條件機率與貝氏定理

Conditional Probability and Bayes' Theorem

Material	Vocabulary
條件機率 當 A, B 為兩事件且 $P(A)>0$ 時，將「在事件 A 發生的條件下，事件 B 發生的機率」稱為條件機率，以符號 $P(B\|A)$ 表示，也就是 $P(B\|A)=\frac{n(A\cap B)}{n(A)}=\frac{P(A\cap B)}{P(A)}$。 符號 $P(B\|A)$ 讀作「在 A 發生的條件下，B 發生的機率」。	1. conditional (條件的), 2. probability (機率), 3. outcome (結果), 4. sample (樣本), 5. event (事件), 6. occur (發生), 7. equally likely (均等), 8. fair (公正), 9. rescale (重新調節), 10. underlying (在…之下), 11. shrunk (縮小/shrink 的動詞過去式), 12. denominator (分母), 13. numerator (分子), 14. freestyle stroke (自由式), 15. backstroke (仰式).

Illustrations I

Conditional[1] Probability[2]

Let's recall what we have learned from book 2 as follows.

1. An **experiment** is a repeatable process that gives rise to a number of **outcomes**[3].

2. A **sample**[4] **space** is the set of all possible outcomes of an experiment.

3. An **event**[5] is a collection (or set) of one or more outcomes.

條件機率

複習第二冊學過的機率：

1. 對於可重複進行觀察的不確定現象，觀察並求出一個結果的過程稱為試驗。

2. 一項試驗中所有可能發生的結果所成的集合，稱為這試驗的樣本空間。

3. 樣本空間 S 的任一子集都稱為一個事件。

The probability of an event is the chance that the event will **occur**[6] as a result of an experiment. Where outcomes are ***equally likely***[7], the **probability of an event** is the number of outcomes in the event A divided by the total number of possible outcomes in the sample space S.

$$P(\text{event})=\frac{\text{Number of equally likely outcomes}}{\text{Total number of possible outcomes}}=\frac{n(A)}{n(S)}$$

設一試驗的樣本空間 S 之樣本點為有限個。當 S 中每個樣本點出現的機會均等時，定義事件 A 發生的機率 $P(A)$ 為：事件 A 的樣本點個數除以樣本空間的樣本點個數

$$P(A)=\frac{n(A)}{n(S)}$$ 。

For example, a **fair**[8] dice is rolled, where the sample space is $\{1,2,3,4,5,6\}$, and event A is "the numbers rolled are greater than 3, which is $\{4,5,6\}$." So, the probability of event A is $\frac{3}{6}=\frac{1}{2}$. However, if the die is rolled and lands on an even number, is the probability still the same as $\frac{1}{2}$?

例如：擲一粒公正骰子一次，其樣本空間為{1 , 2 , 3 , 4 , 5 , 6}，因為擲出點數大於 3 的事件為{4 , 5 , 6}，所以擲出點數大於 3 的機率為 $\frac{3}{6}=\frac{1}{2}$ 。

Consider t rolling a fair die and the events A and B, as defined below.

A: the outcome is an even number (2, 4, or 6)

B: the outcome is greater than 3 (4, 5, or 6).

See figure 1, we knew that the outcome is an even number (event A has happened) so we have a conditional probability which is not the same as $P(B)$. We are looking for the probability of *B* given *A* has happened so we divide $n(B\cap A)$ by $n(A)$ to **rescale**[9] the probability of event *A* to $n(S)$. So, $P(B \text{ given } A)=\frac{n(B\cap A)}{n(A)}=\frac{2}{3}$. You can think of it as the **underlying**[10] sample space *S* "**shrunk**[11]" to the set *A*.

考慮「在擲出點數為偶數的條件下」設 *A* 表示擲出點數為偶數的事件，*B* 表示擲出點數大於 3 的事件，即 A = {2 , 4 , 6} , B = {4 , 5 , 6}。

如圖1 所示。因為已知擲出點數為偶數（即事件A 發生），所以條件機率不等於 $P(B)$。在事件 *A* 發生的條件下，事件 *B* 發生的機率為 $n(B\cap A)$除以 $n(A)$，分母從樣本空間 $P(S)$ 縮放成 $n(A)$。所以， $P(B\mid A)=\frac{n(B\cap A)}{n(A)}=\frac{2}{3}$ 。可以想成將樣本空間由原先的 *S* 限縮到 *A*。

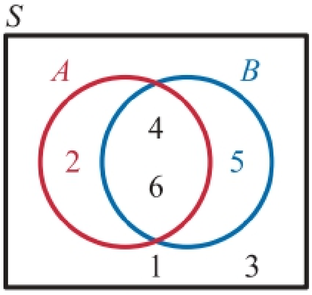


Figure 1

The probability of B given A, written $P(B|A)$, is called the **conditional probability** of B given A and so : $P(B|A)=\dfrac{n(B\cap A)}{n(A)}$. Divide both the **denominator**[12] and the **numerator**[13] by $n(S)$, then we have

$$P(B|A)=\frac{\frac{n(B\cap A)}{n(S)}}{\frac{n(A)}{n(S)}}=\frac{P(B\cap A)}{P(S)}$$

我們把「在事件 A 發生的條件下，事件 B 發生的機率」，稱為條件機率，記作 $P(B|A)$：在事件 A 發生的條件下，事件 B 發生的機率為 $P(B|A)=\dfrac{n(B\cap A)}{n(A)}$，再將分子與分母同時除以 $n(S)$，得

$$P(B|A)=\frac{\frac{n(B\cap A)}{n(S)}}{\frac{n(A)}{n(S)}}=\frac{P(B\cap A)}{P(S)}\text{。}$$

Note:

Conditional probabilities are usually written using the symbol “|” to mean given that. We read $P(B|A)$ as “the probability that B occurs, given that A occurs.”

條件機率用符號「|」表示「在什麼條件下」，$P(B\mid A)$ 讀作「在 A 發生的條件下，B 發生的機率」

Examples I

Two children are selected at random from a group of five boys and seven girls. Find the probability that the second child selected is a boy, given that the first child selected is also a boy.

Solution

Let B stand for the event "the second child selected is a boy" and A the event "the first child selected is also a boy." If a boy is selected first, then the second child is selected from four boys and seven girls.

$$P(B|A)=\frac{4}{11}$$

Examples II

A class of students were each asked whether they can swim in the **freestyle stroke**[14] and whether they can swim in the **backstroke**[15]. There are 28 students who can swim in the freestyle stroke, 20 students who can swim in the backstroke, and 12 students who can do both. Find the probability that a randomly selected student can:

(1) swim in the backstroke, given that they can swim in the freestyle stroke.

(2) swim in the freestyle stroke, given that they can swim in the backstroke.

Solution

Let *A* be the event "a selected student can swim in the freestyle stroke" and *B* be the event "a selected student can swim in the backstroke."

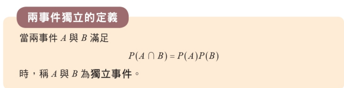
$$n(A)=28,\ n(B)=20,\ n(A\cap B)=12$$

(1) The probability that B occurs, given that A occurs is

$$P(B|A)=\frac{n(A\cap B)}{n(A)}=\frac{12}{28}=\frac{3}{7}$$

(2) The probability that A occurs, given that B occurs is

$$P(A|B)=\frac{n(A\cap B)}{n(B)}=\frac{12}{20}=\frac{3}{5}$$

Material	Vocabulary
 兩事件獨立的定義 當兩事件 A 與 B 滿足 $P(A\cap B)=P(A)P(B)$ 時，稱 A 與 B 為獨立事件。	16. independent (獨立的), 17. indicating (表明), 18. dependent (依賴的).

Illustrations II

Two or more events are **independent**[16] if the occurrence of each event does not affect the occurrence of the other. For instance, suppose *A* and *B* each roll a fair die.

Event *A*: Person *A* rolls a 6.

Event *B*: Person *B* rolls a 6.

The occurrence of Event *A* does not affect the occurrence of Event *B*, and vice versa.

事件B 發生的機率不因事件A 的發生與否而受到影響，此時稱A 與B 為獨立事件。例如：甲、乙兩人各擲一次公正骰子，並令

A 表示「甲擲出6 點」的事件，

B 表示「乙擲出6 點」的事件。

甲是否擲出 6 點並不會影響乙擲出 6 點的機率，反之亦然。

Another example, roll a fair die once.

Event *A*: rolling a prime number.

Event *B*: rolling a 2 or 3.

So, we have $A=\{2,3,5\}$, $B=\{1,2\}$ and $A\cap B=\{2\}$, and

(i) probability of *B* : $P(B)=\frac{2}{6}=\frac{1}{3}$;

(ii) the probability that *B* occurs, given that *A* occurs is: $P(B|A)=\frac{n(A\cap B)}{n(A)}=\frac{1}{3}$.

再例如：擲一粒公正骰子一次，並令

A 表示「出現質數點」的事件，

B 表示「出現1 點或2 點」的事件。

因為 $A=\{2,3,5\}$，$B=\{1,2\}$，且 $A\cap B=\{2\}$，所以

(i) 事件*B*發生的機率為 $P(B)=\frac{2}{6}=\frac{1}{3}$；

(ii) 在事件 *A* 發生的條件下，事件 *B* 發生的機率為 $P(B|A)=\frac{n(A\cap B)}{n(A)}=\frac{1}{3}$。

According to (i) and (ii), we get $P(B)=P(B|A)$. That means the occurrence of *A* does not affect the probability that *B* occurs. We say A and B are **independent events**.

由(i)與(ii)得知，$P(B)=P(B|A)$。也就是說事件 *B* 發生的機率不因事件 *A* 的發生與否而受到影響，此時稱 *A* 與 *B* 為獨立事件。

Moreover, if we define event C be "rolling a 2, 3 or 4," then we have $A \cap C = \{2,3\}$ and $P(C) = \frac{3}{6} = \frac{1}{2}$. However, $P(C|A) = \frac{n(A \cap C)}{n(A)} = \frac{2}{3}$. This demonstrates that $P(C) \neq P(C|A)$, **indicating**[17] that the occurrence of A affects the probability of C occurring. We say A and C are **dependent**[18] **events**.

然而，若令 C 表示「出現1點或2點或3點」的事件，因為 $A \cap C = \{2,3\}$ 所以 $P(C) = \frac{3}{6} = \frac{1}{2}$。而 $P(C|A) = \frac{n(A \cap C)}{n(A)} = \frac{2}{3}$，顯然事件 A 的發生會影響到事件 C 發生的機率。也就是說，A 與 C 並非獨立事件，稱 A 與 C 為相依事件。

When A and B are independent events, and $P(A) > 0$, we can write the definition using the multiplication rule of conditional probability: $P(A \cap B) = P(A)P(B|A) = P(A)P(B)$. Particularly, when $P(A) = 0$, $P(A \cap B) = P(A)P(B)$ still holds.

當 A 與 B 為獨立事件且 $P(A) > 0$ 時，可由條件機率的乘法定理，得

$$P(A \cap B) = P(A)P(B|A) = P(A)P(B) 。$$

特別地，當 $P(A) = 0$ 時，$P(A \cap B) = P(A)P(B)$ 亦成立。

Examples III

Toss a fair coin three times. Consider the following events:

Event A: At least two heads.

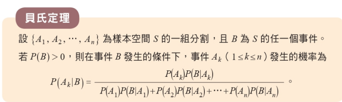
Event B: Same side for three times.

Are events A and B independent? Justify your answer.

Solution

From the info of the question, we have $P(A) = \frac{4}{8} = \frac{1}{2}$, $P(B) = \frac{2}{8} = \frac{1}{4}$ and $P(A \cap B) = \frac{1}{8}$.

Since $P(A \cap B) = \frac{1}{8} = \frac{1}{2} \times \frac{1}{4} = P(A)P(B)$, A and B are independent events.

Material	**Vocabulary**
 貝氏定理 設 $\{A_1, A_2, \cdots, A_n\}$ 為樣本空間 S 的一組分割，且 B 為 S 的任一個事件。若 $P(B) > 0$，則在事件 B 發生的條件下，事件 A_k（$1 \le k \le n$）發生的機率為 $P(A_k\|B) = \frac{P(A_k)P(B\|A_k)}{P(A_1)P(B\|A_1) + P(A_2)P(B\|A_2) + \cdots + P(A_n)P(B\|A_n)}$。	19. scenario (設想), 20. partitioned (分割), 21. mutually exclusive (互斥), 22. alternatively (或者), 23. marbles (彈珠), 24. sequentially (依序地).

Illustrations III

In a class with an equal number of boys and girls, where 40% of the boys and 60% of the girls passed the language assessment. Select any student from the class. What is the probability that the student passed the test, given that she is a girl? Consider the following events:

某班男女生人數各半，男生中有 40%的人通過英檢，女生中有 60%的人通過英檢。班上任選一學生，已知該生通過英檢，那麼此人是女生的機率為多少呢？

Event A_1 : Select a boy.

Event A_2 : Select a girl.

Event B : Select a person who passed the language assessment.

Accordingly, we have

$P(A_1)=0.5$, $P(A_2)=0.5$, $P(B|A_1)=0.4$, $P(B|A_2)=0.6$.

We can represent this both on a Venn diagram and a tree diagram as Figure 2 shown.

A_1 表示選出者是男生的事件

A_2 表示選出者是女生的事件

B 表示選出者通過英檢的事件。

根據上述可得 $P(A_1)=0.5$, $P(A_2)=0.5$, $P(B|A_1)=0.4$, $P(B|A_2)=0.6$ 。

並將這些事件的關係以文氏圖與樹狀圖呈現如圖 2。

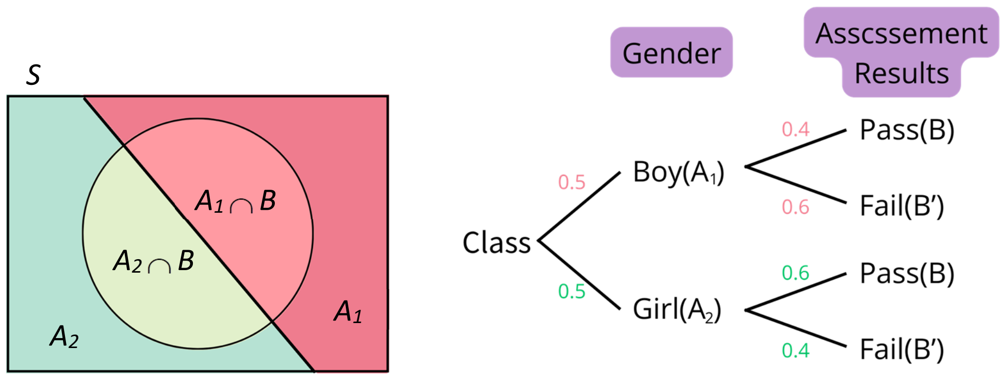


Figure 2

The probability that the student passed the test, given that she is a girl is

$$\frac{0.5\times 0.6}{0.5\times 0.4+0.5\times 0.6}=\frac{3}{5}=0.6.$$

In the **scenario**[19] described above, we determine probabilities using Venn Diagrams and Tree Diagrams. Moreover, we have the support of a fundamental theorem known as Bayes' Theorem.

A sample space *S* is **partitioned**[20] into three by the **mutually exclusive**[21] events A_1, A_2 and A_3. The sample space also contains another event *B*. We can represent this on a Venn diagram as Figure 3 shown.

由文氏圖或樹狀圖，可得某生通過英檢，此人是女生的機率為

$$\frac{0.5\times0.6}{0.5\times0.4+0.5\times0.6}=\frac{3}{5}=0.6.$$

上述的情境我們雖然是透過文氏圖或樹狀圖求得其機率，然而事實上其背後是有著嚴謹的數學理論支持，稱之為貝氏定理。

設 A_1，A_2，A_3 為樣本空間 S 的三個事件。當這三個事件兩兩互斥且 $A_1\cup A_2\cup A_3=S$ 時，我們把 $\{A_1,A_2,A_3\}$ 稱為樣本空間 S 的一組分割，如圖 3 所示。

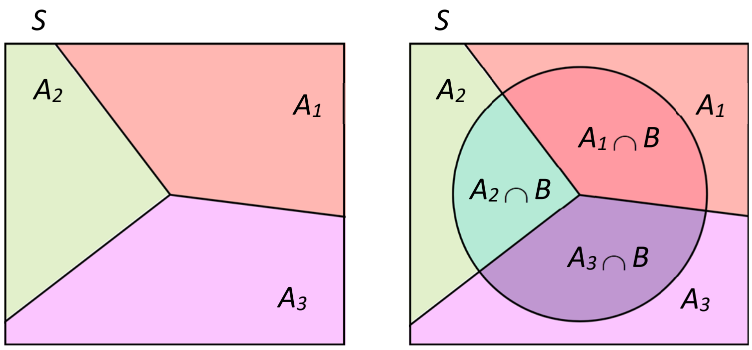


Figure 3

For event *B* in the sample space *S*,

$$P(B)=P(A_1\cap B)+P(A_2\cap B)+P(A_3\cap B).$$

The conditional probability formula:

$$P(A_1|B)=\frac{P(A_1\cap B)}{P(B)}=\frac{P(A_1\cap B)}{P(A_1\cap B)+P(A_2\cap B)+P(A_3\cap B)}.$$

We can express each $P(A_i\cap B)$ by applying the multiplication rules of conditional probability:

$$P(A_i\cap B)=P(A_i)P(B|A_i),\ i=1,2,3.$$

So, the conditional probability can **alternatively**[22] be written as:

$$P(A_1|B)=\frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+P(A_3)P(B|A_3)}.$$

Similarly, we can obtain $P(A_2|B)$ and $P(A_3|B)$.

事件 B 也會被分割成 $A_1\cap B$, $A_2\cap B$, $A_3\cap B$ 兩兩互斥的三個事件，如圖3所示。因此，事件B 發生的機率就是這三個兩兩互斥事件的機率總和，即

$$P(B)=P(A_1\cap B)+P(A_2\cap B)+P(A_3\cap B)\text{。}$$

利用條件機率的乘法定理，可將上式中的每一個$P(A_i\cap B)$ 分別改寫成

$$P(A_i\cap B)=P(A_i)P(B|A_i),\ i=1,2,3\text{。}$$

因此，條件機率可被寫成

$$P(A_1|B)=\frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+P(A_3)P(B|A_3)}\text{。}$$

同理，我們也可求得 $P(A_2|B)$ 與 $P(A_3|B)$。

Bayes' Theorem

Let S be a sample space that is partitioned into n mutually exclusive events $\{A_1,A_2,\cdots,A_n\}$, with an additional event B contained within the sample space. If $P(B)>0$, then the conditional probability of event A_k ($1\le k\le n$) occurring, given that B occurs, is defined as

$$P(A_k|B)=\frac{P(A_k)P(B|A_k)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+\cdots+P(A_n)P(B|A_n)}$$

貝氏定理

設$\{A_1,A_2,\cdots,A_n\}$為樣本空間 S 的一組分割，且 B 為 S 的任一個事件。若$P(B)>0$，則在事件 B 發生的條件下，事件 A_k ($1\le k\le n$) 發生的機率為

$$P(A_k|B)=\frac{P(A_k)P(B|A_k)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+\cdots+P(A_n)P(B|A_n)}\text{。}$$

Examples IV

A can contains 4 blue **marbles**[23] and 2 green marbles. Two marbles are drawn **sequentially**[24] without replacement, and its color is noted. Find the probability that:

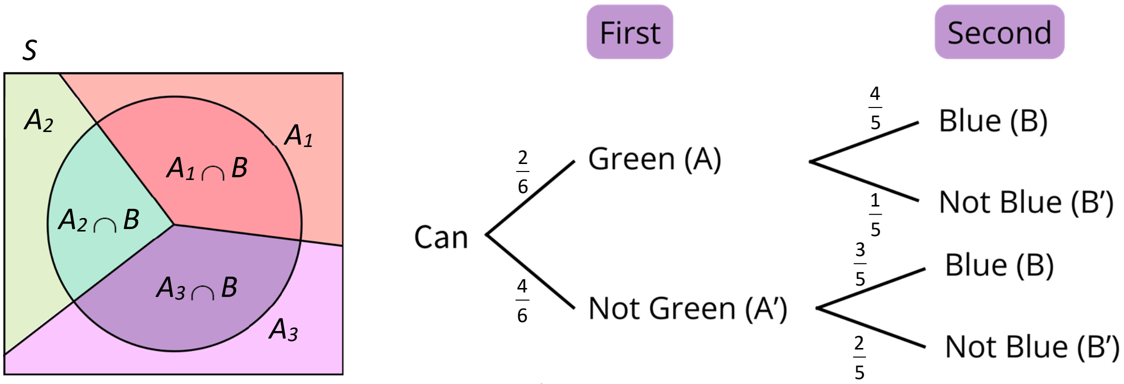
(i) the second marble is blue;

(ii) the first marble was green, given that the second marble is blue.

Solution

Let A be the event that the first marble is green and B the event that the second marble is blue.

From the info, we have $P(A)=\frac{2}{6}$, $P(A')=\frac{4}{6}$, $P(B|A)=\frac{4}{5}$, $P(B|A')=\frac{3}{5}$. We can represent this both on a Venn diagram and a tree diagram as Figure 4 shown.



Figrue 4

(i) The probability of the second marble is blue.

$$P(B)=P(B|A)P(A)+P(B|A')P(A')$$
$$=\frac{4}{5}\times\frac{2}{6}+\frac{3}{5}\times\frac{4}{6}$$
$$=\frac{2}{3}$$

(ii) The probability of the first marble was green, given that the second marble is blue.

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$
$$=\frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A')P(A')}$$
$$=\frac{\frac{4}{5}\times\frac{2}{6}}{\frac{4}{5}\times\frac{2}{6}+\frac{3}{5}\times\frac{4}{6}}$$
$$=\frac{2}{5}$$

References

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