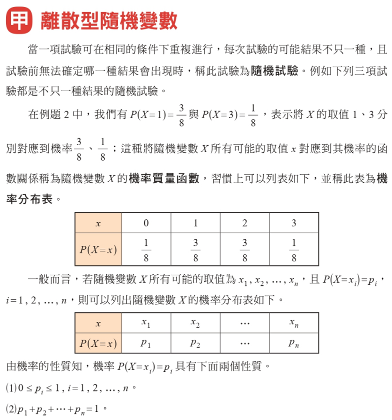
離散型隨機變數

Discrete Random Variables

Material	Vocabulary
	1. discrete (離散), 2. outcome (結果), 3. in advance (事先), 4. die (骰子), 5. toss (擲), 6. fair (公平), 7. character (字符), 8. associate (關聯), 9. assign (分配), 10. obtain (得到), 11. upper case (大寫), 12. lower case (小寫), 13. subscript (下標), 14. tabulation (製表), 15. vertical (垂直), 16. finite (有限的), 17. infinite (無限的), 18. various (各式各樣), 19. exhaustive (詳盡的), 20. occur (發生), 21. defect (缺點), 22. uniform (勻值), 23. without replacement (取後不放回), 24. scenario (設想).

甲 離散型隨機變數

當一項試驗可在相同的條件下重複進行，每次試驗的可能結果不只一種，且試驗前無法確定哪一種結果會出現時，稱此試驗為**隨機試驗**。例如下列三項試驗都是不只一種結果的隨機試驗。

在例題 2 中，我們有 $P(X=1)=\frac{3}{8}$ 與 $P(X=3)=\frac{1}{8}$，表示將 X 的取值 1、3 分別對應到機率 $\frac{3}{8}$、$\frac{1}{8}$；這種將隨機變數 X 所有可能的取值 x 對應到其機率的函數關係稱為隨機變數 X 的**機率質量函數**，習慣上可以列表如下，並稱此表為**機率分布表**。

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

一般而言，若隨機變數 X 所有可能的取值為 $x_1, x_2, \ldots, x_n$，且 $P(X=x_i)=p_i$，$i=1, 2, \ldots, n$，則可以列出隨機變數 X 的機率分布表如下。

x	x_1	x_2	$\cdots$	x_n
$P(X=x)$	p_1	p_2	$\cdots$	p_n

由機率的性質知，機率 $P(X=x_i)=p_i$ 具有下面兩個性質。

(1) $0 \le p_i \le 1$, $i=1, 2, \ldots, n$。

(2) $p_1+p_2+\cdots+p_n=1$。

Illustrations I

Discrete[1] Random Variables

A **random experiment** is a process that can be repeated under the same conditions, and the outcome[2] of the experiment is more than one. The specific outcome is not known in advance[3]. Example of random experiments include throwing a six-sided die[4], tossing[5] a fair[6] coin, or drawing balls from a bag.

當一項試驗可在相同的條件下重複進行，每次試驗的可能結果不只一種，且試驗前無法確定哪一種結果會出現時，稱此試驗為隨機試驗。例如擲骰子、丟硬幣或抽球都是不只一種結果的隨機試驗。

(1) Throwing a Six-Sided Die: The outcome could be any number from 1 to 6, resulting in a total of 6 possible results. 擲一粒骰子，可能出現1，2，3，4，5 或6 點，共有6 種結果。

(2) Tossing a Fair Coin Two times: The outcomes could be (heads, heads), (heads, tails), (tails, heads), or (tails, tails), resulting in a total of 4 possible results. 丟一枚硬幣兩次，可能是(正, 正)、(正, 反)、(反, 正) 或(反, 反)，共有4 種結果。

(3) Drawing a Ball from a bag that contains red balls and blue balls: The outcome could be either

red or blue, resulting in only 2 possible results. 從只裝有紅球與白球的袋中取一球，可能是紅球或白球，共有2 種結果。

The outcomes of a random experiment can be a variety of things, such as real numbers, **characters**[7] (Tail, Head, ...) or symbols ("☺" for Yes, "☹" for No, ...). To make studying easier, we can use a rule where each outcome of a random experiment is **associated**[8] with a real number. For example, considering the experiment of tossing a fair coin two times, and expressing an interest in the occurrences of heads, then we can build a correspondence between the outcomes and the number of times heads appear as follows.

隨機試驗的結果有很多樣式，有些是數值，有些是文字或符號等。為了方便研究，通常會依感興趣的內容，將試驗的結果對應到一些特定的實數值。例如丟一枚硬幣兩次，若感興趣的是正面出現的次數，則會將試驗結果與正面出現的次數對應如下。

outcomes	corresponding to	The occurrences of heads
(tails, tails)	→	0
(tails, heads)	→	1
(heads, tails)	→	1
(heads, heads)	→	2

The relationship that **assigns**[9] each possible outcome (i.e., the sample space) to one and only one real number in a random experiment is called a random variable, usually denoted as *X*.

像這樣將隨機試驗所有可能發生的結果（即樣本空間）對應到一個實數值的函數關係稱為隨機變數，通常以X 表示。

Consider tossing a fair coins two times, where we can **obtain**[10] 0, 1 or 2 heads. The number of heads obtained in each trial, *X*, is a discrete random variable and $X \in \{0,1,2\}$.

令 X 表示丟一枚硬幣兩次正面出現的次數，那麼 X 的值可為 0 , 1 , 2，此時

"$X = 0$" represents "the event of heads appearing 0 times."

"$X = 1$" represents "the event of heads appearing 1 time."

"$X = 2$" represents "the event of heads appearing 2 times."

「$X = 0$」表示「正面出現 0 次的事件」；

「$X = 1$」表示「正面出現1 次的事件」；

「$X = 2$」表示「正面出現 2 次的事件」。

A discrete random variable is usually denoted by an **upper case**[11] letter, such as *X*, *Y*, or *Z*. The values that the variable takes are denoted by **lower case**[12] letters, such as *x, y,* or *z*. Sometimes are given **subscript**[13] x_1, x_2, x_3, … . Thus, $P(X = x_1)$ is the probability that the discrete random variable *X* takes the value x_1.

離散型隨機變數通常以大寫字母表示，例如 *X*、*Y* 或 *Z*。其變數之取值以小寫字母表示，例如 *x*、*y* 或 *z*，有時候會在符號後面加下標，如 x_1, x_2, x_3, …。因此 $P(X = x_1)$ 為離散型隨機變數$X = x_1$ 的機率。

Probability distributions

The **probability mass function** (PMF) is a relationship that defines the probabilities associated with each possible outcome of a discrete random variable. The usual method of display is by **tabulation**[14] in a **probability distribution** table. The probability distribution also can be represented in a **vertical**[15] line graph or in a bar chart. With the previous example of tossing a fair coin two times, we have

機率分布表

機率質量函數用以表示機率與隨機變數之間的關係，通常以表格的形式呈現，也能使用長條圖或直方圖。以擲骰子兩次為例，可得：

$P(X=0)=P(\text{tails, tails})=0.5\times 0.5=0.25$

$P(X=1)=P(\text{heads, tails})+P(\text{tails, heads})=(0.5\times 0.5)+(0.5\times 0.5)=0.5$

$P(X=2)=P(\text{heads, heads})=0.5\times 0.5=0.25$.

The probability distribution for *X* is displayed in the following table.

x	0	1	2
$P(X=x)$	0.25	0.5	0.25

Generally, for a random variable *X*, where all possible values are $x_1, x_2, \ldots, x_n$, and $P(X=x_i)=p_i$ for $i=1,2,\ldots,n$, we can tabulate the probability distribution as follows.

一般來說，隨機變數 *X* 其可能的取值為 $x_1, x_2, \ldots, x_n$ 且 $P(X=x_i)=p_i$，其中 $i=1,2,\ldots,n$，我們可將機率質量函數以表格呈現如下：

x	x_1	x_2	$\cdots$	x_n
$P(X=x)$	p_1	p_2	$\cdots$	p_n

From the properties of probability, the probability $P(X=x_i)=p_i$ has two characteristics: (1) $0\le p_i\le 1$, $i=1,2,\ldots,n$ (2) $p_1+p_2+\cdots+p_n=1$.

根據機率的基本性質知，機率$P(X=x_i)=p_i$具有下面兩個性質：(1)$0\le p_i\le 1$, $i=1,2,\ldots,n$ (2) $p_1+p_2+\cdots+p_n=1$.

Discrete random variable may have a **finite**[16] or an **infinite**[17] number of possible outcomes. Consider that we keep throwing a fair six-sided die until we roll a 6, in which case the distribution may be considered infinite.

隨機變數X也有無限多個取值的情況，例如連續擲一粒骰子，擲到6點才停止，此時隨機變數所有可能的取值就有無限多個。

A infinite discrete random variable has infinite outcomes $x_1, x_2, x_3, \ldots, x_n, \ldots$, with associated probabilities $p_1, p_2, p_3, \ldots, p_n, \ldots$, then the sum of the probabilities must equal 1. Since the **various**[18] outcomes cover all possibilities, they are **exhaustive**[19]. We have

(1) $0\le p_i\le 1, i=1, 2, 3, \ldots, n, \ldots$

(2) $p_1+p_2+\cdots+p_n+\cdots=1$.

若隨機變數X的取值為 $x_1, x_2, x_3, \ldots, x_n, \ldots$ ，且其對應的機率為 $p_1, p_2, p_3, \ldots, p_n, \ldots$ ，則這些機率具有下面兩個性質。

(1) $0\le p_i\le 1, i=1, 2, 3, \ldots, n, \ldots$

(2) $p_1+p_2+\cdots+p_n+\cdots=1$.

A well-known example of an infinite discrete random variable **occurs**[20] in the Poisson distribution (Ex: The number of **defects**[21] in a wire cable can be modeled by the Poisson distribution with a **uniform**[22] rate of 1.5 defects per kilometer. Find the probability that a single kilometer of wire will have at least 5 defects), while the study of discrete random variables in this chapter will limited to finite cases.

其中有名的無限離散隨機變數為卜瓦松分布，例：電纜中的缺陷數量為卜瓦松分布，其中每公里平均有 1.5 個缺陷，求一公里的電線至少有五個缺陷的機率。而此章節的離散型隨機變數則只討論有限多的例子。

Examples I

A bag contains 2 white balls and 3 red balls. Balls are drawn one at a time **without replacement**[23] until a red ball is drawn. The random variable X represents the number of balls are drawn.

(1) Find the probability distribution of X.

(2) Find the value of $P(x \leq 2)$.

Solution

(1) Let X = 1 represent the number of balls drawn as 1, and its color is red. The probability is

$$P(X=1)=\frac{3}{5}.$$

Let X = 2 represent the number of balls drawn as 2, with the first ball being white and the second ball being red. The probability is

$$P(X=2)=\frac{2}{5}\times\frac{3}{4}=\frac{3}{10}.$$

Let X = 3 represent the number of balls drawn as 3, with the colors being white, white, and red in that order. The probability is

$$P(X=3)=\frac{2}{5}\times\frac{1}{4}\times\frac{3}{3}=\frac{1}{10}.$$

Due to the **scenario**[24] of having only two white balls in the bag, It is for certain that the third draw will be a red bal; that is $x \leq 3$, which gives the following probability distribution

The number of balls are drawn x	1	2	3
$P(X=x)$	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{1}{10}$

(2) $P(X\leq 2)=P(X=1)+P(X=2)=\frac{3}{5}+\frac{3}{10}=\frac{9}{10}.$

Material

期望值

設隨機變數 X 的機率分布表如右。定義隨機變數 X 的期望值為

x	x_1	x_2	$\cdots$	x_n
$P(X=x)$	p_1	p_2	$\cdots$	p_n

$$E(X)=x_1p_1+x_2p_2+\cdots+x_np_n=\sum_{i=1}^{n}x_ip_i\text{。}$$

變異數與標準差

設隨機變數 X 的機率分布如右表，且期望值為 $E(X)=\mu$。定義隨機變數 X 的變異數為

x	x_1	x_2	$\cdots$	x_n
$P(X=x)$	p_1	p_2	$\cdots$	p_n

$$\mathrm{Var}(X)=(x_1-\mu)^2p_1+(x_2-\mu)^2p_2+\cdots+(x_n-\mu)^2p_n=\sum_{i=1}^{n}(x_i-\mu)^2p_i\text{，}$$

且 $\sigma(X)=\sqrt{\mathrm{Var}(X)}$ 稱為隨機變數 X 的**標準差**。

Vocabulary

25. expectation (期望值), 26. variance (變異數), 27. estimate (估計), 28. central tendency (集中趨勢), 29. spread (散布), 30. recall (回想), 31. standard deviation (標準差), 32. distinct (相異的), 33. regard (看待), 34. ordinary (普通的), 35. inclusive (包含), 36. alternate (備用), 37. wage (薪資).

Illustrations II

Expectation[25] and Variance[26]

To estimate[27] the probabilities of the various possible outcomes, we can compare the probability distributions by calculating measures of central tendency[28] and a measure of spread[29]. The most useful measure of central tendency is the mean or expectation of the random variable and the most useful measure of spread is the variance.

為了估計各種可能結果的機率，我們透過計算集中趨勢和離散值來比較各個機率分布。針對集中趨勢最有用的測量是隨機變數的平均值或期望值，而對於離散值的測量則是使用變異數。

Recall[30] that when an experiment has n results, $m_1, m_2, \ldots, m_n$, with associated the probabilities $p_1, p_2, \ldots, p_n$ its expectation is given by

當一試驗有 $m_1, m_2, \ldots, m_n$ 等 n 種結果，且每一種結果發生的機率分別為 $p_1, p_2, \ldots, p_n$ 時，此試驗的期望值為

$$E = m_1 p_1 + m_2 p_2 + \cdots + m_n p_n = \sum_{i=1}^{n} m_i p_i .$$

Similarly, the **expected value** or the **expectation** of a discrete random variable X is obtained by adding the products of each possible values, $x_1, x_2, \ldots, x_n$, multiplied by their corresponding probabilities.

同樣地，對於隨機變數 X，將 X 所有可能的取值 $x_1, x_2, \ldots, x_n$ 分別乘上相對應發生的機率之總和，就是隨機變數 X 的期望值。

The Expectation

Let X be a random variable assuming the values $x_1, x_2, \ldots, x_n$ with corresponding probabilities $p_1, p_2, \ldots, p_n$ The mean or expectation of X is defined by

$$E(X) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n = \sum_{i=1}^{n} x_i p_i ,$$

x	x_1	x_2	$\cdots$	x_n
$P(X=x)$	p_1	p_2	$\cdots$	p_n

where the probability distribution is shown on the right.

If it is an infinite discrete random variable X, then

當隨機變數 X 的取值有無限多個時，

$$E(X) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n + \cdots = \sum_{i=1}^{\infty} x_i p_i .$$

Before introducing the concepts of the variance and the standard deviation, let's review mean and variance: there are 6 figures, 4, 5, 7, 5, 4, 5, the mean is

在介紹變異數和標準差的概念之前，我們先回顧一下平均數和變異數：有 6 個數字，4, 5, 7, 5, 4, 5，平均數是

$$\begin{aligned} \mu &= \frac{1}{6}(4+5+7+5+4+5) \\ &= \frac{1}{6}(4\times 2+5\times 3+7\times 1) \\ &= 4\times\frac{2}{6}+5\times\frac{3}{6}+7\times\frac{1}{6}=5 \end{aligned}$$

Recall that when an experiment has n results, $m_1, m_2, \ldots, m_n$, the variance is

當一試驗有 $m_1, m_2, \ldots, m_n$ 等 n 種結果，此試驗的變異數為

$$\sigma^2 = \frac{(m_1-\mu)^2 + (m_2-\mu)^2 + \cdots + (m_n-\mu)^2}{n} = \frac{1}{n}\sum_{i=1}^{n}(m_i-\mu)^2 ,$$

and the **standard deviation**[31] is

且其標準差為

$$\sigma = \sqrt{\frac{(m_1-\mu)^2 + (m_2-\mu)^2 + \cdots + (m_n-\mu)^2}{n}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(m_i-\mu)^2} .$$

Observing these 6 figures, there are 3 **distinct**[32] values: 4, 5, and 7, each appearing 2, 3, and 1 times accordingly. This means they are associated with probabilities of $\frac{2}{6}$, $\frac{3}{6}$, and $\frac{1}{6}$, respectively. Thus, the mean of these 6 figures can be **regarded**[33] as the expectation of a random variable X. Moreover, the variance of these 6 figures is

觀察這 6 個數字，有 3 個不同的值：4、5 和 7，分別出現 2、3 和 1 次。這表示他們的機率分別為 $\frac{2}{6}$、$\frac{3}{6}$ 和 $\frac{1}{6}$。因此，這 6 個數字的平均值可以視為隨機變數 X 的期望值。此外，這 6 個數字的變異數為

$$\sigma^2=\frac{1}{6}\left[(4-5)^2+(5-5)^2+(7-5)^2+(5-5)^2+(4-5)^2+(5-5)^2\right]$$
$$=\frac{1}{6}\left[(4-5)^2\times 2+(5-5)^2\times 3+(7-5)^2\times 1\right]$$
$$=(4-5)^2\times\frac{2}{6}+(5-5)^2\times\frac{3}{6}+(7-5)^2\times\frac{1}{6}=1$$

Hence, following the concept of expectation, where each value is multiplied by its probability, we add the product of the squared deviation multiplied by its corresponding probability. This is denoted as the variance of the random variable *X*.

因此，仿照期望值將每一個數值以其發生的機率加權的概念，將離均差平方乘上其對應的機率之總和，稱為隨機變數 *X* 的變異數。

Variance and Standard Deviation

If *X* is a random variable with mean $E(X)=\mu$, then the variance of *X*, denoted by $Var(X)=\sigma^2$, is defined by

x	x_1	x_2	$\cdots$	x_n
$P(X=x)$	p_1	p_2	$\cdots$	p_n

$$Var(X)=(x_1-\mu)^2p_1+(x_2-\mu)^2p_2+\cdots+(x_n-\mu)^2p_n=\sum_{i=1}^{n}(x_i-\mu)^2p_i$$

or

$$Var(X)=x_1^2p_1+x_2^2p_2+\cdots+x_n^2p_n-\mu^2=\sum_{i=1}^{n}x_i^2p_i-\left(\sum_{i=1}^{n}x_ip_i\right)^2.$$

Beside, $\sigma=\sqrt{Var(X)}$ is the standard deviation of random variable *X*.

The probability distribution is shown on the right.

The second version of the variance is often written as $E(X^2)-[E(X)]^2$, which can be remembered as "the expectation of the squares minus the square of the expectation."

變異數也可以寫成$E(X^2)-[E(X)]^2$，記做「平方期望值減期望值平方」。

Also, if it is an infinite discrete random variable X, then

同樣地，當隨機變數 *X* 的取值有無限多個時，

$$Var(X)=(x_1-\mu)^2p_1+(x_2-\mu)^2p_2+\cdots+(x_n-\mu)^2p_n+\cdots=\sum_{i=1}^{\infty}(x_i-\mu)^2p_i.$$

When calculating the expectation and variance of a discrete probability distribution, you will find it helpful to set your work out systematically in a table.

當計算離散型機率分布的期望值和變異數時，利用表格系統化的計算較容易。

Examples II

The random variable X is given by the sum of the scores when two **ordinary**[34] dice are thrown

(i) Find the probability of *X*.

(ii) Find $E(X)=\mu$ and $Var(X)=\sigma^2$.

(iii) Find the values of $P(X<\mu)$, $P(X>\mu+\sigma)$ and $P(|X-\mu|<2\sigma)$

Solution

(i) The table shows all the possible totals when the two dice are thrown.

		First die					
		1	2	3	4	5	6
Second die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

The probability distribution for X is as follows.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(ii) $E(x)=\mu=\sum_{i=1}^{11} x_i p_i$

$$=2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}+6\times\frac{5}{36}+7\times\frac{6}{36}+8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}+11\times\frac{2}{36}$$

$$+12\times\frac{1}{36}$$

$$=\frac{252}{36}=7$$

$$Var(x)=\sigma^2=\sum_{i=1}^{11}(x_i-\mu)^2 p_i$$

$$=(2-7)^2\times\frac{1}{36}+(3-7)^2\times\frac{2}{36}+(4-7)^2\times\frac{3}{36}+(5-7)^2\times\frac{4}{36}+(6-7)^2\times\frac{5}{36}+(7-7)^2\times\frac{6}{36}$$

$$+(8-7)^2\times\frac{5}{36}+(9-7)^2\times\frac{4}{36}+(10-7)^2\times\frac{3}{36}+(11-7)^2\times\frac{2}{36}+(12-7)^2\times\frac{1}{36}$$

$$=\frac{210}{36}\approx 5.83$$

(iii) $P(X<\mu)=P(X<7)=P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$$=\frac{1}{36}+\frac{2}{36}+\frac{3}{36}+\frac{4}{36}+\frac{5}{36}=\frac{15}{36}=\frac{5}{12}$$

$P(X>\mu+\sigma)=P\left(X>7+\sqrt{5.83}\right)=P(X>9.4)=P(X=10)+P(X=11)+P(X=12)$

$$=\frac{3}{36}+\frac{2}{36}+\frac{1}{36}=\frac{6}{36}=\frac{1}{6}$$

$P(|X-\mu|<2\sigma)=P(\mu-2\sigma<X<\mu+2\sigma)=P(2.2<X<11.8)$

$=P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)+P(X=8)+P(X=9)$

$+P(X=10)+P(X=11)$

$$=\frac{2}{36}+\frac{3}{36}+\frac{4}{36}+\frac{5}{36}+\frac{6}{36}+\frac{5}{36}+\frac{4}{36}+\frac{3}{36}+\frac{2}{36}$$

$$=\frac{34}{36}=\frac{17}{18}$$

The probability of $\mu-2\sigma<X<\mu+2\sigma$ is $\frac{17}{18}$, which is almost 1.

Examples III

Bob earns \$80 per day, Monday to Friday **inclusive**[35]. He works every **alternate**[36] Saturday for which he earns "time and a half" and every fourth Sunday, for which he is paid "double time." Let *X* represents his daily **wage**[37], and *Y* represents his weekly wage.

(i) By considering a typical four-week period of 28 days, find the probability distribution for *X*.

(ii) Calculate the expectation and variance of *X*.

(iii) Show that there are two possible patterns Bob could work over a typical four-week period, depending on which Saturdays and Sunday he works. Hence find the expectation and variance of *Y* under either pattern.

Solution

(i) He earns \$80, and 20 out of 28 are week days.

He earns "time and a half," which means $\$80\times 1.5=120$, and 2 out of 28 are alternate

Saturdays.

He is paid "double time," which means $\$80\times 2=160$, and 1 out of 28 is the fourth Sundays.

The pattern of a typical four-week period of 28 days.

M	Tu	W	Th	F	Sa	Su
80	80	80	80	80	120	X
80	80	80	80	80	X	X
80	80	80	80	80	120	X
80	80	80	80	80	X	160

The probability distribution for *X* gives the following

x (\$)	0	80	120	160
$P(X=x)$	$\frac{5}{28}$	$\frac{20}{28}$	$\frac{2}{28}$	$\frac{1}{28}$

(ii) $E(x)=\mu=\sum_{i=1}^{4}x_i p_i$

$$=0\times\frac{5}{28}+80\times\frac{20}{28}+230\times\frac{2}{28}+160\times\frac{1}{28}$$

$$=\frac{2000}{28}\approx 71.43$$

$Var(x)=\sigma^2=E(X^2)-\mu^2$

$$=0^2\times\frac{5}{28}+80^2\times\frac{20}{28}+120^2\times\frac{2}{28}+160^2\times\frac{1}{28}-\left(\frac{2000}{28}\right)^2$$

$$=\frac{45600}{7}-\frac{250000}{49}=\frac{69200}{49}\approx 1410 \text{ (to 3 s.f.)}$$

Hence, his mean daily wage is \$71.43, with a variance of 1410, resulting in a standard deviation of about \$38.

(iii) **Pattern 1**: he works on the first and third Saturday, and works on the fourth Sunday.

M	Tu	W	Th	F	Sa	Su
80	80	80	80	80	120	X
80	80	80	80	80	X	X
80	80	80	80	80	120	X
80	80	80	80	80	X	160

The probability distribution for Y_1 gives the following

y_1 ($)	400	520	560
$P(Y_1 = y_1)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$$E(y_1) = 400 \times \frac{1}{4} + 520 \times \frac{2}{4} + 560 \times \frac{1}{4} = 500$$

$$Var(y_1) = 400^2 \times \frac{1}{4} + 520^2 \times \frac{2}{4} + 560^2 \times \frac{1}{4} - 500^2 = 3600$$

Pattern 2: he works on the second and fourth Saturday, and works on the fourth Sunday.

M	Tu	W	Th	F	Sa	Su
80	80	80	80	80	X	X
80	80	80	80	80	120	X
80	80	80	80	80	X	X
80	80	80	80	80	120	160

The probability distribution for Y_2 gives the following

y_2 ($)	400	520	680
$P(Y_2 = y_2)$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$$E(y_2) = 400 \times \frac{2}{4} + 520 \times \frac{1}{4} + 680 \times \frac{1}{4} = 500$$

$$Var(y_2) = 400^2 \times \frac{2}{4} + 520^2 \times \frac{1}{4} + 680^2 \times \frac{1}{4} - 500^2 = 13200$$

Hence, his mean weekly wage is $500, with a variance of 500 or 13200.

References

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製作者：臺北市立陽明高中 吳柏萱 教師