雙語教學主題(國中九年級上學期教材): 相似三角形及其應用

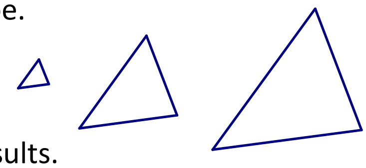
Topic: Triangle similarity theorems and applications

Vocabulary

theorem, similar, similarity, polygon, corresponding sides, corresponding angles, dilate, shrink, enlarge, proportional sides, congruent angles, triangle congruence side length, measure,

We have learned in the last class that when we dilate a geometric shape, we get a new geometric shape that is similar to the original shape.

For instance,



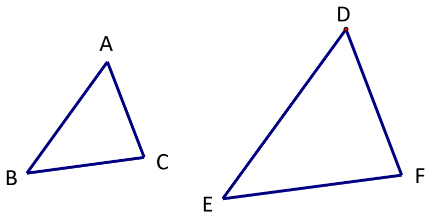
No matter how we dilate (shrink or enlarge) a triangle,
all the outcomes(these triangles) have the following results.
Three pairs of corresponding angles are congruent and
three pairs of corresponding side lengths are proportional.

Now what we are going to discuss is:

If we want to know whether two triangles $\triangle ABC$ and $\triangle DEF$ are similar, do we need to show all the information that

$\angle A=\angle D$, $\angle B=\angle E$, and $\angle C=\angle F$, $\frac{\overline{AB}}{\overline{DE}}=\frac{\overline{AC}}{\overline{DF}}=\frac{\overline{BC}}{\overline{EF}}$

In order to tell $\triangle ABC \sim \triangle DEF$?



The answer is NO. Remember when we learned triangle congruence in the eighth grade, we know we only need one of the theorems like SSS, SAS, AAS, ASA, and RHS to see whether two triangles are congruent. When we discuss triangle similarity, we also have theorems like these.

1. AA similarity theorem
2. SAS similarity theorem
3. SSS similarity theorem

Let's check it out together.

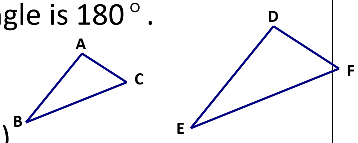
1. AA similarity theorem

   In two triangles, if any two pairs of corresponding angles are congruent, these two triangles are similar. Let's do the explanation here.

   Some information will show the AA similarity theorem as AAA similarity theorem. Let me explain it here.

The AA similarity theorem is the AAA similarity theorem

We know the sum of the three interior angles of a triangle is 180°.



in $\triangle ABC$ and $\triangle DEF$, if $\angle A=\angle D$, and $\angle B=\angle E$,

then $\angle C=180°-(\angle A+\angle B)$

(the sum of the three interior angles is 180°)

$=180°-(\angle D+\angle E)$ (replace $\angle A$ with $\angle D$, and $\angle B$ with $\angle E$)

$=\angle F$

(the sum of the three interior angles is 180°)

When any two corresponding angles are congruent in two triangles, the third corresponding angles in these two triangles must be congruent.

So we always indicate the AA similarity theorem instead of the AAA similarity theorem.

We start with the first similarity theorem.

AA similarity theorem

As shown in Figure 1, in $\triangle ABC$ and $\triangle DEF$,

$\angle A=\angle D$, and $\angle B=\angle E$, then $\triangle ABC \sim \triangle DEF$

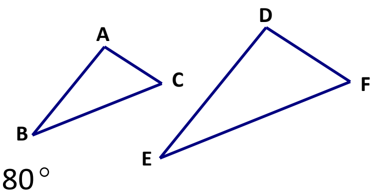


Figure 1

Pf:

The sum of the three inner angles of a triangle is 180°

The third angle

$\angle C=180°-(\angle A+\angle B)$ ($\angle A+\angle B+\angle C=180°$)

$=180°-(\angle D+\angle E)$ (replace $\angle A$ with $\angle D$, and $\angle B$ with $\angle E$)

$=\angle F$ ($\angle D+\angle E+\angle F=180°$)

That is: $\angle A=\angle D$, $\angle B=\angle E$, and $\angle C=\angle F$ …(1)

Now let's look at the next part: the corresponding side lengths are proportional.

Move $\triangle ABC$ onto $\triangle DEF$, and let point A overlap point D, like Figure 2 shown.

Since $\angle A=\angle D$, $\overline{AB}$ and $\overline{AC}$ must lie on $\overline{DE}$ and $\overline{DF}$ respectively.

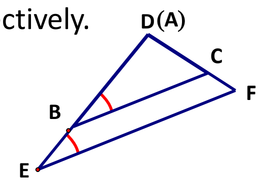


Figure 2

Then

given $\angle ABC=\angle E$, implies that

$\overline{BC} // \overline{EF}$ (congruent corresponding angles)

From what we learned about parallel line properties in the previous classes, we get

$$\frac{\overline{AB}}{\overline{DE}}=\frac{\overline{AC}}{\overline{DF}}=\frac{\overline{BC}}{\overline{EF}} \quad \cdots(2)$$

From (1) and (2), we have proved that $\triangle ABC \sim \triangle DEF$#

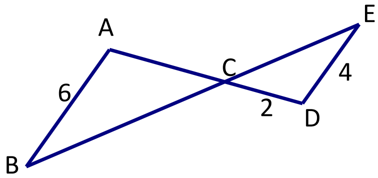
> **Attention:** Something we have to pay attention to is that when we denote two similar triangles, we have to follow the sequence of the corresponding vertices of these two triangles, just like what we do in triangle congruence.

Let's do an example right after the explanation above.

Example:

Given $\overline{AB}//\overline{DE}$, $\overline{AD}$ and $\overline{BE}$ intersect at point C.

If $\overline{AB}$=6, $\overline{CD}$=2, and $\overline{DE}$=4. Find the length of $\overline{AC}$.



Sol:

$\because \overline{AB}//\overline{DE}$,

$\Rightarrow \angle$A=$\angle$D, and $\angle$B=$\angle$E (alternative interior angles are equal) ……*

$\Rightarrow \triangle ABC \sim \triangle DEC$ (AA similarity theorem)

$\Rightarrow \dfrac{\overline{AC}}{\overline{DC}}=\dfrac{\overline{AB}}{\overline{DE}}$ (triangle similarity property)

$\Rightarrow \dfrac{\overline{AC}}{2}=\dfrac{6}{4}$

$\Rightarrow \overline{AC}=3$ #

(*) An alternative way to do the reasoning above by using another angle condition.

You can also say

$\angle$A=$\angle$D (alternative interior angles are equal) and

$\angle$ACB=$\angle$DCE (vertical angles are equal)

The rest is the same.

2. SAS similarity theorem

The second similarity theorem is the SAS similarity theorem.

In two triangles if a pair of corresponding angles are congruent, and two adjacent side lengths of the corresponding angles are proportional, these two triangles are similar.

Let's take a short look at the proof.

SAS similarity theorem

Two triangles ΔABC and ΔDEF, as shown in Figure 1.

If ∠A= ∠D, and $\dfrac{\overline{AB}}{\overline{DE}}=\dfrac{\overline{AC}}{\overline{DF}}$, then ΔABC~ΔDEF.

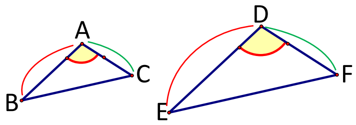
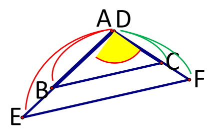


Figure 1

Pf:

Set $\overline{AB}<\overline{DE}$

(Either $\overline{AB}<\overline{DE}$, $\overline{AB}=\overline{DE}$, or $\overline{AB}>\overline{DE}$ will not affect the result. You can try it yourself if you have any doubts)



Move point A onto point D, then ∠A and ∠D will be overlapped completely due to the given ∠A= ∠D. Point A and point D are actually the same point. Figure 2

That is: $\overline{AB}$ lies on $\overline{DE}$ and $\overline{AC}$ lies on $\overline{DF}$. (We can always get a small piece from a big portion, so point B must lie on $\overline{DE}$.)As shown in Figure 2.

$\because \dfrac{\overline{AB}}{\overline{DE}} = \dfrac{\overline{AC}}{\overline{DF}}$ (given) $\Rightarrow \dfrac{\overline{DB}}{\overline{DE}} = \dfrac{\overline{DC}}{\overline{DF}}$ (Point A and point D are overlapped)

$\Rightarrow \overline{BC} // \overline{EF}$ (the property of parallel lines)

In Figure 3, $\overline{BC} // \overline{EF}$ in ΔDEF, from the properties of parallel lines in a triangle, we get

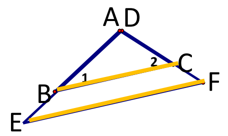


Figure 3

$\dfrac{\overline{DB}}{\overline{DE}} = \dfrac{\overline{DC}}{\overline{DF}} = \dfrac{\overline{BC}}{\overline{EF}}$, ∠1= ∠E, and ∠2= ∠F

(corresponding angles are congruent)

So in ΔABC and ΔDEF, all the corresponding angles are congruent and the corresponding side lengths are proportional

$\Rightarrow$ ΔABC~ΔDEF #

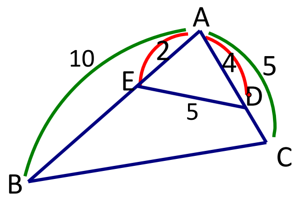
Let's get more familiar with the SAS similarity theorem by doing an example.

Example:

Given ΔABC as shown in the Figure on the right

Point D is on $\overline{AC}$ and point E is on $\overline{AB}$.

$\overline{AB}$=10, $\overline{AC}$=5, $\overline{AE}$=2, $\overline{AD}$=4, and $\overline{DE}$=5.

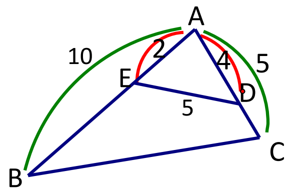


Answer the following questions:

(1) Are ΔABC and ΔADE similar triangles?
Please write down your reasons.

(2) According to your answer to question (1), find the length of $\overline{BC}$

Sol:



(1) Yes.

In ΔABC and ΔADE,

$\dfrac{\overline{AB}}{\overline{AD}} = \dfrac{10}{4} = \dfrac{5}{2} = \dfrac{\overline{AC}}{\overline{AE}}$ (given)

∠A= ∠A (self reflexive property)

$\Rightarrow$ ΔABC~ΔADE (SAS similarity theorem)

(2) Since ΔABC~ΔADE,

$\dfrac{\overline{BC}}{\overline{DE}} = \dfrac{\overline{AB}}{\overline{AD}} = \dfrac{\overline{AC}}{\overline{AE}} = \dfrac{5}{2}$

i.e. $\dfrac{\overline{BC}}{5} = \dfrac{5}{2}$

$\Rightarrow \overline{BC} = \dfrac{25}{2}$ #

Let’s keep moving towards our third and last similarity theorem: SSS

3. SSS similarity theorem

In two triangles, if three corresponding side lengths are proportional, these two triangles are similar.

Generally speaking, if we only have the side lengths of a geometric figure, the shape of the geometric figure is not guaranteed. We need some support from angles to limit the various changing of the geometric shape. In this situation, we are going to create a new triangle that is similar(or congruent) to one of the triangles, meanwhile this new triangle is also congruent(or similar) to the other triangle. From the strong connection among these triangles we create, we can get better support to see if the original two triangles are similar

(一般來說，如果我們只有線段長度的條件，是很難保證幾何圖形的形狀的。我們需要一些角度的條件來支持幾何圖形形狀的確定。所以我們現在要做的事情是創造一個新的三角形，它與其中一個三角形相似(或全等)，同時又與另一個三角形全等(或相似)。我們新增了三角形之間的角度關係， 就有利於討論三角形之間是否存在相似關係。)

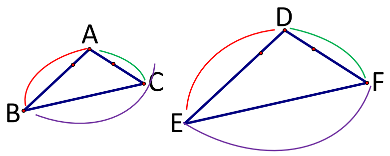
Of course, this is not the only promising way to do the proving. We encourage students to try on. It’s nice to stimulate our brains once in a while!

It will take a little longer time for the explanation. Please be patient for the proving of the last similarity theorem.

SSS similarity theorem

Given two triangles ΔABC and ΔDEF,

If $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{AC}}{\overline{DF}} = \frac{\overline{BC}}{\overline{EF}}$, then $\Delta ABC \sim \Delta DEF$



Pf:

Part A: Create a triangle that is similar to triangle DEF.

Assume $\overline{AB} < \overline{DE}$

Choose a point G on $\overline{DE}$ such that $\overline{DG} = \overline{AB}$,

a point H on $\overline{DF}$ such that $\overline{DH} = \overline{AC}$,(Again, it’s guaranteed to get a small part from a bigger portion.), connect $\overline{GH}$. as shown in Figure 1.

In ΔDGH and ΔDEF,

$\frac{\overline{DG}}{\overline{DE}} = \frac{\overline{AB}}{\overline{DE}} = \frac{\overline{AC}}{\overline{DF}} = \frac{\overline{DH}}{\overline{DF}}$ ($\because \overline{DG} = \overline{AB}$, and $\overline{DH} = \overline{AC}$)

i.e. $\frac{\overline{DG}}{\overline{DE}} = \frac{\overline{DH}}{\overline{DF}}$

$\angle D = \angle D$ (self-reflexive property)

$\Rightarrow \Delta DGH \sim \Delta DEF$ (SAS similarity theorem)

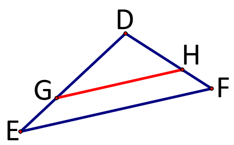


Figure 1

We now create a triangle DGH which is similar to triangle DEF.
Next step, we are going to see if we can get some relationship between ΔDGH and ΔABC.
(我們目前創造了一個新的三角形 DGH 跟三角形 DEF 相似，接著，我們要試試看是不是可以在三角形 DGH 和另一個原來的三角形 ABC 找到相似或者全等的關係)

∵ΔDGH~ΔDE, we know that

$$\frac{\overline{GH}}{\overline{EF}}=\frac{\overline{DG}}{\overline{DE}}=\frac{\overline{DH}}{\overline{DF}}\Rightarrow\frac{\overline{GH}}{\overline{EF}}=\frac{\overline{AB}}{\overline{DE}}=\frac{\overline{AC}}{\overline{DF}}\ (\because \overline{AB}=\overline{DG},\text{ and }\overline{AC}=\overline{DH})$$

But $\frac{\overline{BC}}{\overline{EF}}=\frac{\overline{AB}}{\overline{DE}}=\frac{\overline{AC}}{\overline{DF}}$ (given)

$\Rightarrow \overline{GH}=\overline{BC}$ ⋯(✧)

Now in ΔABC and ΔDGH

$\overline{GH}=\overline{BC}$ ⋯(✧)

$\overline{DG}=\overline{AB}$

$\overline{DH}=\overline{AC}$ (constructed)

$\Rightarrow$ΔABC ≅ΔDGH (SSS triangle congruence theorem)

Now we get ΔABC ≅ΔDGH and ΔDGH~ΔDEF

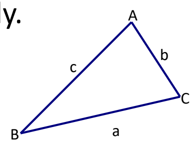
Then ΔABC~ΔDEF#

That proves the third similarity theorem: SSS

Let's do some practice.

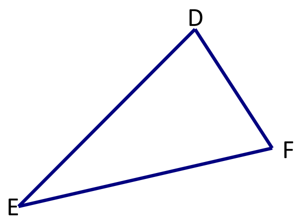
Example:

Given ΔABC, and its three side lengths are a, b, and c correspondingly. 2a=3b, and 3b=2c.



(1) Find a:b:c
(2) If ΔDEF ～ΔABC and the perimeter of triangle DEF is 76, Please find the length of segment EF.

Sol:



(1) From the given 2a=3c and 3b=2c

$\Rightarrow$a:c=3:2 and b:c=2:3

$\Rightarrow$a:b:c=9:4:6#

(2) ΔDEF～ΔABC , let $\frac{\overline{EF}}{\overline{BC}}=\frac{\overline{DE}}{\overline{AB}}=\frac{\overline{DF}}{\overline{AC}}=k$

$\Rightarrow \overline{EF}=k\overline{BC}$, $\overline{DE}=k\overline{AB}$, and $\overline{DF}=k\overline{AC}$

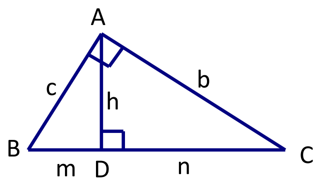
$$\Rightarrow \frac{\overline{EF}}{\overline{EF}+\overline{DE}+\overline{DF}}=\frac{k\overline{BC}}{k\overline{BC}+k\overline{AB}+k\overline{AC}}=\frac{k\overline{BC}}{k(\overline{BC}+\overline{AB}+\overline{AC})}=\frac{a}{a+b+c}=\frac{9}{19}$$

$\therefore \overline{EF}=\frac{9}{19}\times 76=36$#

We have learned three triangle similarity theorems. We are now going to discuss some pretty useful results by applying similarity theorems on right triangles. Right triangles are the most important geometric shape among triangles. So please learn it with heart whenever you confront anything related to right triangles.
(直角三角形是三角形中最重要的幾何圖形，所以請用心學習直角三角形的一切性質。)

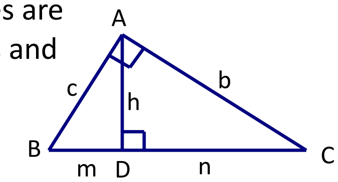
In right triangle ABC, $\angle BAC=90^\circ$, and $\overline{AD}\perp\overline{BC}$.
Point D is on segment BC. Let $\overline{AB}=c$, $\overline{AC}=b$, $\overline{AD}=h$, $\overline{BD}=m$, and $\overline{CD}=n$.
(for the sake of convenience)
Then

1. $c^2=m(m+n)$
2. $b^2=n(m+n)$
3. $h^2=mn$



pf:

1. the equation contains c and m, so we look at ΔABD, and the equation contains c and (m+n), so we look at ΔABC. If these two triangles are similar, it will lead us to the proportion of side lengths and probably we can get equation 1. Let's see.
   In ΔABD and ΔABC,



$\angle ADB=90^\circ$ ($\overline{AD}\perp\overline{BC}$)
so $\angle ADB=90^\circ=\angle BAC$ (given $\angle BAC=90^\circ$)
and $\angle B=\angle B$ (self reflexive property)
$\Rightarrow \Delta ABD \sim \Delta CBA$ (AA similarity theorem)

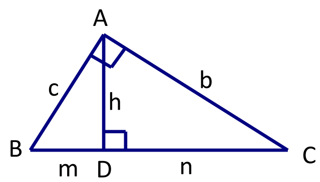
Attention: We need to follow the corresponding sequence of vertices when we talk about triangle congruence and triangle similarity.

$\Rightarrow \dfrac{\overline{AB}}{\overline{BC}}=\dfrac{\overline{BD}}{\overline{AB}}$ (similarity property)

$\Rightarrow \dfrac{c}{(m+n)}=\dfrac{m}{c}$

$\Rightarrow c^2=m(m+n)$ # (cross multiplying)

2. When we look at the equation $b^2=n(m+n)$, we need to do the same analysis as we do above, we will look at ΔABC(b and (m+n)) and ΔACD(b and n).
   Let's see if these two triangles are similar.
   In Δ ACD and Δ ABC,



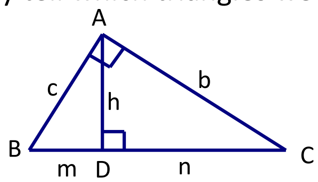
$\angle ADC=90^\circ$ ($\overline{AD}\perp\overline{BC}$)
$\angle BAC=90^\circ$ (given)
$\Rightarrow \angle ADC=90^\circ=\angle BAC$
and $\angle C=\angle C$ (self reflexive property)
$\Rightarrow \Delta ACD \sim \Delta BCA$ (AA similarity theorem)

$\Rightarrow \frac{\overline{AC}}{\overline{BC}} = \frac{\overline{CD}}{\overline{AC}}$ (similarity property)

$\Rightarrow \frac{b}{(m+n)} = \frac{n}{b}$

$\Rightarrow b^2=n(m+n)$ # (cross multiplying)

3. Now we are going to prove the third equation $h^2=mn$.
   Please do the analysis yourself and hope you can easily tell which triangles we are looking at now.



   In ΔABD and ΔACD,

   $\angle ADC=90° = \angle ADB \cdots(1)$ ($\overline{AD} \perp \overline{BC}$)

   And

   $\angle ABD+\angle BAD=90°$ (Two acute angles are complementary in a right triangle.)

   $\angle BAD+\angle CAD=90°$ (given $\angle BAC=90°$)

   $\angle ABD+\angle BAD=\angle BAD+\angle CAD$

   $\Rightarrow \angle ABD=\angle CAD \cdots(2)$ (subtractive axiom of equality)

   $\Rightarrow \Delta ABD \sim \Delta CAD$ (from (1) and (2), AA similarity theorem)

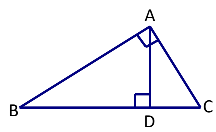
   $\Rightarrow \frac{\overline{BD}}{\overline{AD}} = \frac{\overline{AD}}{\overline{CD}}$ (similarity property)

   $\Rightarrow \frac{m}{h} = \frac{h}{n}$

   $\Rightarrow h^2=mn$ # (cross multiplying)

Let's wind up this session by doing the last example.

Remember we learned to find the measure of the height of the hypotenuse in a right triangle in eighth grade?



A right triangle ABC is shown on the right. $\angle BAC=90°$ and $\overline{AD}$ is the height of the hypotenuse $\overline{BC}$.

we need to find the measure of $\overline{AD}$.

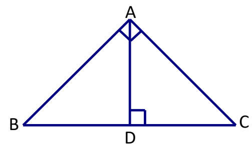
Of course we had a very easy and good way to find the measure of $\overline{AD}$ only by using the concept of Formula for Area of the Triangle (If you suddenly forget it, please look it up and do a review. We won't do the explanation here.). The result is:

$\overline{AD} = \frac{\overline{AB} \cdot \overline{AC}}{\overline{BC}}$ (Does it ring the bell now?)

But it's always fun to do math in different ways. Here we can also find the measure of the height of the hypotenuse in right triangles with the third equation we just proved above.

Example 1:

An isosceles right triangle ABC as shown.



$\angle BAC = 90^{\circ}$ and $\overline{AB} = \overline{AC}$, $\overline{AD} \perp \overline{BC}$. Prove $\overline{AD} = \frac{1}{2}\overline{BC}$.

Pf:

Set $\overline{BD} = \overline{CD} = r$

(An isosceles triangle is a symmetric shape, and $\overline{AD} \perp \overline{BC}$, o $\overline{BD} = \overline{CD}$)

Then $\overline{AD}^2 = r \bullet r$

$\overline{AD}^2 = r^2$

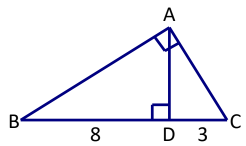
$\overline{AD} = \pm r$

$\overline{AD} = r$ ($\overline{AD}$ is the measure of a side length, it has to be positive)

i.e. $\overline{AD} = \overline{BD} = \overline{CD} = \frac{1}{2}\overline{BC}$ #

Example 2:

In right triangle ABC as shown.



$\angle BAC = 90^{\circ}$ and, $\overline{AD} \perp \overline{BC}$.

If $\overline{BD} = 8$, $\overline{CD} = 3$, find $\overline{AD}$.

Sol:

$\overline{AD}^2 = \overline{BD} \bullet \overline{CD}$

$= 8 \bullet 3$

$= 24$

So $\overline{AD} = \pm 2\sqrt{6}$

$\overline{AD} = 2\sqrt{6}$ # (The measure of a segment is always positive.)

That's enough for this lesson and we will explore some applications of triangle similarities and solve some word problems in the next class. See you guys next time.

Reference:

教育部國民中學數學 108 課綱

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