Topic: Sum and Difference Formulas part 2

1. The sum and difference formula of $\tan\theta$

a. Fill in the blanks to complete the sum formula of $\tan\theta$

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

=____________________(Expand)

=____________________(Divide both the numerator and denominator by $\cos\alpha\cos\beta$)

=____________________(Simplify the fraction to get the terms like $\frac{\sin\alpha}{\cos\alpha}$, $\frac{\sin\beta}{\cos\beta}$)

=____________________(Express in terms of $\tan\alpha$, $\tan\beta$)

b. Use the sum formula of $\tan\theta$ then substitute β into $-\beta$ to get the difference formula.

$\tan(\alpha-\beta)=\tan(\alpha+(-\beta))=$ ____

使用建議	
教學活動安排	從之前學過的正餘弦的和角公式去嘗試推導正切和角公式。

使用建議	
英文提問/開場	Today we are going to learn the sum and difference formula of $\tan\theta$ What's the relationship between $\sin\theta$, $\cos\theta$ and $\tan\theta$? Good! So we're going to use the sum formula we've learned to get the sum formula of $\tan\theta$. Now complete the following blanks, we'll check-in in five minutes. Anyone wants to share your work? For the difference formula of $\tan\theta$, use the sum formula of $\tan\theta$ then substitute β into $-\beta$ to get the difference formula. Go and give it a try. We'll check-in in five minutes. Anyone want to share your work? Excellent! 底下的提問是為了加深學生對公式的印象，教師可以自行決定是否運用。 Now let's have a discussion to have a deeper impression about this formula. How is the process of getting formulas connected to what you already learned? Share with your partner. Anyone wants to share your thoughts or your partner's opinion? What part of the formulas do you find similar to what you already learned? How do the formulas or the process open up to what you already learned? What challenges or difficulties came to your mind? Do you think it's a challenge to memorize this formula?
參考資料	泰宇版高中數學（三）A。

使用建議	
參考答案	$\tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$ ← 商數關係：$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ $=\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$ $=\dfrac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$ ← 同除以 $\cos\alpha\cos\beta$，來製造出 $\dfrac{\sin\alpha}{\cos\alpha}$ 及 $\dfrac{\sin\beta}{\cos\beta}$ $=\dfrac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}}$ $=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$。 將上式中的 β 換成 $-\beta$，再利用負角性質 $\tan(-\beta)=-\tan\beta$，可以得到 $\tan(\alpha-\beta)=\tan(\alpha+(-\beta))=\dfrac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$。

2. Practice of using the formulas

a. Fill in the blanks to evaluate $\tan 75°$ and $\tan 15°$

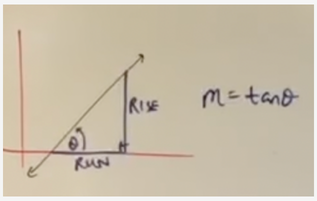
$\tan 75° = \tan(\square + \square) =$ ____________________=________

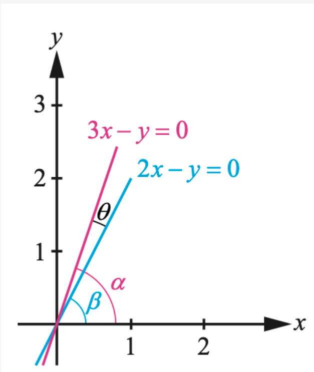
$\tan 15° = \tan(\square - \square) =$ ____________________=________

b. Verify the identity $\tan(\pi+\theta)=\tan\theta$

使用建議	
教學活動安排	練習使用公式及其應用

使用建議	
英文教學過程	a. Fill in the blanks to evaluate $\tan 75^\circ$ and $\tan 15^\circ$ $\tan 75^\circ = \tan(\square + \square) =$ _______________=______ $\tan 15^\circ = \tan(\square - \square) =$ _______________=______ Let's use the sum and difference formula of $\tan\theta$ to find the exact value of $\tan 75^\circ$ and $\tan 15^\circ$. We'll check-in in five minutes. Anyone wants to share your work? Excellent! b. Verify the identity $\tan(\pi+\theta)=\tan\theta$ Let's see another application of the sum and difference formula. It's really useful to verify some kind of identity like this one. Have a try, expand it by the sum formula of $\tan\theta$ then simplify it. We'll check-in in five minutes. Anyone want to share your work? Excellent!
參考答案	a. $\tan 75^\circ = \tan(45^\circ+30^\circ) = \dfrac{\tan 45^\circ + \tan 30^\circ}{1-\tan 45^\circ \tan 30^\circ} = \dfrac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = 2+\sqrt{3}$ $\tan 15^\circ = \tan(45^\circ-30^\circ) = \dfrac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ} = \dfrac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = 2-\sqrt{3}$ b. $\tan(\pi+\theta) = \dfrac{\tan\pi + \tan\theta}{1-\tan\pi\tan\theta} = \dfrac{0+\tan\theta}{1-0} = \tan\theta$
參考資料	泰宇版高中數學（三）A。

<table>
<tr><th colspan="2">使用建議</th></tr>
<tr><td>教學活動</td><td>用一個例子來講解如何應用斜率及正切差角公式來求兩直線夾角。</td></tr>
<tr><td>英文教學過程</td><td>
Another application of the difference formula of $\tan\theta$ is to find the angles between two lines.
We've learned that the angle of inclination of a line is the angle between the line and positive direction of the x-axis.


Using rise over run equals m for slope($\frac{rise}{run} = m$).
With the angle of inclination θ , we get $m = \tan\theta$.
Let's do this example together.


There are two lines $L_1 : 3x - y = 0$ and $L_2 : 2x - y = 0$.
So, let α is the angle of inclination of $L_1 : 3x - y = 0$, then we get $\tan\alpha = 3$.
Let β is the angle of inclination of $L_2 : 2x - y = 0$, then we get $\tan\beta = 2$.
The angle θ is the angle between L_1 and L_2, then $\theta = \alpha - \beta$
So, $\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{3-2}{1+3\times 2} = \frac{1}{7}$
Using the calculator, we get $\theta \approx 8.13^\circ$.
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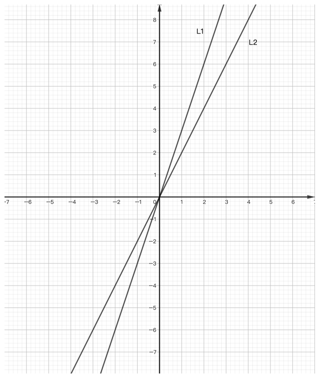
3. Find the angle between two lines

Use the difference formula of $\tan\theta$ to find the acute angle between $L_1 : 3x - y = 0$ and $L_2 : 2x - y = 0$ (Hint: We've learned to use the slope of a line to find its angle of inclination.)



使用建議	
參考資料	1.泰宇版高中數學（三） A 。 2.https://youtu.be/XnEW60rYqLE?si=16XRJM1QNYAiDDAT 3. The angle inclination of a line https://www.brightstorm.com/math/trigonometry/advanced-trigonometry/angle-inclination-of-a-line/

5. Applications of the sum formulas-double angle formula

Fill in the blank to complete the proof of double-angle formulas

a. $\sin 2\theta = \sin(\theta + \theta) = \sin\square\cos\square + \cos\square\sin\square =$ ___________

b. $\cos 2\theta = \cos(\theta + \theta) = \cos\square\cos\square - \sin\square\sin\square =$ ___________=___________=_________

c. $\tan 2\theta = \tan(\theta + \theta) =$ _______________

使用建議	
教學活動安排	利用和角公式推導倍角公式

使用建議	
英文提問/開場	Another application of the sum formulas is the double-angle formulas. What exactly are these double-angle formulas? As the name suggests, we expand the double-angle expression into the sum of two angles and then apply the sum formulas to derive the result Fill in the blank on the worksheet to complete the proof of double-angle formulas. Go and have a try. We'll check-in in five minutes. Anyone want to share your work? Good job!
參考資料	泰宇版高中數學（三）A。
答案	a. $\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$ b. $\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$ $= (1-\sin^{\theta}) - \sin^2\theta = 1 - 2\sin^2\theta$ $= \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$ c. $\tan 2\theta = \tan(\theta+\theta) = \dfrac{\tan\theta + \tan\theta}{1-\tan\theta\tan\theta} = \dfrac{2\tan\theta}{1-\tan^2\theta}$

6. Practice the double-angle formula

Find the exact value of $\sin 15^\circ$ and $\cos 15^\circ$ by using $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

使用建議	
教學活動安排	倍角公式應用練習

使用建議	
英文提問/開場	The significance of learning the double-angle formulas becomes evident in various applications when solving trigonometric equations, particularly as you advance in your engineering courses at university. We won't dive into those advanced and complex trigonometric equations just yet. Let's take a look at a simpler application. The double-angle formulas offer an alternative method to determine the exact values of $\sin 15^{\circ}$ and $\cos 15^{\circ}$. Go and have a try. We'll check-in in five minutes. Anyone want to share your work? Good job!
參考資料	泰宇版高中數學（三）A。
答案	a. $\cos 30^{\circ} = 1 - 2\sin^2 15^{\circ}$ $\Rightarrow \sin^2 15^{\circ} = \dfrac{1-\cos 30^{\circ}}{2}$ $\Rightarrow \sin 15^{\circ} = \sqrt{\dfrac{1-\cos 30^{\circ}}{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ b. $\cos 30^{\circ} = 2\cos^2 15^{\circ} - 1$ $\Rightarrow \cos^2 15^{\circ} = \dfrac{1+\cos 30^{\circ}}{2}$ $\Rightarrow \cos 15^{\circ} = \sqrt{\dfrac{1+\cos 30^{\circ}}{2}} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

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