期望值

Expected value

<table>
<tr><th colspan="2">第 1 節
1st Period</th></tr>
<tr><th>Material</th><th>Note</th></tr>
<tr><td>
我們來看看這個例子：百貨公司將福袋 200 份包裝後讓來賓抽獎，其中價值 1000 元的 A 獎有 20 份，價值 500 元的 B 獎有 80 份，價值 300 元的 C 獎有 100 份。那麼，每份福袋的平均價值是多少呢？
| 福袋 | A | B | C |
| 價值（元） | 1000 | 500 | 300 |
| 數量 | 20 | 80 | 100 |
計算每份福袋的平均價值：
$$\frac{1000\times 20+500\times 80+300\times 100}{200}=450$$（元）。
上式可以改寫成
$$1000\times\frac{20}{200}+500\times\frac{80}{200}+300\times\frac{100}{200}=450$$（元）。</td>
<td>Word：Expected value (期望值), Mean (平均值), Weighted Arithmetic Mean (加權平均值), outcome (結果), mystery grab bag(福袋)。
Sentence：
1. We can compute the weighted mean of the mystery grab bags. (我們可以計算福袋的加權平均數。)
2. We can rewrite this formula into… (將其改寫成…)
3. We can list all of our outcomes and the probability associated with each of those outcomes. (將結果及其機率列出來。)</td></tr>
<tr><td>期望值
設 S 為一試驗的樣本空間，$A_1, A_2, \ldots, A_n$ 為兩兩交集為空集合的事件，且 $S=A_1\cup A_2\cup\cdots\cup A_n$。若對每個 $i=1,2,\ldots,n$，事件 A_i 發生的機率為 p_i，且此時可得對應值 m_i，其中 m_i 為實數，則稱
$$E=m_1p_1+m_2p_2+\cdots+m_np_n$$
為此試驗的數學期望值，簡稱期望值。</td>
<td>Word: discrete (離散的), random (隨機), variable (變數), converge (收斂), distribution (分配), for short (簡稱).
Translation:
Set S is a sample space of a trial. $A_1, A_2, \ldots, A_n$ are mutually exclusive events and $S=A_1\cup A_2\cup\cdots\cup A_n$. For any $i=1,2,\ldots,n$, each A_i is associated with its probability p_i, and it is corresponding to m_i, which m_i is a real number.</td></tr>
</table>

Note that

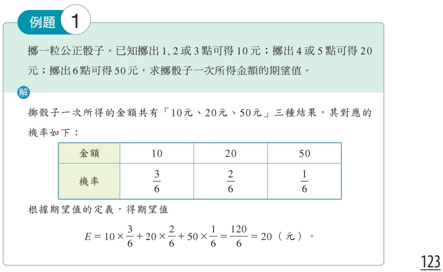
$$E = m_1 p_1 + m_2 p_2 + \cdots + m_n p_n$$

is the **expected value** of this trial.

Another Definition:

Let X be a numerically-valued discrete random variable with sample space Ω and distribution function $m(x)$. The *expected value* $E(x)$ is defined by $E(x) = \sum_{x \in \Omega} x m(x)$, provided this sum converges absolutely. We often refer to the expected value as the *mean*, and denote $E(x)$ by μ for short.

Note: $A \cup B$ is read as A union B.



例題 1

擲一粒公正骰子。已知擲出 1, 2 或 3 點可得 10 元；擲出 4 或 5 點可得 20 元；擲出 6 點可得 50 元，求擲骰子一次所得金額的期望值。

解

擲骰子一次所得的金額共有「10元、20元、50元」三種結果，其對應的機率如下：

金額	10	20	50
機率	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

根據期望值的定義，得期望值

$$E = 10 \times \frac{3}{6} + 20 \times \frac{2}{6} + 50 \times \frac{1}{6} = \frac{120}{6} = 20（元）。$$

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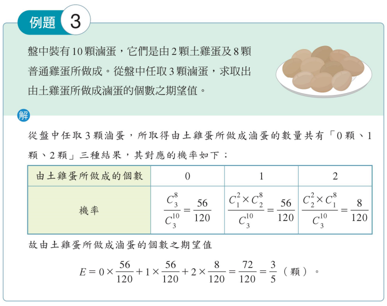
Translation:

There are three outcomes 10, 20 and 30 dollars by rolling a die one time. The probability associated with each of those outcomes is as follows:

Money	10	20	50
Prob.	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

By the definition of expected value, we've got E of x is equal to 10 dollars times the probability $\frac{3}{6}$ add 20 dollars times the probability $\frac{2}{6}$ add 50 dollars times the probability $\frac{1}{6}$ is equal to 20 dollars.



例題 3

盤中裝有10顆滷蛋，它們是由2顆土雞蛋及8顆普通雞蛋所做成。從盤中任取3顆滷蛋，求取出由土雞蛋所做成滷蛋的個數之期望值。

解

從盤中任取3顆滷蛋，所取得由土雞蛋所做成滷蛋的數量共有「0顆、1顆、2顆」三種結果，其對應的機率如下：

由土雞蛋所做成的個數	0	1	2
機率	$\frac{C_3^8}{C_3^{10}}=\frac{56}{120}$	$\frac{C_1^2\times C_2^8}{C_3^{10}}=\frac{56}{120}$	$\frac{C_2^2\times C_1^8}{C_3^{10}}=\frac{8}{120}$

故由土雞蛋所做成滷蛋的個數之期望值

$$E=0\times\frac{56}{120}+1\times\frac{56}{120}+2\times\frac{8}{120}=\frac{72}{120}=\frac{3}{5}\text{（顆）。}$$

Word: braised eggs(滷蛋), free-range chicken(土雞)

Translation:

We take any 3 braised eggs. There are three outcomes That these braised eggs were produced by free-range chicken 1, 2 and 3. The corresponding probability is as follows:

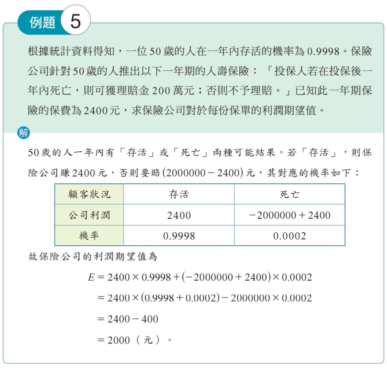
The number of braised eggs produced by free-range chicken	0	1	2
Probability	$\frac{C_3^8}{C_3^{10}}=\frac{56}{120}$	$\frac{C_1^2\times C_2^8}{C_3^{10}}=\frac{56}{120}$	$\frac{C_2^2\times C_1^8}{C_3^{10}}=\frac{8}{120}$

Hence, the expected value for the number of braised eggs produced by free-range chicken is

$$E=0\times\frac{56}{120}+1\times\frac{56}{120}+2\times\frac{8}{120}=\frac{3}{5}.$$

Note:

1. C_k^n can be written in ${}^nC_k$, ${}_nC_k$, nCk or $C(n,k)$.
2. C_k^n can be read as "n C k" or "n choose k".



例題 5

根據統計資料得知，一位50歲的人在一年內存活的機率為0.9998。保險公司針對50歲的人推出以下一年期的人壽保險：「投保人若在投保後一年內死亡，則可獲理賠金200萬元；否則不予理賠。」已知此一年期保險的保費為2400元，求保險公司對於每份保單的利潤期望值。

解

50歲的人一年內有「存活」或「死亡」兩種可能結果。若「存活」，則保險公司賺2400元，否則要賠$(2000000-2400)$元，其對應的機率如下：

顧客狀況	存活	死亡
公司利潤	2400	$-2000000+2400$
機率	0.9998	0.0002

故保險公司的利潤期望值為

$$\begin{aligned}E&=2400\times0.9998+(-2000000+2400)\times0.0002\\&=2400\times(0.9998+0.0002)-2000000\times0.0002\\&=2400-400\\&=2000\text{（元）。}\end{aligned}$$

Translation:

There are two possible outcomes within 1 year, "Life" or "death", of a 50-year-old man. If the circumstances is "Life", then the net incomes of insurance company will be 2,400 dollars. Otherwise, the company will pay 2,000,000 – 2,400 (2 million minus 2 thousand and 4 hundred) dollars. The corresponding probability is as follows:

Customer	Life	Death
Net incomes	+2,400	+2,000,000 – 2,400
Probability	0.9998	0.0002

Hence, the expected value for the profit of insurance company is

$$\begin{aligned} E &= 2400 \times 0.9998 + (-2000000 + 2400) \times 0.0002 \\ &= 2400 \times (0.9998 + 0.0002) - 2000000 \times 0.0002 \\ &= 2400 - 400 \\ &= 2000 \ (dollars). \end{aligned}$$

補充題

Material

In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Solution 1:

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" $-\frac{\$2}{\$38} \approx -\$0.053$. In other words, on average you lose 5.3 cents per space you bet on

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

Solution 2:

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

outcome	\$35	-\$1
Probability of outcome	$\frac{1}{38}$	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get $\$35\cdot\frac{1}{38}\approx 0.9211$ and $-\$1\cdot\frac{37}{38}\approx -0.9737$, and if we add these numbers we get $0.9211+(-0.9737)=-0.053$, which is the expected value we computed above.

Note

Word: Roulette (輪盤), Spun (spin 旋轉的過去分詞), Summarizing (總結), Complement (補集), Corresponding (對應).

Sentence:

1. On average, how much money should a player expect to win or lose if they play this game repeatedly? (如果他們重覆的玩這個遊戲，一個玩家平均來說會得到多少錢?)
2. Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. (如果你在輪盤的 38 個格子都下 1 元的注，總共下注 38 元。)
3. The complement, the probability of losing, is $\frac{37}{38}$. (它的補集是$\frac{37}{38}$，也就是輸的機率。)

參考資料

References

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