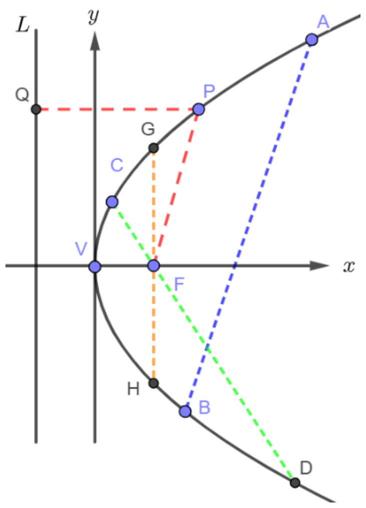
Parabola

I. Key mathematical terms

<table>
<tr><th>Terms</th><th>Symbol</th><th>Chinese translation</th></tr>
<tr><td>Parabola</td><td rowspan="6">
</td><td></td></tr>
<tr><td>Vertex</td><td></td></tr>
<tr><td>Chords</td><td></td></tr>
<tr><td>Focus and focal length</td><td></td></tr>
<tr><td>Axis of symmetry</td><td></td></tr>
<tr><td>Directrix</td><td></td></tr>
</table>

II. Definition of the parabola

We have learned the properties of quadratic functions ($f(x) = ax^2 + bx + c$) in grade

10. The graphs of these quadratic functions are parabolas. In our daily lives, parabolas can be observed in various fields of applications, such as:

1. **Projection Motion**

   The path of a projectile (Throwing an object or launching a rocket) follows a parabolic trajectory under the influence of gravity. (重力) Engineers and physicists use the equations of parabolas to predict and analyze the motion of projectiles.

2. **Building Design**

   Some architectural structures, such as the Sydney Opera House, incorporate parabolic shapes for both aesthetic and structural reasons.

Sydney Opera House

https://unsplash.com/photos/opera-house-sydney-australia-xdGCrwu2CWM

3. **<u>Telescope Mirrors</u>**

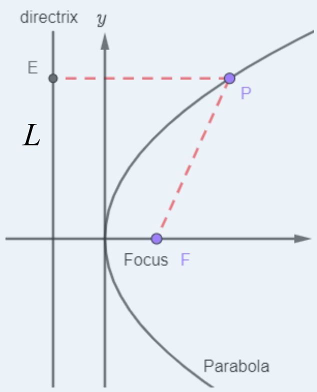
   The mirrors in reflecting telescopes are often shaped as segments of a parabola. This shape allows incoming parallel light rays to gather at a single point, improving the quality of the image.

With the examples above, you may understand that we'll use parabolas in many different fields' applications. How do these applications work? They actually use the definition and properties of the parabola. Let's take a closer look at the definition of a parabola.

Definition of the parabola

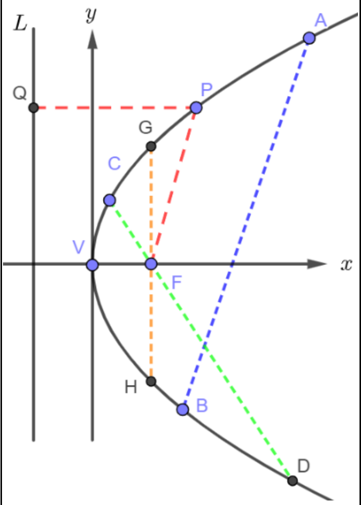
A parabola is a curve where any point is at an equal distance from:

1. A fixed point (focus), and
2. A fixed straight line (directrix)



P is the point on the parabola.
F is the focus of the parabola.
L is the directrix of the parabola.

<key>



(1) Vertex: (point V on the graph) The intersection of the parabola and its axis of symmetry. Also, it's the point at which the parabola makes its sharpest turn.
(2) Focal length: (distance $\overline{VF}$) Focal length is the distance between the vertex and the focus of the parabola. (Along the axis of symmetry.)
(3) Chords: (segment $\overline{AB}$, $\overline{CD}$, $\overline{GH}$) A chord of a parabola refer to the line segment that connects two distinct points on the parabola.
(4) Focal chord ($\overline{CD}$, $\overline{GH}$) and focal diameter: The focal chord is the chord that passes through the focus. The focal diameter is the segment that passes through the focus and is parallel to the directrix.

Example 1

Prove that :focal diameter of a parabola = 4* focal length (正焦弦長=4 倍焦距)

Hint: You can draw an arbitrary parabola with its directrix and latus rectum.

Example 2

Given the parabola with focus $F(1,-1)$ and directrix $L: y=x+2$. Find:

(1) The function of the symmetric axis,

(2) The vertex of the parabola,

(3) The focal length of the parabola.

III. Equation of the parabola

The equation of a parabola can help us solve many more complex problems. We use the definition to derive the equation of circle $C:(x-h)^2+(y-k)^2=r^2$ (This circle is centered at (h,k) and has a radius of r). To derive the equation of a parabola, we use the definition "parabola is a curve generated by a point moving such that its distance from a fixed point is equal to its distance from a fixed line." Let's take a look at the following example

Example 3

Use the definition of the parabola to find the equation of a parabola with focus $F(1,1)$ and directrix $L: x+y+3=0$.

<illustration>

Let point $P(x,y)$ be the point on the parabola, by definition we have:

$$\sqrt{(x-1)^2+(y-1)^2}=\frac{|x+y+3|}{\sqrt{2}} \quad (\overline{PF}=d(P,L))$$

Square both sides, and we have: $x^2-2xy+y^2-10x-10y-5=0$ ◆

This is the equation of a parabola with a symmetric axis not parallel with the coordinate axis. However, in our high school curriculum, the symmetric axes of parabolas we are mainly concerned with are either parallel or perpendicular to the coordinate axes. Let’s move on to further discussion.

Example 4

Use the definition of a parabola to find the equation of a parabola with:

(1) Focus $F_1(0,4)$ and directrix $L_1: y=-4$

(2) Focus $F_2(0,-4)$ and directrix $L_1: y=4$

After reviewing the above examples, you may wonder if there is a general method for solving quadratic equations and what commonalities exist among them. Next, let’s take a look at the explanatory example together.

Example 5

Find the equation of the parabola with focus $F(0,c),\ c\neq 0$ and directrix $L: y=-c$

<illustration>

Let point $P(x,y)$ be the point on the parabola, by definition we have:

$$d(P,L)=\overline{PF}$$

Then we use the distance formula to get:

$$d(P,L)=|y+c|=\sqrt{x^2+(y-c)^2}=\overline{PF}$$

Square both sides, and we have:

$$(y+c)^2=x^2+(y-c)^2$$

Then we can simplify to get the final result:

$$x^2=4cy \blacklozenge$$

Conversely, if a point $P(x,y)$ satisfy the equation $x^2=4cy$, then the point P lies on the parabola. $x^2=4cy$ is a parabola with the x-axis as the axis of symmetry. Now, what if the parabola has the y-axis as the axis of symmetry? Let’s look at the next example.

Example 6

Find the equation of the parabola with focus $F(c,0),\ c \neq 0$ and directrix $L: x = -c$

<illustration>

We can use the same process in Example5.

Let point $P(x,y)$ be the point on the parabola, by definition we have:

$$d(P,L) = \overline{PF}$$

Then we use the distance formula to get:

$$d(P,L) = |x+c| = \sqrt{(x-x)^2 + y^2} = \overline{PF}$$

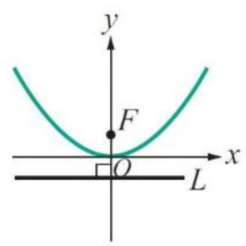
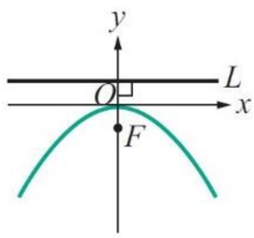
Square both sides, and we have:

$$(x+c)^2 = (x-c)^2 + y^2$$

Then we can simplify to get the final result:

$$y^2 = 4cx \blacklozenge$$

With the examples above, we can draw the following table:

<table>
<tr><th>Equation</th><th>Vertex</th><th colspan="2">Symmetric axis</th><th>Focal length</th></tr>
<tr><td rowspan="3">$x^2 = 4cy$</td><td>$(0,0)$</td><td colspan="2">y-axis</td><td>$|c|$</td></tr>
<tr><td colspan="2">$c > 0$ (concave upward)</td><td colspan="2">$c < 0$ (concave downward)</td></tr>
<tr><td colspan="2">
</td><td colspan="2">
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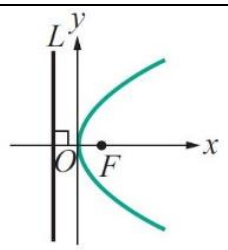
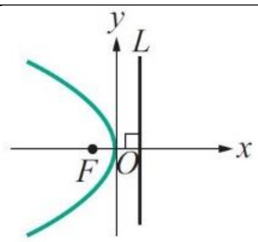
<tr><td rowspan="3">$y^2 = 4cx$</td><td>$(0,0)$</td><td colspan="2">x-axis</td><td>$|c|$</td></tr>
<tr><td colspan="2">$c > 0$ (concave to the right)</td><td colspan="2">$c < 0$ (concave to the left)</td></tr>
<tr><td colspan="2">
</td><td colspan="2">
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</table>

Hence, if we already have the vertex $(0,0)$ and symmetric axis $(y = 0 \text{ or } x = 0)$ of a parabola, we only need to know the focal length and the direction of opening, we can write the equation of the parabola. Let's try some more examples.

Example 7

A parabola has vertex at $(0,0)$ and a symmetric axis at $x=0$ and passed through the point $M(-2\sqrt{2},2)$. Find the following:

(1) The equation of the parabola
(2) The concave direction (Upward/downward/to the right/to the left)
(3) Focal length
(4) Focus
(5) Directrix

Now, we know how to find a parabola's equation when the parabola has a vertex at the origin and symmetry to the coordinate axis. What if the vertex of our parabola is not at the origin, and the symmetry axis is not the x-axis or the y-axis? In fact, we can achieve the desired result by applying the concept of translation (similar to what we've learned in the operations on linear functions, quadratic functions, and cubic functions). Using the same method, we can determine the equation we need. Let's try the following examples:

Example8

Given a parabola $\Gamma_1 : x^2 = 4cy \ (c \neq 0)$, translating Γ_1 along the vector $\vec{v} = (-2,3)$ results in a new parabola Γ_2. Find the following:

(1) The vertex of Γ_2
(2) The equation of Γ_2

<illustration>

(1) The vertex of Γ_1 is $(0,0)$. All the points on Γ_1 move along the vector $\vec{v} = (-2,3)$, hence the vertex of Γ_2 is $(0,0)+(-2,3)=(-2,3)$

(2) The equation of Γ_2 is $(x+2)^2 = 4c(y-3)$

Example 9

(1) Find the equation of the parabola with the vertex at $V(5,1)$ and the directrix L at $x=1$

(2) Find the equation of the parabola with the focus at $F(-2,3)$ and the vertex at $V(-2,5)$

(3) Find the equation of parabola with the focus at $F(4,5)$ and the vertex at $V(-2,5)$

(What are the differences between question (2) and question (3)? What similarities are there?)

<資料來源>

1. **Definition of parabola/ Equation of the parabola**
   https://www.mathsisfun.com/geometry/parabola.html
   https://byjus.com/jee/parabola/
   https://phys.libretexts.org/Bookshelves/Astronomy__Cosmology/Celestial_Mechanics_(Tatum)/02%3A_Conic_Sections/2.03%3A_The_Parabola
   https://www.cuemath.com/geometry/parabola/

2. **南一書局數學甲上冊**

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