2022.08.02

Matrix IV

I. Key mathematical terms

Terms	Symbol	Chinese translation
Shear matrix		
Reflection matrix		

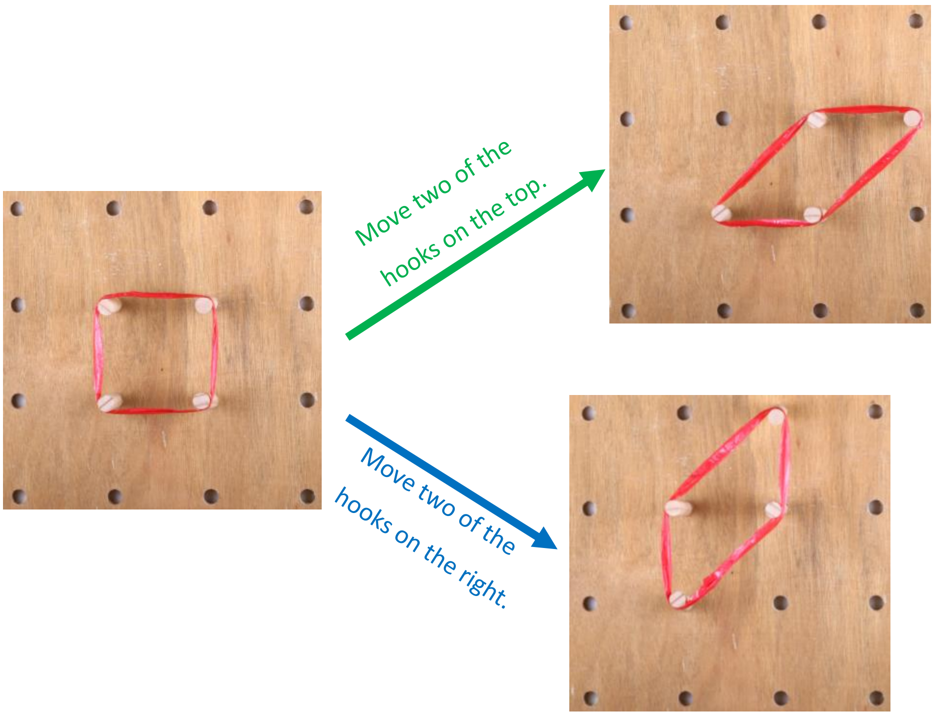
II. 2×2 Linear transformation

Definition:(Shear matrix)

Pegboard in the bathroom

Pegboard in the classroom

Pegboard(釘板) is a useful tool for us to organize items in the bathroom or tools in the living technology classroom. We can plug in peg hooks to rearrange our items.



In the above picture, we have four hooks and a red rope which forms a red square on the pegboard. If we move two of the hooks on the top from left to right by 2 units, we'll have a new parallelogram. If we move two of the pen hooks on the right by 4 units up, we'll have another new parallelogram. We can describe these operations using matrix multiplications. We say these kinds of matrices are "shear matrices".

Given constant k.

"Shear" of matrix moves every point by an amount that is proportional to its distance from the x-axis or y-axis. A two-dimensional shear matrix is defined by the following:

(1) Shear parallel to the x-axis (y-axis invariant)

If we want to shear a point P(x,y) with factor k parallel to the x-axis, and keep the y-coordinate invariant to get a new point P'(x',y'). We have:

$$\begin{cases} x' = x + ky \\ y' = y \end{cases}$$

We can also represent it by the matrix multiplication:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is the shear transformation. $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is a shear matrix parallel to the x-axis.

(2) Shear parallel to the y-axis (x-axis invariant)

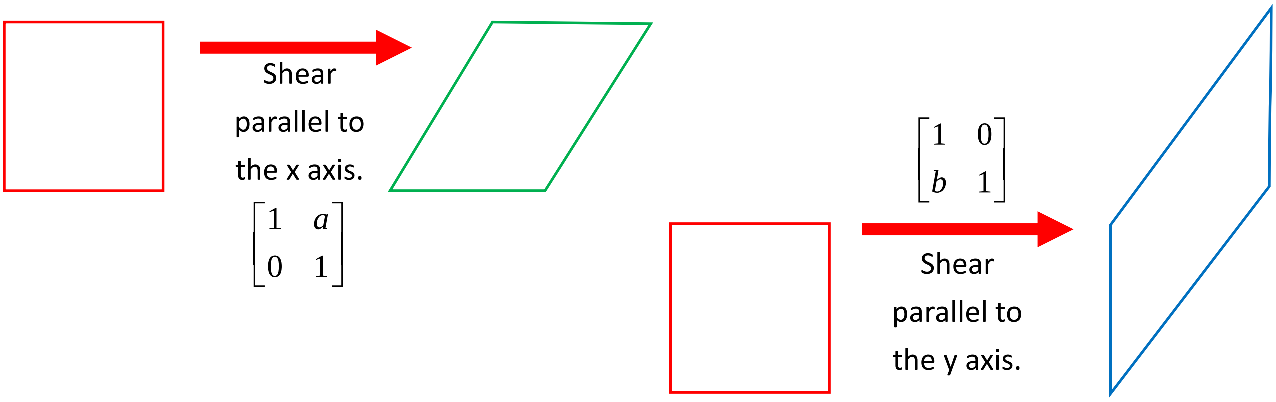
If we want to shear a point P(x,y) with factor k parallel to the y-axis, and keep the x-coordinate invariant to get a new point P'(x',y'). We have:

$$\begin{cases} x' = x \\ y' = y + kx \end{cases}$$

We can also represent it by the matrix multiplication:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is the shear transformation. $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ is a shear matrix parallel to the x-axis.

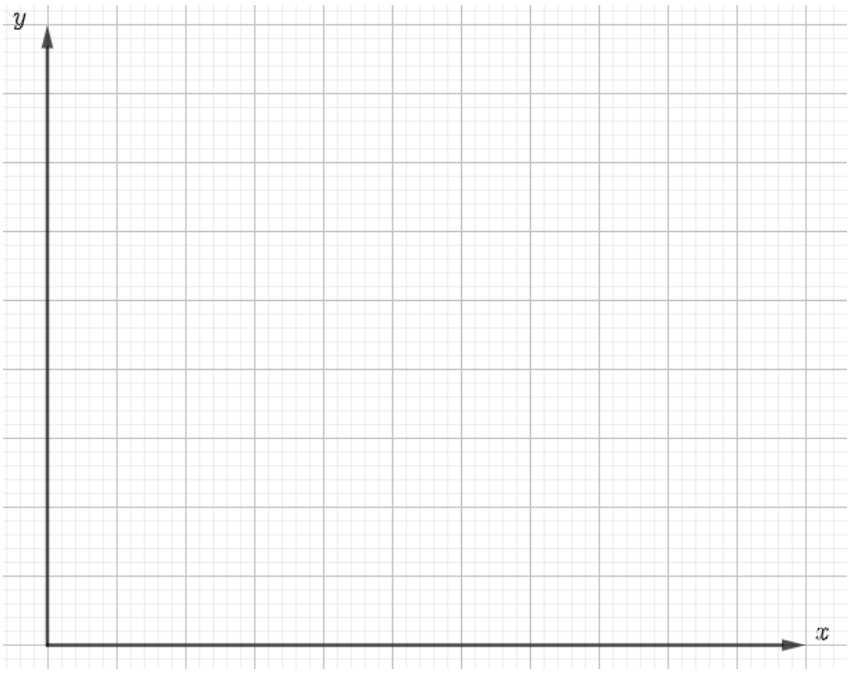


<key> The area of polygon under shear transformation will remain the same.
(Why? Try to explain this by examples.)

Example1.

Given a parallelogram OABC. The coordinates of each points are O(0,0), A(2,0), B(3,2), C(1,2). If we keep y-invariant and shear each point along x-axis by 2 units of y-coordinate to get O', A', B', C'. Find:

(1) The shear matrix.
(2) The coordinates of O', A', B', C'
(3) Draw the quadrilateral of O'A'B'C' on the coordinate plane.
(4) The ratio of the area of OABC and O'A'B'C'.



Example2.

Suppose a region on the coordinate is defined by $G = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 4\}$.

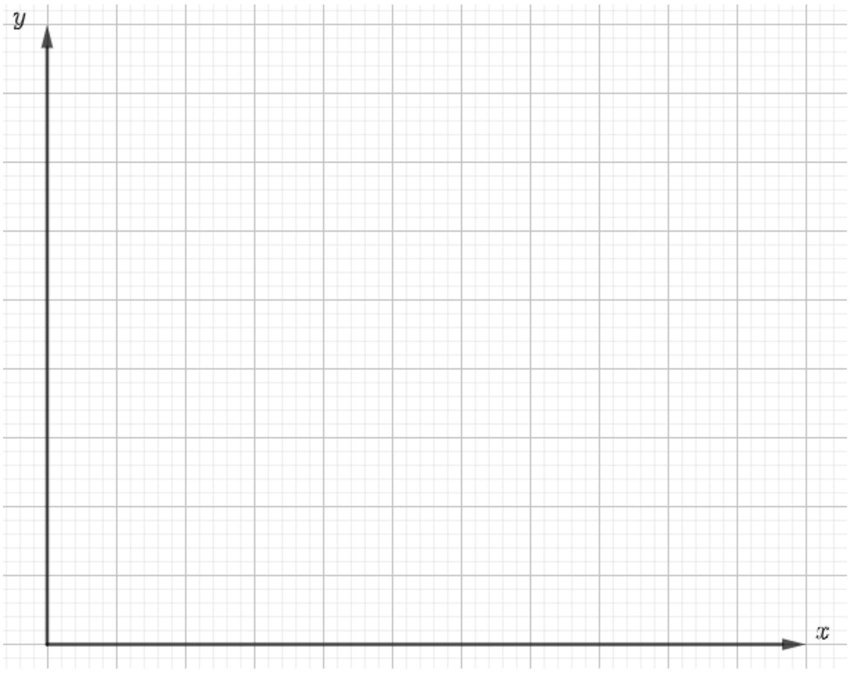
If we keep y-invariant and shear each point along x-axis by $\frac{3}{4}$ units of y-coordinate to get a new region G'. Find:

(1) The shear matrix.

(2) The coordinates of four vertices of G'.

(3) Describe the region of G' by set notation.

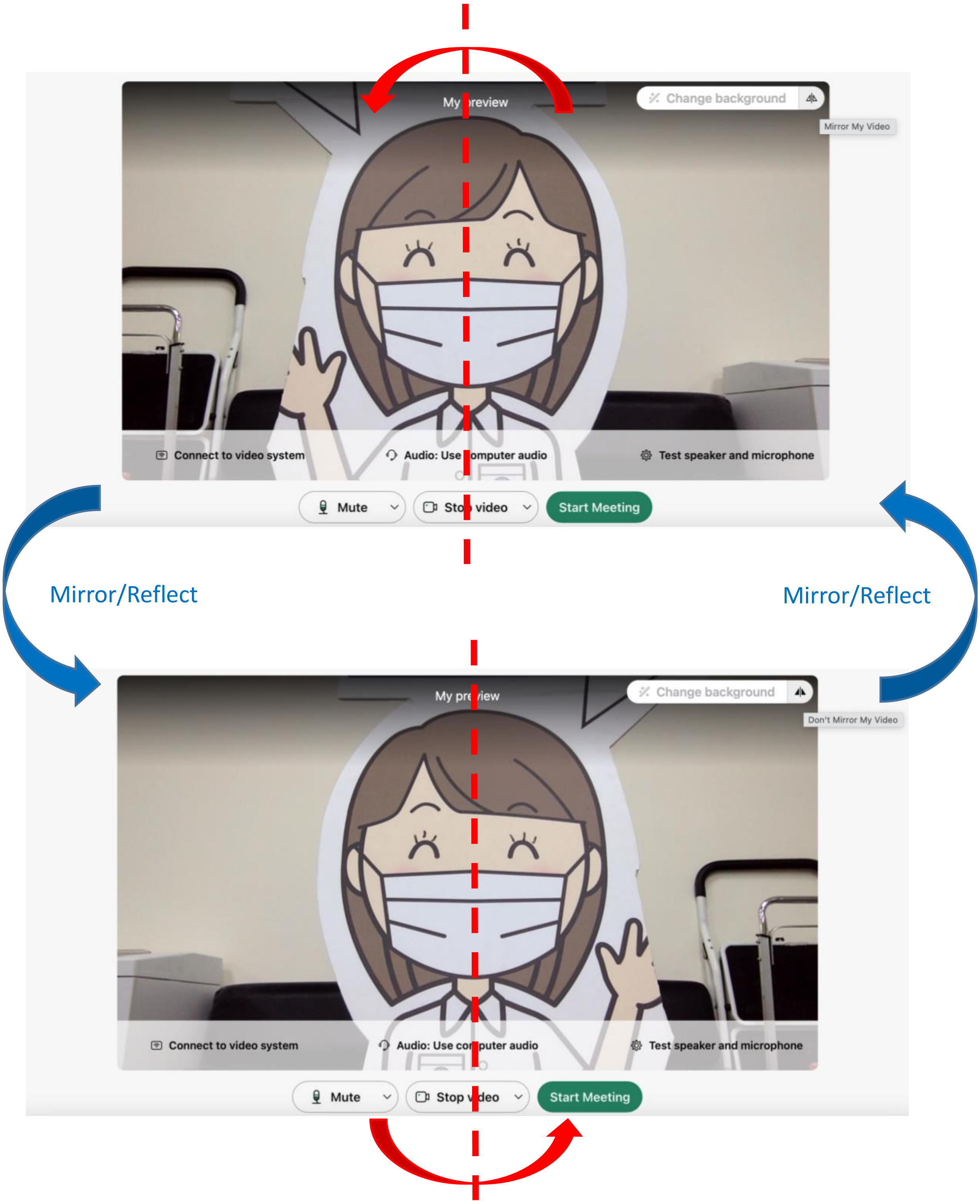
(4) The ratio of the area of region G and region G'.



2022.08.02

Definition:(Reflection matrix)

When we hold an online meeting, some of the meeting software can choose to mirror the video. If we click the bottom "Mirror My Video". The video will flip over along the y-axis. We can describe this operation using matrix multiplication. We say this kind of matrix "reflection matrix".



2022.08.02

Given a straight line passing through the origin with angle θ.
A point $P(x,y)$ "reflect" or "mirror" through this line to get $P'(x',y')$.
We should consider the following condition:

(1) Suppose the angle between the straight line and the x-axis is θ.

(2) Suppose $\overline{OP}=r$, and $P(x,y)$ satisfiese $x=r\cos\alpha, y=r\sin\alpha$.

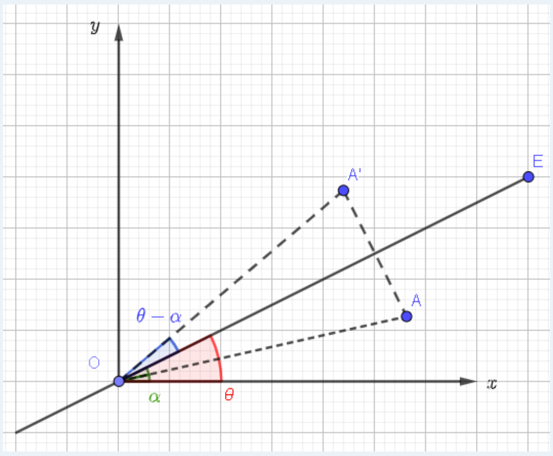
Then we can have the following:

$$\begin{cases} x'=r\cos(2\theta-\alpha) \\ y'=r\sin(2\theta-\alpha) \end{cases} \Rightarrow \begin{cases} x'=r(\cos2\theta\cos\alpha+\sin2\theta\sin\alpha)=x\cos2\theta+y\sin2\theta \\ y'=r(\sin2\theta\cos\alpha-\cos2\theta\sin\alpha)=x\sin2\theta-y\cos2\theta \end{cases}$$

We can represent it by the matrix multiplication:

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} \cos2\theta & \sin2\theta \\ \sin2\theta & -\cos2\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

This is the reflection/mirror transformation. $\begin{pmatrix} \cos2\theta & \sin2\theta \\ \sin2\theta & -\cos2\theta \end{pmatrix}$ is a reflection/ mirror matrix along a straight line which passes through the origin with angle θ.



<key> Some special cases about reflection:

Reflection over the:	x-axis	y-axis	Line $x=y$
Multiply the pont by (Matrix) :			

<key> The area of polygon under reflection transformation will remain the same.
(Why? Try to explain this by examples.)

Example3.

Suppose $L: y=2x$.

(1) Find the corresponding matrix for reflection about the line $L: y=2x$.

(2) Find the reflection of the point $P(-2,1)$ in the line $L: y=2x$.

Example4.

Mirror a circle $C:(x-5)^2+(y-10)^2=9$ through the line $L: x-2y=0$ to get a new circle C'.

(1) Find the center and radius of circle C'.

(2) Find the equation of C'.

<資料來源>

1. **Shear Transformations Using Matrices**
   https://www.youtube.com/watch?v=Z2bR0Mb1Jj0
2. **Matrices and Transformations IV|SHEAR|STRETCH|ISOMETRIC AND NON-ISOMETRIC TRANSFORMATIONS**
   https://www.youtube.com/watch?v=hKLn-y5UfEA
3. **Transformations and Matrices**
   https://www.mathsisfun.com/algebra/matrix-transform.html
4. **2d shear matrix**
   https://www.youtube.com/watch?v=cxbTV5Jjm6s
5. **Linear Transformations on the Plane**
   https://highscope.ch.ntu.edu.tw/wordpress/?p=51374

製作者：國立臺灣師範大學附屬高級中學　蕭煜修