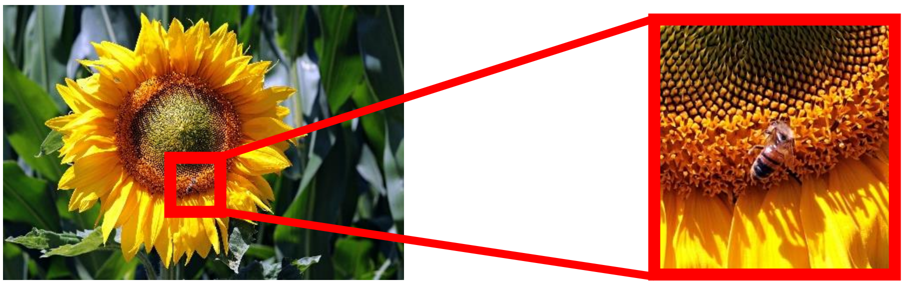
Matrix III

I. Key mathematical terms

Terms	Symbol	Chinese translation
Scaling matrix		
Rotation matrix		
Reflection matrix		
Shear matrix		
Markov chain		

II. 2×2 Linear transformation

In our daily life. We can enlarge(放大), shrink(縮小), rotate(旋轉) or mirror(鏡射) graphs with computers or cell phones. Matrices are very useful for describing these different transformations. We can define the transformation in two dimensions by describing how a general point with position (x,y) is transformed. The new point is called the image.



pixabay

Example 1.

Three transformations S, T and U are defined as follows. Find the point (1,2) under each of these transformation

$$S:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x+1\\2y-1\end{pmatrix},\ T:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}2x+y\\x-y\end{pmatrix},\ U:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}3y\\x^2\end{pmatrix}$$

(1) S (2) T (3) U

<key>
A “linear” transformation has the special properties that the transformation only involves linear terms in x and y.
(In the example above, S, T and U,which is the linear transformation?)

Definition:(Linear transformation on the plane)

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $2x2$ square matrix. Then the point on the plane $P(x, y)$ can be transformed to $P'(x', y')$ by the matrix A with matrix multiplication.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

We say $P'(ax+by, cx+dy)$ is the linear transformation of $P(x, y)$ under the matrix multiplication of A.
<key> Linear transformations always map the origin onto itself.
<key> Any linear transformation can be represented by matrix multiplication.

Example 2.
Find matrices to represent these linear transformations.

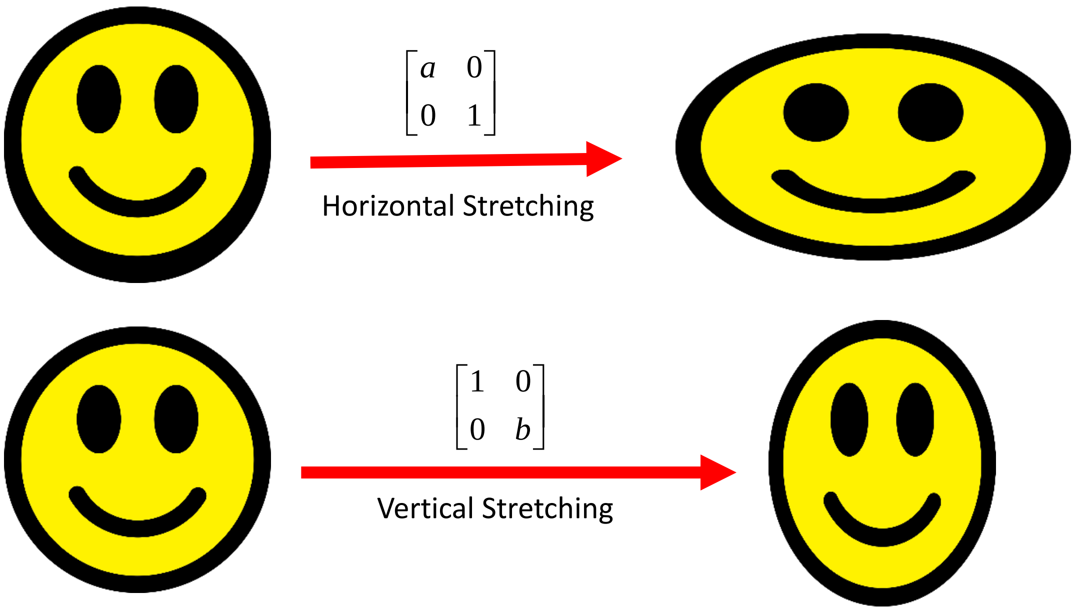
(1) $T:\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x-y \\ 3y \end{pmatrix}$

(2) $U:\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -3y \\ 5x-y \end{pmatrix}$

Example 3.
A triangle ABC has vertices ABC at (0,0), (5,0) and (2,7).
(1) Find the area of triangle ABC
(2) Find the vertices of A’B’C’ under the transformation represented by the matrix
$A=\begin{pmatrix} 1 & 2 \\ 0 & -\frac{1}{2} \end{pmatrix}$
(3) Find the area of triangle A’B’C’
(Hint: After you finish, please try to compare the answer you got from(1) and (3).)

Definition:(Scaling matrix on the plane)

In our daily life, we'll use copy machine or computer to duplicate, enlarge or stretch graphs. We can describe these operations using matrix multiplications. We say these kinds of matrices are "scaling matrix".



Given positive numbers h,k.

If we want to "stretch" point $P(x,y)$ along x-axis by a factor h (horizontally) and along y-axis by a factor k (vertically). We'll have the following relation:

$$\begin{cases} x' = hx \\ y' = ky \end{cases}, \quad \text{where } P'=(x',y')$$

We can also represent this by the matrix multiplication:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is the scaling transformation. $\begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$ is the scaling matrix.

<key> For a stretch parallel to the x-axis only, points on the y-axis are <u>invariant</u>.(   )

For a stretch parallel to the y-axis only, points on the x-axis are <u>invariant</u>.(   )

<key> For stretches in both directions, there are no invariant lines and the only invariant point is the origin.

Example 4.

The unit square is transformed using the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(1) Describe fully the transformation represented by the matrix.

(2) Write down the coordinates of any invariant points.

Example 5.

A triangle with vertices at $P(1,1)$, $Q(3,2)$ and $R(2,3)$ is transformed by a matrix A. We got a new triangle with vertices at $P'(2,2)$, $Q'(6,4)$ and $R'(4,6)$.

(1) Find the scaling matrix A.

(2) Find the area of the new triangle.

(3) Draw the graph of triangle PQR and $P'Q'R'$ on the coordinates plane.

Example 6.

A triangle T has vertices at the points $A(k,1)$, $B(4,1)$, $C(4,k)$ where k is an integer constant. We transform this triangle by the matrix $\begin{pmatrix} 4 & -1 \\ k & 2 \end{pmatrix}$.

Given that the triangle has a right angle at B and the area of the image triangle T' is 10, find the value of k.

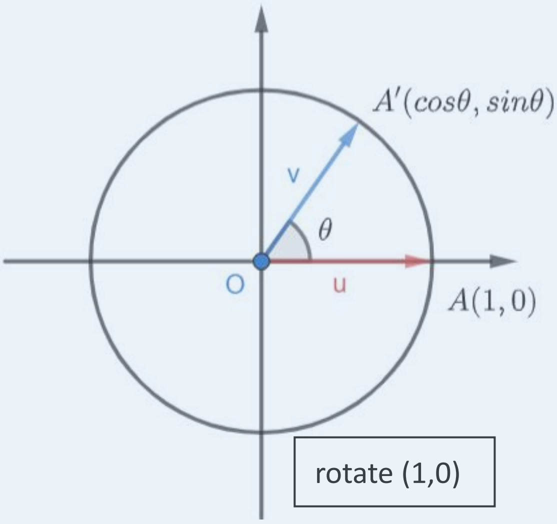
2022.07.01

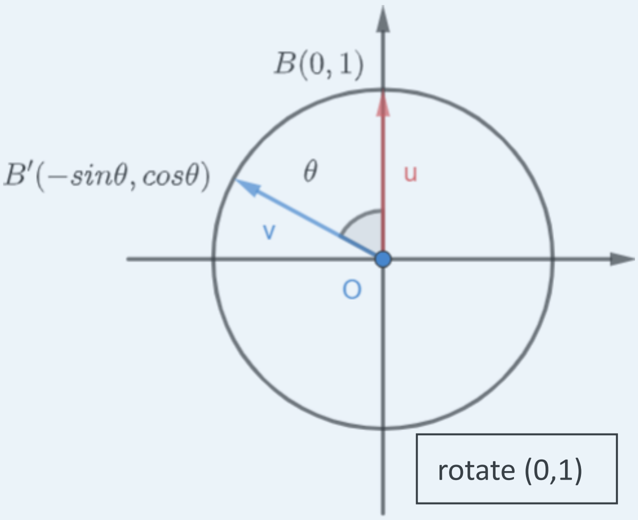
Definition:(Rotation matrix on the plane)

Sometimes when we use cell phone to take a picture. We'll rotate the picture to fit the screen and make it more reader-friendly. We can also do the operation by matrix multiplication. We say these kinds of matrices are "rotation matrix".

Rotate
90 degrees

To "rotate" a point P(x, y) through angle θ anticlockwise about the origin.
We first consider the unit vectors *(1, 0)* and *(0, 1)*.





With this two graphs, we have the following relation:

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{cases}, \text{ where } P' = (x', y')$$

We can also represent by the matrix multiplication:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is the rotation transformation. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the rotation matrix.

Example 7.

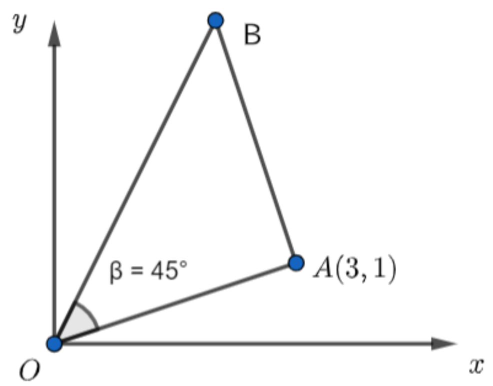
$$M=\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

(1) Describe geometrically the rotation represented by $\boldsymbol{M}$.

(2) A square $\boldsymbol{S}$ has vertices at (1,0), (2,0), (2,1) and (1,1). Find the coordinates of the vertices of the images of $\boldsymbol{S}$ under the transformation described by $\boldsymbol{M}$.

Example 8.

In the coordinate. An isosceles triangle OAB satisfies $\angle A=90°$, O(0,0), A(3,1) and B in the first quadrant. Find the coordinate B(x,y).



製作者：國立臺灣師範大學附屬高級中學 蕭煜修