Matrix

I. Key mathematical terms

Terms	Symbol	Chinese translation
Matrix		
Column		
Row		

II. What is a matrix?

Jonathan is trying to run a small business. He is selling hand-made cookies, cakes and pies for a bake sale. He records his sales during a five-day period – Monday through Friday with a table. He also records the unit prices with another table.

	Hand-made cookies	cakes	pies
Monday	3	5	2
Tuesday	7	2	1
Wednesday	2	1	5
Thursday	5	0	8
Friday	4	7	6

Table1

	Unit prices
Hand-made cookies	4
Cakes	10
pies	8

Table2

We can store these data in two different matrices easily:

$$S_1=\begin{bmatrix}3&5&2\\7&2&1\\2&1&5\\5&0&8\\4&7&6\end{bmatrix}\text{ (matrix about goods sold), }\quad P=\begin{bmatrix}4\\10\\8\end{bmatrix}\text{ (matrix about unit prices)}$$

Definition:(Matrix)

A matrix is a rectangular array of numbers (or symbols, or functions) arranged into rows and columns. For example, the following are all matrices:

$$A=\begin{bmatrix}1&0\\2&1\\-1&3\end{bmatrix},\quad B=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix},\quad C=\begin{pmatrix}a&b\\c&d\end{pmatrix},\quad D=\begin{pmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{pmatrix}$$

$$E=\begin{pmatrix}0&1&-2&3\end{pmatrix},\quad F=\begin{pmatrix}0\\-1\\5\end{pmatrix},\quad G=\begin{pmatrix}1\\0\end{pmatrix},\quad H=\begin{bmatrix}0\\1\end{bmatrix}$$

A matrix with m rows and n columns can be represented by the following:

$$A_{m\times n}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{matrix} 1^{st} \\ 2^{nd} \\ \vdots \\ m^{th} \end{matrix} \qquad =[a_{ij}]_{m\times n}$$

$$\begin{matrix} 1^{st} & \cdots & \cdots & n^{th} \end{matrix}$$

We can say "the **order** of matrix A is m by n" or "A is a matrix of dimension m×n".
The individual **entries** in the matrix are referred to as its **elements**. The entry in ith row and jth column of a matrix is referred to as the (i,j)th element of a matrix M and is usually denoted a_{ij} .

<Key> The entries can be arranged in a square brackets or a parentheses.

Example1. (1) Matrix A is a ___________ matrix. (Describe the order of the matrix.)

(2) (2,3)th element of matrix A above is ___________.

III. Some special matrices

Terms	Symbol	Explanation	Chinese translation
Square matrix			
Row matrix			
Column matrix			
Zero matrix			
Identity matrix			
Diagonal matrix			
Symmetric matrix			

Example2. (1) Matrix B, C, D are ___________ matrices.

(2) Matrix E is a ___________ matrix.

(3) Matrix F, G, H are ___________ matrices.

IV. Equality of matrices

Two matrices A and B are said to be equal if both of the following conditions hold true:

(1) A and B have the same order.

(2) All corresponding elements of A and B are equal.

Then we can say A and B are equal and can be denoted by A=B.

Example3. Find a, b, x, y with the given condition.

$$\begin{pmatrix} a+2 & 4 \\ 3b & 3 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 6 & 2y-1 \end{pmatrix}$$

V. Matrix Operations I: matrix addition and scalar multiplication

Jonathan has recorded the number items sold in the three-week period in the following matrices :

$$S_1 = \begin{bmatrix} 3 & 5 & 2 \\ 7 & 2 & 1 \\ 2 & 1 & 5 \\ 5 & 0 & 8 \\ 4 & 7 & 6 \end{bmatrix}, S_2 = \begin{bmatrix} 3 & 5 & 2 \\ 7 & 2 & 1 \\ 2 & 1 & 5 \\ 5 & 0 & 8 \\ 4 & 7 & 6 \end{bmatrix}, S_3 = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 6 & 5 \\ 8 & 7 & 7 \\ 10 & 5 & 11 \\ 12 & 1 & 16 \end{bmatrix}$$

First-week   Second-week   Third-week

The total sales of the first two weeks are:

$$S_1 + S_2 = \begin{bmatrix} 3 & 5 & 2 \\ 7 & 2 & 1 \\ 2 & 1 & 5 \\ 5 & 0 & 8 \\ 4 & 7 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 2 \\ 7 & 2 & 1 \\ 2 & 1 & 5 \\ 5 & 0 & 8 \\ 4 & 7 & 6 \end{bmatrix} = 2 \times \begin{bmatrix} 3 & 5 & 2 \\ 7 & 2 & 1 \\ 2 & 1 & 5 \\ 5 & 0 & 8 \\ 4 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 3\times2 & 5\times2 & 2\times2 \\ 7\times2 & 2\times2 & 1\times2 \\ 2\times2 & 1\times2 & 5\times2 \\ 5\times2 & 0\times2 & 8\times2 \\ 4\times2 & 7\times2 & 6\times2 \end{bmatrix} = \begin{bmatrix} 6 & 10 & 4 \\ 14 & 4 & 2 \\ 4 & 2 & 10 \\ 10 & 0 & 16 \\ 8 & 14 & 12 \end{bmatrix}$$

The total sales of these three weeks are:

$$S_1 + S_2 + S_3 = (S_1 + S_2) + S_3 = \begin{bmatrix} 6 & 10 & 4 \\ 14 & 4 & 2 \\ 4 & 2 & 10 \\ 10 & 0 & 16 \\ 8 & 14 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 0 \\ 2 & 6 & 5 \\ 8 & 7 & 7 \\ 10 & 5 & 11 \\ 12 & 1 & 16 \end{bmatrix} = \begin{bmatrix} 6+1 & 10+5 & 4+0 \\ 14+2 & 4+6 & 2+5 \\ 4+8 & 2+7 & 10+7 \\ 10+10 & 0+5 & 16+11 \\ 8+12 & 14+1 & 12+16 \end{bmatrix} = \begin{bmatrix} 7 & 15 & 4 \\ 16 & 10 & 7 \\ 12 & 9 & 17 \\ 20 & 5 & 27 \\ 20 & 15 & 28 \end{bmatrix}$$

These two operations of matrix are **matrix addition** and **scalar multiplication**.

Matrix addition

$A=[a_{ij}]_{m\times n}, B=[b_{ij}]_{m\times n}$ are matrices which has the same order.

The addition/subtraction of A and B is given by:

$$A+B=[a_{ij}+b_{ij}]_{m\times n} \quad \text{(adding corresponding entry)}$$
$$A-B=[a_{ij}+b_{ij}]_{m\times n} \quad \text{(subtracting corresponding entry)}$$

Properties of matrix addition

For matrices A, B, C has the same order, then the following holds true:

(1) **commutativity**: $A+B=B+A$

(2) **associativity**: $(A+B)+C=A+(B+C)$

(3) **existence of additive identity**: $A+O=A$

(4) **existence of additive inverse**: $A+(-A)=O$

Example4. For the matrices A and B,

$$A=\begin{pmatrix} 5 & 3 \\ -1 & 2 \\ 2 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} -2 & 2 \\ 5 & 1 \\ -7 & 10 \end{pmatrix}, \text{ find:}$$

(1) $A+B$ and $B+A$

(2) $A+(-A)$

(3) $A-B$ and $B-A$

Example5. For the matrices A, B, C, O,

$$A=\begin{pmatrix} 5 & 3 \\ -1 & 2 \\ 2 & 0 \end{pmatrix}, \quad A=\begin{pmatrix} 5 & 3 \\ -1 & 2 \\ 2 & 0 \end{pmatrix}$$

Scalar multiplication

$A=[a_{ij}]_{m\times n}$ is a matrix and k is a real number.

The scalar multiplication of A is given by:

$$kA=k[a_{ij}]_{m\times n}=[ka_{ij}]_{m\times n} \text{ (multiply every entry of the matrix by } k\text{)}$$

Properties of scalar multiplication

For matrices A, B has the same order and *r,s* are real numbers. The following holds true:

(1) **associativity**: $(rs)A=r(sA)=s(rA)$
(2) **distributivity**: $(r+s)A=rA+sA,\ r(A+B)=rA+rB$
(3) $0A=O$
(4) $1A=A$

Example6. For the matrices A and B,

$$A=\begin{pmatrix}2 & 4\\ -3 & 0\end{pmatrix},\ B=\begin{pmatrix}1 & -1\\ 0 & 5\end{pmatrix}, \text{ find: } 3(2A+B)-2(A-B)$$

Example7. For the matrices A and B,

$$A=\begin{pmatrix}2 & -1\\ 1 & -4\end{pmatrix},\ B=\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}, \text{ find matrix } X \text{ satisfies : } 2(X-A)=3X+2B$$

VI. Matrix Operations II: matrix multiplication

Jonathan wants to know the total income of three weeks. We've already known that the unit prices can be recorded in Table 2 and total salling $S_1 + S_2 + S_3$ can be recorded in the Table 3 below:

	Unit prices
Hand-made cookies	4
Cakes	10
pies	8

Table2

	Hand-made cookies	cakes	pies
Monday	7	15	4
Tuesday	16	10	7
Wednesday	12	9	17
Thursday	20	5	27
Friday	20	15	28

Table 3 (total sailing)

The total income of each day can be calculated by the following:

$$\begin{array}{lll} \text{Monday} & 7\times4+15\times10+4\times8 & =(7,15,4)\cdot(4,10,8) \\ \text{Tuesday} & 16\times4+10\times10+7\times8 & =(16,10,7)\cdot(4,10,8) \\ \text{Wednesday} & 12\times4+9\times10+17\times8 & =(12,9,17)\cdot(4,10,8) \\ \text{Thursday} & 20\times4+5\times10+27\times8 & =(20,5,27)\cdot(4,10,8) \\ \text{Friday} & 20\times4+15\times10+28\times8 & =(20,15,28)\cdot(4,10,8) \end{array}$$

If we try to represent the tables into matrices we'll find that the calculation is just like the inner product of rows and columns of two matrices. That is the multiplication of matrices.

$$S = S_1 + S_2 + S_3 = \begin{bmatrix} 7 & 15 & 4 \\ 16 & 10 & 7 \\ 12 & 9 & 17 \\ 20 & 5 & 27 \\ 20 & 15 & 28 \end{bmatrix}, \ P = \begin{bmatrix} 4 \\ 10 \\ 8 \end{bmatrix}$$

$$S \times P = \begin{bmatrix} 7 & 15 & 4 \\ 16 & 10 & 7 \\ 12 & 9 & 17 \\ 20 & 5 & 27 \\ 20 & 15 & 28 \end{bmatrix} \begin{bmatrix} 4 \\ 10 \\ 8 \end{bmatrix} = \begin{bmatrix} 7\times4+15\times10+4\times8 \\ 16\times4+10\times10+7\times8 \\ 12\times4+9\times10+17\times8 \\ 20\times4+5\times10+27\times8 \\ 20\times4+15\times10+28\times8 \end{bmatrix} = \begin{bmatrix} 210 \\ 220 \\ 274 \\ 346 \\ 454 \end{bmatrix}$$

Matrix multiplication

$A=[a_{ij}]_{m\times n}$ is a $m\times n$ matrix and $B=[b_{ij}]_{n\times p}$ k is a $n\times p$ matrix.

Then we can define the matrix multiplication of A and B:

$AB=C=[c_{ij}]_{m\times p}$ is a $m\times p$ matrix and satisfies

$$c_{ij}=(a_{i1}b_{1j})+(a_{i2}b_{2j})+...+(a_{in}b_{nj}) \quad \text{(inner product of } i\text{-th row and } j\text{-th column)}$$

<Key> The numbers of columns of A must equal the numbers of rows of B, then we can form the product matrix AB.

Example8. For the matrices A and B,

$A=\begin{pmatrix}3 & -2\\ 1 & 4\\ 2 & 5\end{pmatrix}$, $B=\begin{pmatrix}1 & 4 & 3\\ 2 & 2 & 1\end{pmatrix}$, find matrix multiplication (1) AB (2)BA

製作者：國立臺灣師範大學附屬高級中學 蕭煜修