Topic: Sum and Difference Formula (or Compound Angle Formula)

1. Think –pair–share

Without using a calculator, evaluate $\sin 75^\circ$. Compare your answer with a calculator.

(Hint: $75^\circ = 45^\circ + 30^\circ = 90^\circ - 15^\circ = 60^\circ + 15^\circ$)

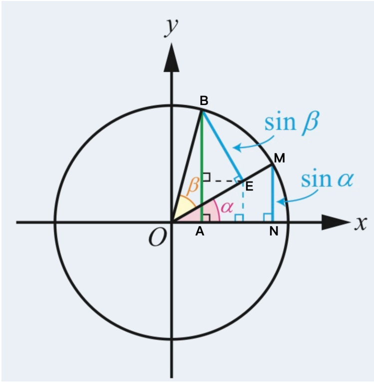
	使用建議
教學活動安排	1.利用$75^\circ = 45^\circ + 30^\circ = 90^\circ - 15^\circ = 60^\circ + 15^\circ$可拆成各種特殊角的和去引發學生連結角度的和與角度正餘弦值的和的關係 。如果有學生想起可以之前高一學過的方法，可鼓勵學生除了這個方法可以求以外還有沒有別的方法？ 2.關於和差角公式的英文名稱，比較常見的是Sum and difference formula, 不過也有名稱Compound Angle Formula,使用這兩個名稱都可以在網路上找到和差角公式的內容。
英文提問/開場	Today we are going to learn the sum and difference formula. By the name of this topic, you might think it's a formula talking about addition and subtraction. It's about trigonoemtric ratios addition and subtraction. Let's explore its idea through an example. How many ways do you have to evaluate the value of $\sin 75^\circ$ without using a calculator? Try to think of one way to do this, then share it with your partner. Then compare your answer with a calculator. From this activity, what ideas or insights come to your mind? Anyone? [Revised version by Chatgpt] Today, we're diving into something called the "sum and difference formula." Don't let the name confuse you; it's not about regular addition and subtraction, but it's all about adding and subtracting trigonometric ratios. Let's break it down with an example. So, here's a question: How many different ways can you figure out the value of sin75 without using a calculator? Take a moment to come up with one way, and then discuss it with your partner. After that, we'll check it against the calculator. Now, after this activity, what thoughts or insights did you come up with? Anyone want to share their findings?
參考資料	泰宇版高中數學（三）A。

2. Investigate the relationship between $\sin(\alpha+\beta)$ and $\sin\alpha+\sin\beta$

With the definition of trigonometric ratios for any angle, we

know that $\overline{AB}=\sin(\alpha+\beta)$, $\overline{MN}=\sin\alpha$, and $\overline{BE}=\sin\beta$

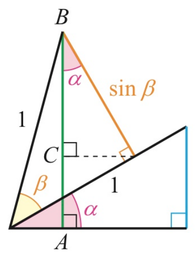
Compare the value of $\sin(\alpha+\beta)$ and $\sin\alpha+\sin\beta$.



使用建議	
教學活動安排	連結廣義三角比的定義，讓學生去看到$\sin(\alpha+\beta)\neq\sin\alpha+\sin\beta$
英文提問／開場	From the activity of evaluation of $\sin 75°$, we notice that $\sin(\alpha+\beta)\neq\sin\alpha+\sin\beta$. But why? We've learned the definition of trigonometric ratios for any angle, let's have a look to why $\sin(\alpha+\beta)\neq\sin\alpha+\sin\beta$, and compare the two values. Which one is the larger? What makes you say that? Anyone? More chanllenging questions, Is it always that $\sin(\alpha+\beta)<\sin\alpha+\sin\beta$? On what condition, $\sin(\alpha+\beta)>\sin\alpha+\sin\beta$?
參考答案	此圖觀察可以看出$\sin(\alpha+\beta)>\sin\alpha+\sin\beta$ 。若考慮此圖中的角度是變動，則答案為不一定。
參考資料	泰宇版高中數學（三） A。

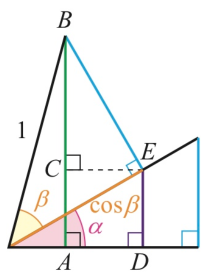
3. Investigate $\sin(\alpha+\beta)$

Read the following proof and fill in the blanks.



As shown in the graph, $\sin(\alpha+\beta)$ corresponds to the green-colored segment $\overline{AB}$, and the length of this segment is $\overline{AC}+\overline{BC}$, indicating that $\overline{AB}=\overline{AC}+\overline{BC}$.

$\overline{BC}$ can be represented by__________.(Hint: trigonometric ratio)



As shown in the graph, $\overline{AC}=\overline{DE}$ (purple-colored segment)

$\overline{DE}$ can be represented by__________.(Hint: trigonometric ratio)

So, $\sin(\alpha+\beta)=$ ______________.

使用建議	
教學活動安排	從前面的例子讓學生去利用提示去探索$\sin(\alpha+\beta)$到底等於什麼。
英文提問/開場	Now we know that $\sin(\alpha+\beta)\neq\sin\alpha+\sin\beta$. Then we wonder what $\sin(\alpha+\beta)$ is exactly equal to. Let's read the following proof and fill in the blanks to investgate what $\sin(\alpha+\beta)$ is exactly equals to. We'll check-in in five minutes. Any questions about the proof? Anyone wants to share your answer?
參考答案	$\overline{BC}=\sin\beta\cos\alpha$ $\overline{DE}=\cos\beta\sin\alpha$ $\sin(\alpha+\beta)=\overline{BC}+\overline{DE}=\sin\beta\cos\alpha+\cos\beta\sin\alpha$

使用建議	
英文提問/開場	Now we know that $\sin(\alpha+\beta) \neq \sin\alpha + \sin\beta$. Then we wonder what $\sin(\alpha+\beta)$ is exactly equal to. Let's read the following proof and fill in the blanks to investgate what $\sin(\alpha+\beta)$ is exactly equals to. We'll check-in in five minutes. Any questions about the proof? Anyone wants to share your answer?
參考答案	$\overline{BC} = \sin\beta\cos\alpha$ $\overline{DE} = \cos\beta\sin\alpha$ $\sin(\alpha+\beta) = \overline{BC} + \overline{DE} = \sin\beta\cos\alpha + \cos\beta\sin\alpha$
參考資料	泰宇版高中數學（三）A。

4. The sum and difference formula

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

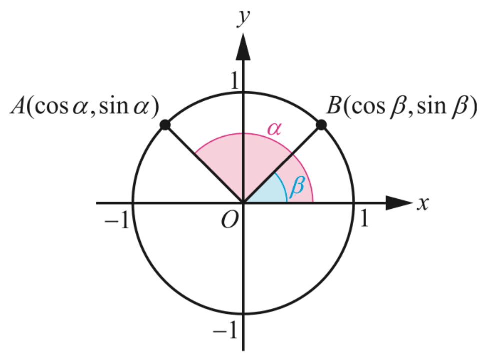
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

使用建議	
教學活動安排	介紹和差角公式。

使用建議	
英文提問/開場	From the investigation, we know that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. We would wonder that if there is a similar identity for $\sin(\alpha-\beta)$? YES. It is $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ Also, there are cosine versions. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ These four equations are called “the sum and difference formula”. And we definitely have a tangent version, it will be introduced in the next class.
參考資料	泰宇版高中數學（三）A。

5. The proof of the difference formula



As shown, consider A and B as any two points on the unit circle. $\angle AOB=\alpha-\beta$.

Using the distance formula, we get

$\overline{AB}^2=(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2$

$=\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta$ Expand the bracket.

$=2-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$ …(1) Simplify

And, by the cosine rule in $\triangle OAB$,

$\overline{AB}^2=1^2+1^2-2\times1\times1\times\cos(\alpha-\beta)=2-2\cos(\alpha-\beta)$…(2)

Comparing (1) and (2), we get $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.

Also, $\sin(\alpha-\beta)=\cos[\frac{\pi}{2}-(\alpha-\beta)]=\cos[(\frac{\pi}{2}-\alpha)-(-\beta)]$

$=\cos(\frac{\pi}{2}-\alpha)\cos(-\beta)+\sin(\frac{\pi}{2}-\alpha)\sin(-\beta)$

$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

6. The alternative proof of the sum formula

Read the proof and fill in the blanks.

By the difference formula $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$, let’s substitute β into $-\beta$.

$\cos(\alpha+\beta)=\cos(\alpha-(-\beta))=$ ____.

By the difference formula $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$, let’s substitute β into $-\beta$.

$\sin(\alpha+\beta)=\sin(\alpha-(-\beta))=$ ____.

使用建議	
教學活動安排	前面的例子及證明都只是銳角的情形，現在要用公式證明對於任何角度都成立。
英文提問／開場	In this activity, we just proved the formula for acute angles. How do we prove the formula for any type of angles? Here’s what we are going to do. Let’s start with the easy one $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$. Let’s check the proof. Now we finish the proof of the difference formula, it’s your turn to prove the sum formula. Don’t worry, read the proof and fill in the blanks then you’ll complete the proof. We’ll check-in in five minutes. Let’s check the answer. Any questions? Good!
參考資料	泰宇版高中數學（三）A。

7. Use the sum and difference formula to find the exact value of $\sin 75^\circ$ and $\sin 15^\circ$.

使用建議	
教學活動安排	練習代公式
英文提問/開場	Let's use the sum and difference formula to find the exact value of $\sin 75^\circ$ and $\sin 15^\circ$. We'll check-in in five minutes.
參考答案	$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

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