科學記號與常用對數

Scientific Notation and Common Logarithm

Material	Vocabulary
科學記號表示法 每一個正實數 a 都可表成 $a = b \times 10^n$ ， 其中 $1 \le b < 10$ ，且 n 是一個整數，並稱 $b \times 10^n$ 為 a 的科學記號。	1. physics (物理), 2. round (四捨五入), 3. significant (有效的), 4. figure (數字), 5. scientific (科學的), 6. notation (符號), 7. approximately (近似), 8. exponent (指數), 9. population (人口).

Illustrations I

Frequently numbers that occur in **physics**[1] and other sciences are either very large or very small. For example, the speed of light in a vacuum is 299,792,458 m/s, which can be **rounded**[2] to 299,800,000 m/s. When we use 299,800,000 m/s to represent the speed of light, it is referred to as "rounding the speed of light to 4 **significant**[3] **figures**[4]," or "we use 4 significant figures to represent the speed of light, which is 299,800,000 m/s." Additionally, we can express the number as 2.998×10^8, which is referred to as **scientific**[5] **notation**[6].

在科學的領域中，常有很大或很小的數，舉例來說光速為 299792458 公尺/秒，將其四捨五入到萬位可得 299800000。當我們使用 299800000 來表示光速時，稱之為「將光速取 4 位有效數字」或稱「以 299800000 來表示光速的有效位數為 4」。再將其以 2.998×10^8 表示，這樣的表示方式稱為科學記號。

Scientific Notation

Every positive real number a can be expressed as

$$b = a \times 10^n,$$

where $1 \le a < 10$, and n is an integer. $a \times 10^n$ is called a scientific notation of b.

It's important to note that rounding the speed of light to 3 significant figures gives 300,000,000 m/s, which can be expressed as 3.00×10^8 m/s.

Scientific notation is utilized to represent not only large but small numbers. For example, the wavelength of red light, **approximately**[7] 0.000000650 meters, can be expressed as 6.5×10^{-7}; the diameter of the SARS virus is 85 nanometers, and since 1 nanometer equals 10^{-7} centimeters, it can be converted to 8.5×10^{-6} centimeters.

特別值得注意的是，當我們取 3 位有效數字時，必須寫成 3.00×10^8 。

科學記號除了能表示很大的數字之外，很小的數字也常用以表示之，例如紅光的波長為 0.000000650 公尺，可表示為6.5×10^{-7}公尺；SARS 的病毒直徑為 85 奈米，又因 1 奈米等於10^{-7}公分，所以其病毒直徑約為8.5×10^{-6}公分。

Note:

Scientific notation is sometimes informally referred to as "e-notation." In "e-notation," the number is represented as a coefficient multiplied by 10 raised to a certain power. The letter "e" stands for "**exponent**[8]." For example, the number 6.5×10^{-7} in scientific notation is equivalent to 6.5e-7 in "e-notation." This notation is commonly used in scientific and mathematical contexts.

科學記號也常被稱為 E 記號，其數字代表 10 的次方，則字母 E 為（Exponent）指數的字首。舉例來說科學記號6.5×10^{-7}在 E 記號裡為 6.5e-7。此記號方式常用於科學與數學。

Examples I

The **population**[9] of the USA in 2018 was 329,976,959

(1) express the population in scientific notation with 2 significant figures.

(2) express the population in scientific notation with 3 significant figures.

Solution

(1) To express the number with 2 significant figures, it needs to be rounded to the nearest ten million, resulting in 330,000,000. This can be written in scientific notation as 3.3×10^{8}.

(2) To express the number with 3 significant figures, it needs to be rounded to the nearest million, resulting in 330,000,000. This can be written in scientific notation as 3.30×10^{8}.

Examples II

Compute the following values and express them in scientific notation with 2 significant figures.

(1) $(3.0 \times 10^{3}) \times (4.5 \times 10^{-7})$ (2) $(3.2 \times 10^{15}) - (7.1 \times 10^{13})$

Solution

(1) $(3.0 \times 10^{3}) \times (4.5 \times 10^{-7})$

$= (3.0 \times 4.5) \times (10^{3} \times 10^{-7})$ multiply the coefficients and add the exponents

$= 13.5 \times 10^{-4}$

$= 1.35 \times 10^{-3}$ express it in scientific notation with a coefficient less than 10.

$\approx 1.4 \times 10^{-3}$ express it with 2 significant figures

(2) $(3.2 \times 10^{15}) - (7.1 \times 10^{13})$

$= (3.2 \times 10^{15}) - (0.071 \times 10^{15})$ ensures that both terms have the same power of 10

$= (3.2 - 0.071) \times 10^{15}$ subtract the coefficients

$= 3.129 \times 10^{15}$ express it in scientific notation.

$\approx 3.1 \times 10^{15}$ express it with 2 significant figures

Material	Vocabulary
常用對數的定義 設 $b > 0$，當實數 a 滿足 $b = 10^a$ 時，這個指數的值 a 可用符號 $\log b$ 表示，即 $b = 10^{\log b}$， 並稱 $\log b$ 為 b 的**常用對數**。	10. common (常用), 11. logarithm (對數), 12. observe (觀察), 13. accurate (準確), 14. raise (升起), 15. fundamental (基本), 16. property (性質), 17. frequency (頻率), 18. adjacent (鄰近的).

Illustrations II

Scientific notation, expressed as $b = a \times 10^n$, is a tool for representing both very large and very small numbers. The exponent "n" of 10 helps us figure out how big or small b is.

Any positive number "b" can be expressed in the form of "an exponent with base 10," and the value of this exponent (also known as the index) can be denoted by using the symbol "log b" as $b = 10^{\log b}$, where log b is called the **common**[10] **logarithm**[11] of b. Ex: $2 = 10^{\log 2}$, $0.5 = 10^{\log 0.5}$.

科學記號 $b = a \times 10^n$ 的形式表示很大或很小的數，其中「10 的次方 n」幫助我們看出此數的大小。

任意正數 b 都可以化成「10 的次方」的形式，而這個次方（也稱指數）的值我們以符號 log b 表示，即 $b = 10^{\log b}$，並稱 log b 為 b 的常用對數。例如：$2 = 10^{\log 2}$, $0.5 = 10^{\log 0.5}$。

Logarithms in Base 10

Considering the graph of $y = 10^x$ which is shown in figure 1, it's easy to solve the equation $10^x = 10$, which yields x equals 1. But, when dealing with the equation $10^x = 20$, we could make an approximation by **observing**[12] that $10^1 = 10$ and $10^2 = 100$. Hence, we can estimate that x lies within the range from 1 to 2.

考慮 $y = 10^x$ 的圖形（如圖 1），我們能輕易解出方程式 $10^x = 10$ 的解為 1。然而若要解方程式 $10^x = 20$，我們以 $10^1 = 10$ 與 $10^2 = 100$ 找估計值，略估 x 的值落在區間 1 到 2。

Now, to get a more **accurate**[13] value for x, we can experiment with different values of x within the range from 1 to 2 until we find the exact value that satisfies $10^x = 20$.

若要找到更準確的 x 值，我們可以在區間 1 到 2 之間找更多的值來計算，直到找出精確的值以滿足方程式 $10^x = 20$。

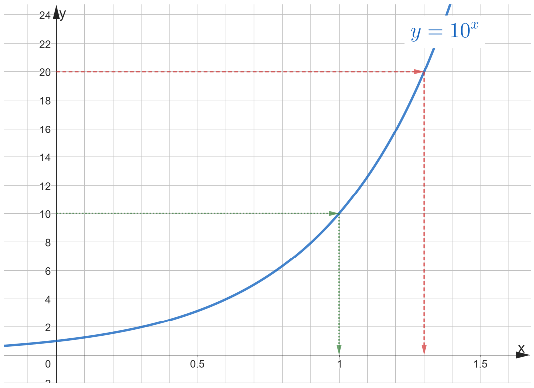


Figure 1

When we express a positive number y as 10^x, we refer to x as the logarithm in base 10 of y. The base-10 logarithm of a positive number is the power to which 10 must be **raised**[14] to produce that number.

當 10^x 表示為一正數 y 時，我們將 x 表示為以 10 為底，y 的對數。而底數為 10 的對數為 10 的次方。

The Definition of Common Logarithm

The logarithm in base 10 of a positive number is the power that 10 must be raised to yield that number. For any $b > 0$, $10^x = b$ if and only if $\log_{10} b = x$.

For example:

1. The base-10 logarithm of 1000 is 3 because $1000 = 10^3$. This can be written as $\log_{10} 1000 = 3$ or $\log 1000 = 3$.
2. $\log (0.01) = -2$ since $0.01 = 10^{-2}$.
3. To solve the equation $10^x=20$, we can approximate x equals 1.3 by using a calculator, denoted in logarithmic notation as $x = \log 20 \approx 1.3$.

Key points:

The rule for base 10 is:

If $y = 10^x$ then $x = \log_{10} y$

This rule can be described in words as:

$\log_{10} y$ is the power that 10 must be raised to obtain y.

Note:

1. log is short for logarithm.

2. $\log_{10} b$ can be read as "log base 10 of b," "the logarithm in base 10 of b," "log b to the base of 10," or it's shortened to "log 10 b."
3. A common logarithm is a logarithm with base 10. You can write a common logarithm $\log_{10} x$ simply as $\log x$, without showing the 10.

Examples III

(a) Express 2.7 in the form of an exponent with base 10 using the common logarithm.

(b) Find the value of $10^{\log 3.14}$.

Solution

(a) The rule for base- 10 logarithm is: if $y = 10^x$ then $x = \log_{10} y$.

Plug in $y = 2.7$, which yields $2.7 = 10^x$, leading to $x = \log_{10} 2.7 = \log 2.7$.

Hence, $2.7 = 10^x = 10^{\log 2.7}$.

The answer is $10^{\log 2.7}$.

(b) From question 1, we know that $y = 10^x = 10^{\log y}$.

Plug in y = 3.14, which results in $3.14 = 10^x = 10^{\log 3.14}$.

The answer is 3.14.

Examples IV

Given $b = \log 13$, find the following values: (a) 100^b (b) $10^b + 10^{-b}$.

Solution

By applying the **fundamental**[15] **property**[16] of logarithms, we know that $13 = 10^{\log 13} = 10^b$.

(a) $100^b = (10^2)^b = (10^b)^2 = 13^2 = 169$.

(100 raised to the power of "*b*" equals 10 squared raised to the power of "*b*," also equals 10 raised to the power of "*b*" squared, and equals 13 squared, which is 169.)

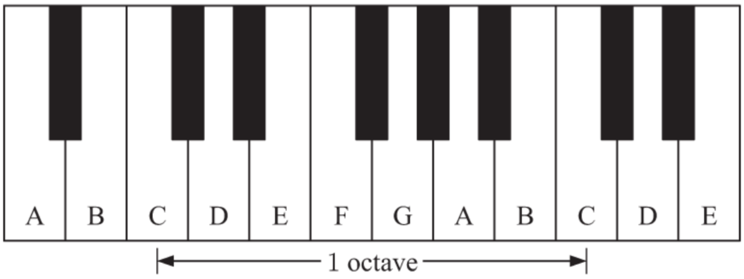
(b) $10^b + 10^{-b} = 10^b + \left(10^b\right)^{-1} = 13 + 13^{-1} = 13 + \frac{1}{13} = \frac{170}{13}$.

(10 to the power of "*b*" plus 10 to the power of "negative b" equals 10 to the power of "*b*" plus 10 to the power of "*b*" whole quantity to the power of negative 1 equals 13 plus 13 to the power of negative 1, and equals 13 plus 1 over 13, which is 170 over 13.)

Examples IV

Musical notes are named according to the **frequency**[17] of their sound waves. They are labeled with letters of the alphabet(字母). A note which has twice the frequency of another is said to be one octave(八度) higher than it. So, one C is an octave below the next C.

A note n octave above "Middle C" has a frequency of f Hz. The variables n and f are related by the equation $n \approx 3.322\log\left(\frac{f}{261.6}\right)$.



(a) Find the frequency of "Middle C".

(b) How many octaves above "Middle C" is a note with a frequency of 784 Hz?

(c) Find the frequency of the note: (i) 3 octaves above "Middle C" and (ii) 1 octave below "Middle C."

(d) There are 12 different notes in an octave. They are equally spaced on the logarithmic scale. Find the ratio of frequencies between two adjacent notes.

Solution

(a) Middle C is neither above nor below itself, so n is 0, which is 0 octaves from itself.

Plugging in n = 0 in equation: $0 \approx 3.322\log\left(\frac{f}{261.6}\right)$

$\Rightarrow \log\left(\frac{f}{261.6}\right) \approx 0$ divided by 3.322

$\Rightarrow \frac{f}{261.6} \approx 10^0 = 1$ the definition of common logarithm

$\Rightarrow f \approx 261.6$ Hz.

(b) Plugging in f = 784 in the equation gives $3.322\log\left(\frac{784}{261.6}\right) \approx 3.322 \times \log 3 \approx 1.58$ octaves.

(c) (i) Since a note that is considered 1 octave higher than another has double the frequency, a note that is 3 octaves above "Middle C" should have a frequency that is multiplied by 2, three times over. The frequency of the note 3 octaves above "Middle C" is $261.6 \times 2 \times 2 \times 2 \approx 2093$ Hz.

(ii) A note that is 1 octave below "Middle C," implies that the frequency of "Middle C" is 1 octave higher than that note, clearly indicating that the note's frequency is half that of "Middle C." The frequency of the note 1 octave below "Middle C" is

$$f \times 2 = 261.6 \Rightarrow f = \frac{261.6}{2} \approx 130.8 \text{ Hz.}$$

(d) Given that there are 12 notes in an octave, the note neighboring "Middle C" is $\frac{1}{12}$ of an octave higher or lower than it. So, we can substitute $n = \frac{1}{12}$ in the equation to determine the frequency of the note that is next to Middle C:

$$\frac{1}{12} = 3.322 \log \frac{f_{\text{neighbor}}}{261.6}$$

$$\Rightarrow \log \frac{f_{\text{neighbor}}}{261.6} = 0.025$$ divided by 3.322

$$\Rightarrow \frac{f_{\text{neighbor}}}{261.6} = 10^{0.025} \approx 1.059$$ the definition of common logarithm

$$\Rightarrow f_{\text{neighbor}} \approx 261.6 \times 1.059 \approx 277.0 \text{ Hz.}$$ multiplied by 261.6.

This results in the ratio of frequencies between two **adjacent**[18] notes is

$$\frac{f_{MiddleC}}{f_{neighbor}} = \frac{277.0}{261.1} \approx 1.06 .$$

References

1. 許志農、黃森山、陳清風、廖森游、董涵冬（2019）。*數學 1：單元 4 科學記號與常用對數*。龍騰文化。
2. Michael Haese, Mark Humphries, Chris Sangwin, and Ngoc Vo (2019). *Mathematics: Applications and Interpretation SL.* Haese Mathematics.
3. Randall I. Charles, Basia Hall, Dan Kennedy, Allan E. Bellman, Sadie Chavis Bragg, William G. Handlin, Stuart J. Murphy, & Grant (2012). *Pearson Algebra 2 Common Core.* Pearson.

製作者：臺北市立陽明高中 吳柏萱 教師