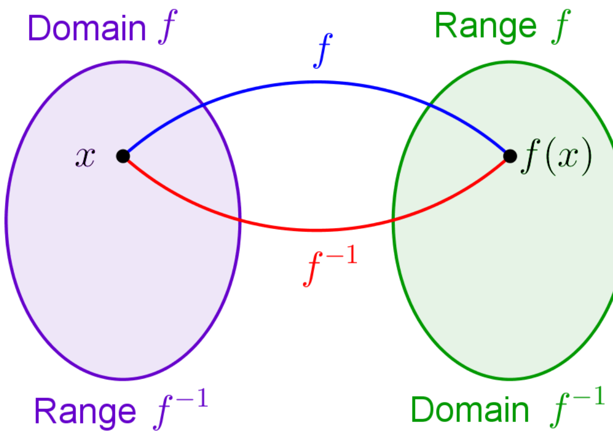
(2) For all $x \in A,\ g(f(x)) = x.$

Then $g(x)$ is the inverse of $f(x)$ and $f(x)$ is the inverse of $g(x)$.

We can represent the function $f(x)$ and $g(x)$ as follows:

$$f(x)=g^{-1}(x)\ \ (g \text{ inverse of } x),\ \ g(x)=f^{-1}(x)\ (f \text{ inverse of } x)$$

<key>The mapping process can be represented by the following graph:



https://olvereducation.weebly.com/6d---inverse-functions.html

<key> The domain of inverse function

The domain of $f(x)$ equals the range of $f^{-1}(x)$, the domain of $f^{-1}(x)$ equals the range of $f(x)$.

<key> The existence of inverse function

Not all functions have inverse functions. For example, $f(x)=x^2$ has domain $\mathbb{R}$ and range $\{y \mid y \geq 0\}$. For $f(2)=4,\ f(-2)=4$, but $f^{-1}(4)=2$ or -2 has two different values, which doesn't satisfy the definition of a function. Only one-to-one and onto function's inverse exists.

Example 8

Find the inverse and domain of the following functions:

(Hint: In some cases, the inverse function may not exists.)

(1) $f(x)=2x+1$

(2) $g(x)=\sqrt{x-1}$

(3) $h(x)=(g\circ f)(x)$

<資料來源>

1. **The concept of functions**
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   Calculus 9/e Metric Version-by-James-Jtewart
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3. **Arithmetic of functions and composite functions**
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4. **Inverse function**
   https://unacademy.com/content/cbse-class-11/study-material/mathematics/what-is-the-inverse-function-graph-like/
5. **南一書局數學甲下冊**

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