圓方程式

Equations of Circles

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1st Period</th></tr>
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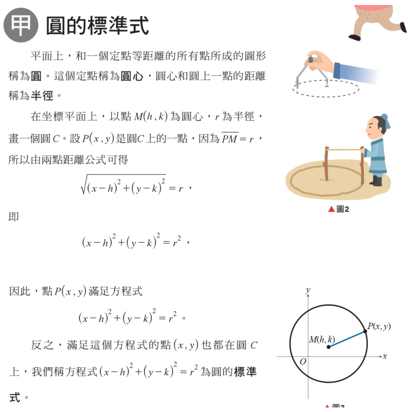

甲 圓的標準式

平面上，和一個定點等距離的所有點所成的圖形稱為**圓**。這個定點稱為**圓心**，圓心和圓上一點的距離稱為**半徑**。

在坐標平面上，以點 $M(h,k)$ 為圓心，r 為半徑，畫一個圓 C。設 $P(x,y)$ 是圓 C 上的一點，因為 $\overline{PM}=r$，所以由兩點距離公式可得

$$\sqrt{(x-h)^2+(y-k)^2}=r，$$

即

$$(x-h)^2+(y-k)^2=r^2，$$

▲圖2

因此，點 $P(x,y)$ 滿足方程式

$$(x-h)^2+(y-k)^2=r^2。$$

反之，滿足這個方程式的點 (x,y) 也都在圓 C 上，我們稱方程式 $(x-h)^2+(y-k)^2=r^2$ 為圓的**標準式**。

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Vocabulary: Compass (圓規), Standard Equation of a Circle (圓的標準式), Center (中心), Radius (半徑), Distance Formula (距離公式), Diameter (直徑).

Sentences:

1. A circle is the set of all points in a plane that are a fixed distance from a given point called the center of a circle. (平面上，和一個定點等距離的所有點所成的圖形稱為圓。這個定點稱為圓心。)
2. The distance from the center to a point on the circle is called the radius of a circle. (圓心和圓上一點的距離稱為半徑。)
3. Let r be the radius of the circle C. (設 r 是圓 C 的半徑。)
4. Let point P(x, y) be any point on the circle C. (設 P(x, y)是圓 C 上的任意點。)
5. By the Distance Formula, you can get… (由距離公式可得…)
6. We call this $(x-h)^2+(y-k)^2=r^2$ is the **Standard Equation of a Circle**.

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Video: BYJU'S - *Circles : Introduction*

https://youtu.be/m9dpeG2rKdY

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Vocabulary: Diameter (直徑), Chord (弦), Arc (弧), Sector (扇形), Segment (弓形), Circumference (圓周), Exterior (外部), Interior (內部).

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圓的標準式

以 $M(h,k)$ 為圓心，r 為半徑的圓方程式為

$$(x-h)^2+(y-k)^2=r^2 \text{。}$$

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Translations:

The standard equation of a circle with center at M(h, k) and radius r is $(x-h)^2+(y-k)^2=r^2$.

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例題 1

求下列各圓的方程式：

(1) 以點 $(2,-3)$ 為圓心，半徑為 4 的圓。

(2) 以點 $M(2,-3)$ 為圓心，又通過點 $A(5,1)$ 的圓。

解

(1) 因為圓心 $(2,-3)$，半徑 4，所以由標準式得

$$(x-2)^2+(y-(-3))^2=4^2 \text{，}$$

即圓方程式為

$$(x-2)^2+(y+3)^2=16 \text{。}$$

(2) 圓的半徑

$$r=\overline{AM}=\sqrt{(5-2)^2+(1-(-3))^2}=5 \text{，}$$

由標準式得圓方程式為

$$(x-2)^2+(y+3)^2=25 \text{。}$$

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Translations:

Example 1

Find the equation of the circle:

(1) The circle with the center $(2,-3)$ and a radius 4.

(2) The circle with the center $M(2,-3)$ and passes through point $A(5,1)$.

Solution

(1) From the standard equation of a circle, we know that

$$(x-2)^2+(y-(-3))^2=4^2$$

So that the equation of a circle is

$$(x-2)^2+(y+3)^2=16$$

(2) The radius of the circle is

$$r=\overline{AM}=\sqrt{(5-2)^2+(1-(-3))^2}=5$$

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	From the standard equation of a circle, we get the equation of a circle is $(x+2)^2+(y-3)^2=25$
乙 **圓的一般式** 將圓的標準式 $(x-h)^2+(y-k)^2=r^2$ 展開，可得 $x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$， 這種形如 $x^2+y^2+dx+ey+f=0$ 的二元二次方程式，稱為圓的**一般式**。 **圓的一般式** 圓的方程式都可表示成底下二元二次方程式的形式： $x^2+y^2+dx+ey+f=0$。	**Vocabulary:** General Form of a Circle (圓的一般式), Quadratic Equation in Two Variables (二元二次方程式) **Translations:** Expand the standard form of a circle $(x-h)^2+(y-k)^2=r^2$ and we get $x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$. This form of equation is similar to quadratic equation in two variables $x^2+y^2+dx+ey+f=0$, which is the "General Form of a Circle". **General Form of a Circle** All of the equations of circles can be expressed as the form of two-variable quadratic equations: $x^2+y^2+dx+ey+f=0$.
例題 3 已知圓 $C:x^2+y^2-2x+6y+6=0$，求圓 C 的圓心與半徑。 解 將 $x^2+y^2-2x+6y+6=0$ 分別對 x,y 配方，得 $(x^2-2x+1)+(y^2+6y+9)=-6+1+9$， 即 $(x-1)^2+(y+3)^2=2^2$。 依標準式知圓 C 的圓心為 $(1,-3)$，半徑為 2。	**Translations:** Example 3 Find the center and the radius of the circle $C:x^2+y^2-2x+6y+6=0$ Solution Complete the square for the x terms, and similarly for

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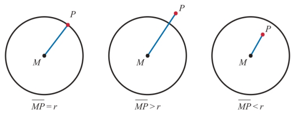
$\left(x^2-2x+1\right)+\left(y^2+6y+9\right)=-6+1+9$
so that
$\left(x-1\right)^2+\left(y+3\right)^2=2^2$
From the standard equation of a circle we know that the center of the circle is $\left(1,-3\right)$ and the radius is 2.</td></tr>
<tr><td>丙 點與圓的關係
在圓上的點與圓心的距離恰等於半徑。那麼，如何知道一個點是在圓的內部還是外部呢？透過幾何觀點，觀察一個以 M 為圓心，r 為半徑的圓和一點 P，它們的關係可歸納為下列三種情形：


▲圖5
(1) 當 $\overline{MP}=r$ 時，P 點在圓上。
(2) 當 $\overline{MP}>r$ 時，P 點在圓外。
(3) 當 $\overline{MP}<r$ 時，P 點在圓內。
利用以上的方法判斷點與圓的關係。</td><td>Vocabulary: Interior Point (內部點), Exterior Point (外部點).
Illustrations:
Look at figure 5, point M is the center of the circle. The radius of the circle is r and a point P.
(1) If point P lies on the circle, then $\overline{MP}=r$.
(2) If point P lies exterior of the circle, then $\overline{MP}>r$.
(3) If point P lies interior of the circle, then $\overline{MP}<r$.</td></tr>
<tr><td>例題 8
已知圓 C 的方程式為 $(x-2)^2+\left(y+3\right)^2=25$，分別判斷 $P(6,0)$, $Q(-2,-1)$, $R(0,2)$ 三點與圓的關係（即點是在內部、外部還是圓上）。
解
圓 $C:(x-2)^2+\left(y+3\right)^2=5^2$ 的圓心為 $M(2,-3)$，半徑為 5。
分別計算 P,Q,R 三點與圓心 $M(2,-3)$ 的距離，得
$\overline{PM}=\sqrt{(6-2)^2+(0+3)^2}=5;$
$\overline{QM}=\sqrt{(-2-2)^2+(-1+3)^2}=\sqrt{20}<5;$
$\overline{RM}=\sqrt{(0-2)^2+(2+3)^2}=\sqrt{29}>5$。
因此，P 在圓上，Q 在圓內，R 在圓外。</td><td>Example 8
Determine if these the three points $P\left(6,0\right)$, $Q\left(-2,-1\right)$ and $R\left(0,2\right)$ lie on inside, outside or on the circle of this equation: $C:\left(x-2\right)^2+\left(y+3\right)^2=25$.
Solution
From standard form of a circle we know the center is $M\left(2,-3\right)$, and the radius is 5.
Calculate the distance from point P, Q and R to center</td></tr>
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$M(2,-3)$ respectively, and we get

$$\overline{PM}=\sqrt{(6-2)^2+(0+3)^2}=5$$

$$\overline{QM}=\sqrt{(-2-2)^2+(-1+3)^2}=\sqrt{20}<5$$

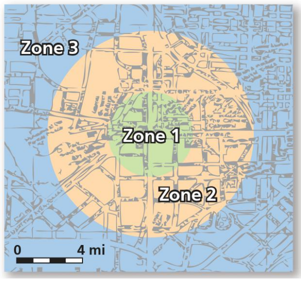
$$\overline{RM}=\sqrt{(0-2)^2+(2+3)^2}=\sqrt{29}>5$$

Thus, point P is on the circle, point Q is inside of the circle and point R is outside of the circle.

補充題

Material 1

A city's commuter system has three zones. Zone 1 serves people living within 3 miles of the city's center. Zone 2 serves those between 3 and 7 miles (included) from the center. Zone 3 serves those over 7 miles from the center. (Shown in the right figure.)



Determine which zone serves people whose homes are represented by the points $A(3,4)$, $B(6,5)$, $C(1,2)$, $D(0,3)$ and $E(1,6)$.

Solution

We set the center of the city as $O(0,0)$. Calculate the distance from point A, B, C, D and E to the center O respectively.

$\overline{AO}=\sqrt{(3-0)^2+(4-0)^2}=5$, and $3<\overline{AO}<7$. A is in zone 2.

$\overline{BO}=\sqrt{(6-0)^2+(5-0)^2}=\sqrt{61}$, and $\overline{BO}>7$. B is in zone 3.

$\overline{CO}=\sqrt{(1-0)^2+(2-0)^2}=\sqrt{5}$, and $\overline{CO}<3$. C is in zone 1.

$\overline{DO}=\sqrt{(0-0)^2+(3-0)^2}=3$, and $\overline{DO}=3$. D is in zone 2.

$\overline{EO}=\sqrt{(1-0)^2+(6-0)^2}=\sqrt{37}$, and $3<\overline{EO}<7$. E is in zone 2.

Material 2

The epicenter of an earthquake is the point on Earth's surface directly above the earthquake's origin. A seismograph can be used to determine the distance to the epicenter of an earthquake. Seismographs are needed in three different places to locate an earthquake's epicenter.

Use the seismograph readings from locations A, B, and C to find the epicenter of an earthquake.

- The epicenter is 7 miles away from A(−2, 2.5).
- The epicenter is 4 miles away from B(4, 6).
- The epicenter is 5 miles away from C(3, −2.5).

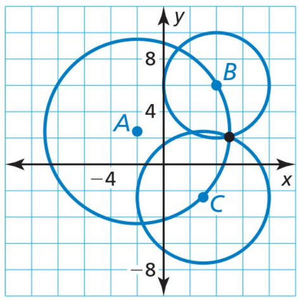
Solution

The set of all points equidistant from a given point is a circle, so the epicenter is located on each of the following circles.

⊙A with center (−2, 2.5) and radius 7

⊙B with center (4, 6) and radius 4

⊙C with center (3, −2.5) and radius 5



To find the epicenter, graph the circles on a coordinate plane where each unit corresponds to one mile. Find the point of intersection of the three circles.

The epicenter is at about (5, 2).

Note

Word: Commuter (通勤者), Serve (服務), Represent(表示), Epicenter (震央), Earthquake (地震), Surface (表面), Origin (起源), Seismograph (地震儀), Intersection Point (交點).

Sentence:

1. Zone 1 serves people living within 3 miles of the city's center. (第一區服務距離市中心 3 英

里以內的人民。)

2. Graph the circles on a coordinate plane. (將圓畫在坐標平面上。)
3. Each unit corresponds to one mile. (每單位為 1 英里。)

參考資料

References

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2. Big ideas math (2022). *Circles in the Coordinate Plane.* https://reurl.cc/qNm0Kg.

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