The limit of a function

I. Key mathematical terms

Terms	Symbol	Chinese translation
right-hand limit/ left-hand limit		
Continuous function		
Intermediate value theorem		
location of roots theorem		

II. Definition of the limit of a function

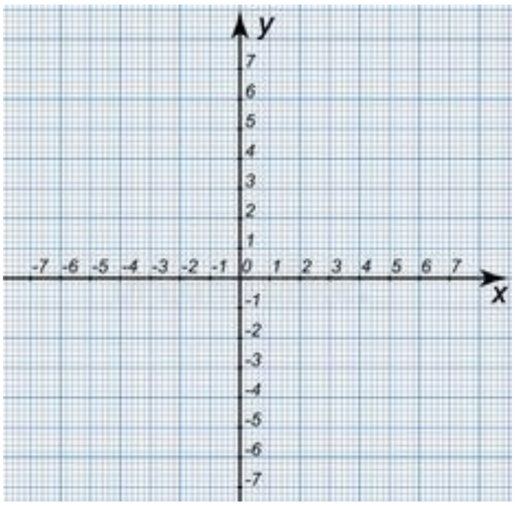
Before we start discussing the limits of functions, let's try an example first:

Example 1

1. Draw the graph of the function $f(x)=\frac{x^2-1}{x+1}$ and $g(x)=x-1$
2. Use the calculator to complete the following table:

x	0	0.9	0.99	0.999	1	1.001	1.01	1.1	2
$f(x)$									
x	0	0.9	0.99	0.999	1	1.001	1.01	1.1	2
$g(x)$									

3. For the result in the previous question's table, what do you observe?



In the previous example, you might have found that as the value of x gets closer to -1 (where x doesn't equal -1), the value of the function $f(x)$ tends to the value of $g(-1)=0$. That is to say, we can control the value of $f(x)$, making it as close to 0 as we want (but not equal to 0), within a specified range of x, ensuring that the difference between $f(x)$ and 0 is limited. Why does this conclusion arise? Let's take a look at the definition of the limit of a function.
(也就是說，在上面的例題中，我們可以控制函數 $f(x)$ 與 0 之間的差距，我們想要讓函數 $f(x)$ 與 0 有多靠近就可以有多靠近，但就是不可以等於 0。)

Definition of the limit of a function

Let $f(x)$ be a function defined on an open interval that contains $x=a$ (but $f(x)$ may not be defined at $x=a$). Then we say that the limit of $f(x)$ as x approaches a is L, denoted as:

$$\lim_{x \to a} f(x) = L$$

This definition means that as x gets arbitrarily close to a (but not equal to a), the values of $f(x)$ gets arbitrarily close to L. (Simply speaking, the limit L is the value the function 'wants to be' at that point.)

<key> If the limit of a function at a specific point exists, then the limit is unique.
(The uniqueness of the limit.)
<key> Whether the function $f(x)$ is defined at $x=a$ or not, we can still discuss the limit of the function $f(x)$ at $x=a$.

Example 2

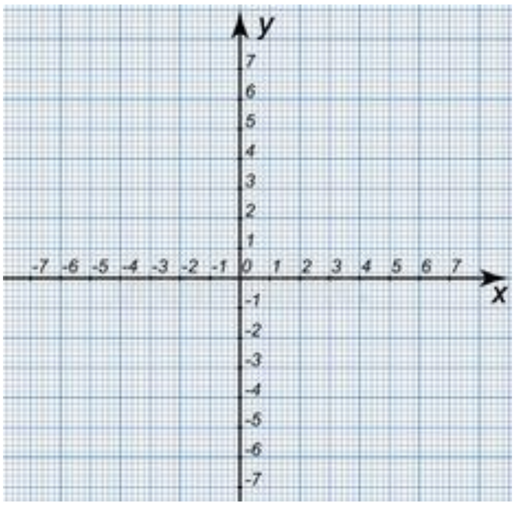
There are three different functions:

$$f(x)=x^2+x+1,\ g(x)=\frac{x^3-1}{x-1},\ h(x)=\begin{cases}\dfrac{x^3-1}{x-1}, & x\neq 1\\ 2, & x=1\end{cases}$$

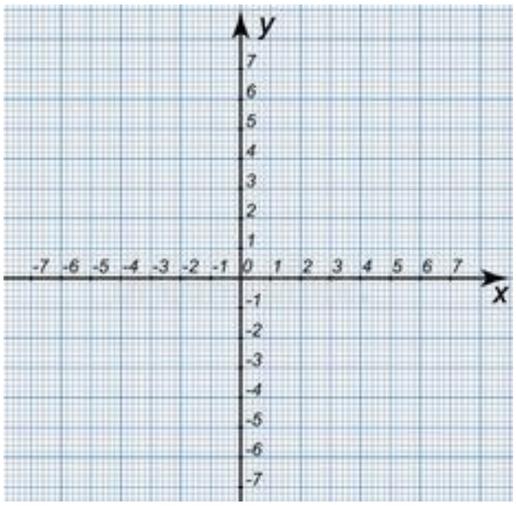
(1) Draw the graph of these three functions.
(2) With the graph you've drawn, find the following limits:

$$\lim_{x\to 1} f(x),\ \lim_{x\to 1} g(x),\ \lim_{x\to 1} h(x)$$

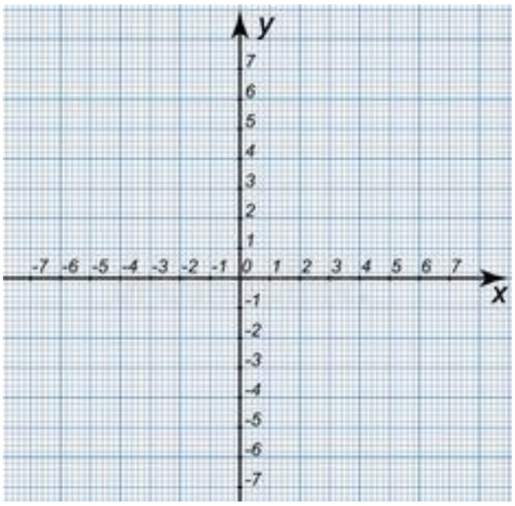
<Hint>
When plotting the graphs of $g(x),\ h(x)$, you can simplify first, but pay attention to the situation at $x=1$.



$f(x) = x^2 + x + 1$



$g(x) = \frac{x^3 - 1}{x - 1}$



$$h(x) = \begin{cases} \frac{x^3 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

Example 3

Let $f(x) = \frac{|x-2|}{x-2}, x \neq 2$, find the limit $\lim_{x \to 2} f(x) = ?$

(You should consider the approach from the left and the approach from the right. Do they meet at the same point, or do they have different values?)

In the previous example, you might have found that the function is defined different around $x=2$. Now, we'll introduce the concept of left and right limits. The left and right limits are definitions used to describe how a function behaves as it approached a specific point from the left and right side. We'll use the following symbols to represent it.

Left limit: $x \to a^-$ indicates that we are approaching a from the left side of a.

$\lim_{x \to a^-} f(x) = L_1$ (The left limit of function $f(x)$ at $x=a$ equals L_1)

Right limit: $x \to a^+$ indicates that we are approaching a from the right side of a.

$\lim_{x \to a^+} f(x) = L_2$ (The right limit of function $f(x)$ at $x=a$ equals L_2)

Conditions for the existence of a limit

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

The limit of $f(x)$ at $x=a$ exists if and only if the left limit and right limit at $x=a$ exists and the same.

Example 4

Let $f(x)=\begin{cases} x^2+5, & x \geq 1 \\ -x+7, & x<1 \end{cases}$ be a piecewise function (分段函數), find the limits:

(1) $\lim_{x \to 1^+} f(x)$ (2) $\lim_{x \to 1^-} f(x)$ (3) $\lim_{x \to 1} f(x)$

Example 5

Let $g(x)=\dfrac{x^2-4}{x+2}$, find the limits:

(2) $\lim_{x \to -2^+} f(x)$ (2) $\lim_{x \to -2^-} f(x)$ (3) $\lim_{x \to -2} f(x)$

III. Arithmetic of the limit of functions

Given functions $f(x),\ g(x)$, if the limits of both $f(x)$ and $g(x)$ exists at $x=a$, how about the limits of the addition, subtraction, multiplication and division of these two functions? Do they exist? Let's take a look at the arithmetic of limit of functions:

Arithmetic of the limit of functions

Given two functions $f(x)$ and $g(x)$, if $\lim_{x\to a} f(x)=L$, $\lim_{x\to a} g(x)=M$, then the following holds true:

(1) $\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)=L\pm M$

(2) $\lim_{x\to a}\left[f(x)g(x)\right]=\lim_{x\to a} f(x)\lim_{x\to a} g(x)=LM$

(3) When $\lim_{x\to a} g(x)=M\neq 0$, $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{L}{M}$

Let's apply the properties above to solve the examples.

Example 6

Let $\lim_{x\to a} k=k,\ \lim_{x\to a} x=a$, for $a,\ k$ are constants, find the following limits:

(1) $\lim_{x\to a} xk$ (2) $\lim_{x\to a} k^2$ (3) $\lim_{x\to a} x^2$

(4) $\lim_{x\to a}(x+1)(x-2)$ (5) $\lim_{x\to a} 2x^2-x+6$ (6) $\lim_{x\to a} x^n$

Example 7

Find the following limits (You should simplify the fraction before you find the limit.)

(1) $\lim_{x\to 3}\frac{x^2-5x+6}{x-3}$ (2) $\lim_{x\to 3}\frac{1}{x-3}$ (3) $\lim_{x\to 3}\frac{x^2-5x+6}{x+3}$

By following the same approach as in the previous problems and applying basic properties of limits, we can obtain the following result.

The limit of polynomial functions and rational functions

Let $p(x)=a_n x^n + a_{n-1}x^{n-1} + ... a_1 x + a_0$ be a polynomial with degree n.

$q(x)=b_m x^m + b_{m-1}x^{m-1} + ... b_1 x + b_0$ be a polynomial with degree m.

Then the following properties hold true:

(1) $\lim_{x\to a} p(x) = p(a),\ \lim_{x\to a} q(x) = q(a)$.

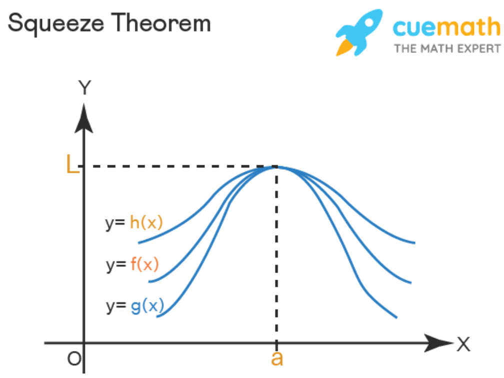
(2) The rational function $f(x)=\dfrac{p(x)}{q(x)}$ and any real number a,

$\lim_{x\to a} f(x) = \lim_{x\to a}\dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}$ if $q(a)\neq 0$.

(If $q(a)=0$, then the function may or may not have any outputs.)

IV. The pinching theorem/squeeze theorem

A very useful argument used to find the limits is called the "pinching/squeeze" theorem. It essentially said that if we can "pinch" a function's limit between two other limits which have a common value, then the common value is the value of the limit we want.



We can use the value of $h(x),\ g(x)$ to "pinch" the value of $f(x)$.

The pinching theorem/squeeze theorem

Let $f(x),\ g(x),\ h(x)$ are three functions that are defined over an open interval containing a, and satisfy the following conditions:

(1) $g(x)\le f(x)\le h(x)$ (x is near a, except possibly at a)

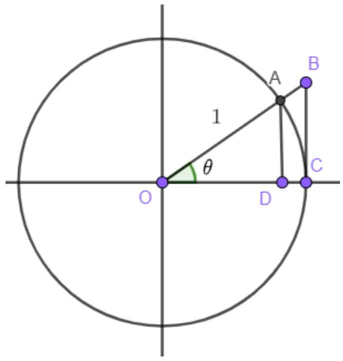
(2) $\lim_{x\to a} h(x) = L = \lim_{x\to a} g(x)$

Then we have $\lim_{x\to a} f(x) = L$

Example 8

Use pinching theorem to prove that $\lim_{x \to 0} \frac{\sin x}{x}$

(Hint, you can use the graph on the left for help.)



Example 9

We know that $x \neq 0$ and the inequality $-1 \leq \cos\frac{1}{x^2} \leq 1$ holds true.

Find the limit $\lim_{x \to 0} x^2 \frac{1}{\cos x^2}$. (Hint, you will use the pinch theorem.)

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V. Definition of a continuous function

A continuous function is a kind of function where there are no changes with jump. Alternatively, you can imagine that if an ant is crawling on a function, it can crawl to any point on the function without obstacles. (No sudden drops or breakpoints) Let's take a look at the mathematical definition of "continuous" below.

Definition of continuity at $x = a$

Let function $f(x)$ is defined at $x = a$ and satisfy the following conditions:

(1) $\lim_{x \to a} f(x)$ exits (2) $\lim_{x \to a} f(x) = f(a)$

Then we said that $f(x)$ is continuous at $x = a$.

Now let's use the definition of continuity to verify the following example.

Example 10

Let $f(x)=\begin{cases}\dfrac{x^2-16}{x+4}, & x\neq 4\\ -8, & x=-4\end{cases}$, determine

(1) $\lim_{x\to -4} f(x)$ =?

(2) The function is continuous at $x=-4$ or not?

Definition of a continuous function

1. A function $f(x)$ is continuous on an interval if it is continuous at every number in the interval.
2. If a function $f(x)$ is continuous at every points on its domain, then we said that $f(x)$ is a continuous function.

Properties of continuous functions

Let $f(x)$ and $g(x)$ be continuous functions. Then the following functions with arithmetic operations are still continuous functions.

(1) $f(x)+g(x)$

(2) $f(x)-g(x)$

(3) $f(x)g(x)$

(4) $\dfrac{f(x)}{g(x)}$

Example 11

Let $f(x)=\begin{cases}x^2, & x<-1\\ x, & -1\leq x<1,\\ 1/x, & x\geq 1\end{cases}$ $g(x)=\begin{cases}1+x^2, & x\leq 0\\ 2-x, & 0<x\leq 2\\ (x-2)^2, & x>2\end{cases}$ determine whether function $f(x)$, $g(x)$ are continuous functions.

Intermediate Value Theorem

Let $f(x)$ is continuous on $[a,b]$ and let k be any number between $f(a)$ and $f(b)$. Then there exists a number c such that

(1) $a < c < b$ (2) $f(c) = k$

For more details and the proof of this theorem, you can scan the following QR code and watch the online video.

Khan Academy

https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-new/ab-1-16/v/intermediate-value-theorem

Example 12

Let $f(x) = x^4 - 4x^3 - x + 2$, prove that there exists a constant c in the interval $(-1, 2)$ such that $f(c) = 5$

<資料來源>

1. **The limit of functions**
   Calculus-9th-Edition-by-Ron-Larson
   Calculus 9/e Metric Version-by-James-Jtewart
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2. **Definition of limit of functions**
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3. **Pinching theorem/squeeze theorem**
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   https://www.cuemath.com/calculus/squeeze-theorem/

4. 南一書局數學甲下冊

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