Topic: Geometric sequences and series

1. Check these words

English	中文	圖示
sequence/ sequences		
Series		
Sum		
Geometric sequence		
Geometric series		

2. Wheat and chessboard problem

There is an ancient legend that the game of chess was invented in India by a man named Sissa ibn Dahir.

The king, Shihram, was so pleased with the game that he offered Sissa any reward that he wanted. Sissa said that he would take this reward: the king should put one grain of wheat on the first square of a chessboard, two grains of wheat on the second square, four grains on the third square, and so on, doubling the number of grains of wheat with each square.

The king thought it was easy and the amount seemed not very much.

a. Do you think the reward Sissa requested is too little? Why?

b. From this story, did you find any mathematical patterns?

c. What else in the story do you wonder about?

使用建議	
教學活動安排	用一個古老有名的故事當作引入，複習國中學過的等比數列的相關概念，並引出要如何求和。
英文提問/開場	Sequences and patterns often provide powerful tools for modeling real-life situations and making predictions. Can anyone recall the special sequences we've learned before? We've covered arithmetic sequences and their sums, as well as geometric sequences. Does the sum of geometric sequences have a formula that we can work with? Today, we'll start with revisiting the properties of geometric sequences, and then we'll proceed to explore the sum of geometric sequences. Let's explore an ancient legend by working through this example to uncover the magic of math. Anyone would like to read the legend on the worksheet for us? Good job! Did this story remind you of any math concepts you've learned before? From this story, did you find any mathematical patterns? Anyone? Great! Now, let's identify the pattern of the number of grains on each chessboard square. Let $a_1 = 1$, $a_2 = 2$, and a_n represents the number of grains on the nth chessboard square. Find a_n. So, What is the formula for the nth term of a geometric sequence? Given a_1 and r. Anyone? Good job! Do you think the reward Sissa requested is too little? Why? What else in the story would you wonder about? Are there any real-life situations that make you think of geometric sequences?
參考資料	1. https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem 2. Oxford IB Diploma Programme: IB Mathematics: applications and interpretation, Standard Level. http://www.redefiningthesacred.com/math4.html 3. 教師可考慮故事部分改放影片https://en.etudes.ru/etudes/geometric-progression-chess/

3. Investigate the sum of a geometric sequence

Following the ancient legend, the numbers of the grains on the chessboard form a geometric sequence with $a_1 = 1$ and $r = 2$

a. Find the sum $S_{10} = a_1 + a_2 + a_3 + ... + a_{10}$

b. Find the sum $S_{64} = a_1 + a_2 + a_3 + ... + a_{64}$

	使用建議
教學活動安排	利用此例子求和，教師先示範求前10項的和，接著讓學生試著求64項的和，最後推導等比級數求和公式。
英文提問/教學	Now we know that the numbers of the grains on the chessboard form a geometric sequence with $a_1 = 1$ and $r = 2$. So, How many grains should the king pay Sissa as a reward? To answer this, we have to explore how to calculate the sum of geometric sequences. Let's start with a modest number and calculate the sum of the first ten terms. Also, to help us find the sum of the first ten terms, we'll introduce a new symbol, we denote the sum of the first ten terms as S_{10}. So, we are going to find the sum of the first ten terms $S_{10} = a_1 + a_2 + a_3 + ... + a_{10}$ $= 1 + 2 + 4 + ... + 2^9$ (Which equals one plus two plus four and dot dot dot plus two to the power of nine.) Then, let's try a little trick, we multiply both sides of the equation by two. We get $2S_{10} = 2 + 4 + 8 + ... + 2^{10}$ Then we subtract these two equations to eliminate identical terms. $S_{10} = 1 + \cancel{2} + \cancel{4} + \cancel{\cdots} + \cancel{2^9}$ $-)\ 2S_{10} = \cancel{2} + \cancel{4} + \cancel{8} + \cancel{\cdots} + 2^{10}$ $S_{10} = 2^{10} - 1$ We get two to the power of ten minus one equals one thousand and twenty three. So the sum of the first ten terms is 1023.

使用建議	
英文提問／教學	Now it's your turn, follow the trick "multiply both sides of the equation by two" to find the sum of the grains on the chessboard. Another reminder, you can keep your anwser in an exponetial form. Anyone wanna share? $S_{64} = 1 + \cancel{2} + \cancel{4} + \cancel{\cdots} + \cancel{2^{63}}$ $-)\ 2S_{64} = \cancel{2} + \cancel{4} + \cancel{8} + \cancel{\cdots} + 2^{64}$ $S_{64} = 2^{64} - 1$ Does anyone know how big the number of all the grains on the chessboard? We learned about logarithms to determine the magnitude of a number, whether it's large or small. https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem# To save time, let's check what wikipedia said.
參考資料	1.有關等比級數和推導可參考 https://youtu.be/MBY4WqbOkJg?si=mfEsrUiq_O_2-0PV 2. https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem# On the entire chessboard there would be $2^{64} - 1$ = 18,446,744,073,709,551,615 grains of wheat, weighing about 1,199,000,000,000 metric tons. This is over 1,600 times the global production of wheat (729 million metric tons in 2014 and 780.8 million tonnes in 2019).[9]

4. The sum of the first nth term of a geometric series

The sum of the first nth term of a geometric series a_1, a_1r , a_1r^2, a_1r^3, a_1r^4,…a_1r^{n-1} with a common ratio $r \neq 1$ is given by $S_n = \dfrac{a_1(1-r^n)}{1-r}$

使用建議	
教學活動安排	推導等比級數求和公式。

使用建議	
英文提問/開場	Now, let's try to generalize the result for a geometric series which has n terms and common ratio r. In the example of grains, the key is to multipy each term by two, two is the common ratio. So we are going to mutiply both sides of the equations by the common ratio r. Then we subtract these two equations to eliminate identical terms.. $S_n = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cdots + \cancel{a_1 r^{n-1}}$ $-)\ r S_n = \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \cdots + a_1 r^n$ $(1-r) S_n = a_1 - a_1 r^n$ $\Rightarrow S_n = \frac{a_1(1-r^n)}{1-r}$ We get the formula $S_n = \frac{a_1(1-r^n)}{1-r}$ So, let's wrap up the discussion above. How can we find the sum of the first nth term of a geometric sequence? Are there any specific formulas for this?

5. Practice makes perfect

Find the sum of the following geometric sequences

a. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + (\frac{1}{2})^9$

b. $2^{12} - 2^{11} + 2^{10} - 2^9 + \ldots + 1$

使用建議	
教學活動安排	練習
英文提問/開場	Now it's your turn to practice! And the key is to make sure you get all the correct information. We'll check-in in five minutes.

使用建議	
答案	a. $\frac{511}{512}$ b. 2731

6. Challenge

Find the sum of a finite geometric series given that a_1 and the common ratio is 1.

使用建議	
教學活動安排	給學生思考公比是1時，該如何計算？
英文提問/開場	Finally, here’s chanllenge for you to think. Find the sum of a finite geometric series where the common ratio is 1 and the first term is a_1 .
答案	$S_n = n \times a_1$

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