

廣義三角比與極坐標

Trigonometric Ratios of General Angles and Polar Coordinates

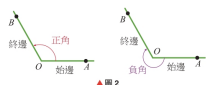
第 1 節

1st Period

Material

甲 廣義角

平面上， $\angle AOB$ 為將射線 OA 繞著 O 點旋轉到射線 OB 所成的角（可以超過一圈），如圖 2 所示，其中射線 OA 稱為角的始邊，射線 OB 稱為角的終邊；並規定逆時針方向旋轉出的角為正角，順時針方向旋轉出的角為負角，像這種有正負方向的角，稱為有向角。

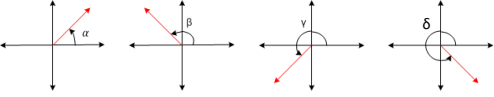


Note

Vocabulary: General Angle (廣義角), Ray (射線), Vertex (頂點), Initial Side (始邊), Terminal Side (終邊), Positive Angle (正向角), Negative Angle (負向角), Directed Angle (有向角), Counterclockwise (逆時針), Clockwise (順時針), Designate (指定的).

Translations:

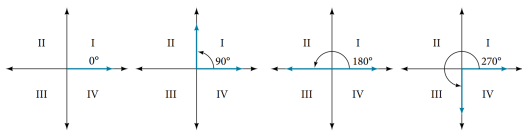
1. The angle AOB is formed by rotating the ray OA with **vertex** O to the ray OB . The fixed ray OA is the **initial side**, and the rotated ray OB is the **terminal side**. (平面上， $\angle AOB$ 為將射線 OA 繞著 O 點旋轉到射線 OB 所成的角，其中射線 OA 稱為角的始邊，射線 OB 稱為角的終邊。)
2. If the angle is measured in a **counterclockwise** direction from the initial side to the terminal side, the angle is called a **positive angle**. (逆時針方向旋轉出的角為正角。)
3. If the angle is measured in a **clockwise direction**, the angle is called a **negative angle**. (順時針方向旋轉出的角為負角。)
4. A directed angle is one that shows a **designated** direction, such as a positive and negative

	<p>direction. (像這種有正負方向的角，稱為有向角。)</p>
<p>同界角的性質 任意兩同界角的度數相差 360° 的整數倍。</p> <p>將廣義角置於坐標平面上，角的頂點與原點重合，角的始邊在x軸正向上，這樣的角稱為標準位置角。當角的終邊落在第一、二、三或四象限時，分別稱這個角為第一、二、三或四象限角。舉例如下。</p>	<p>Vocabulary: Coterminal Angles (同界角), Standard Position (標準位置), Origin (原點).</p> <p>Translations:</p> <ol style="list-style-type: none"> 1. The measures of any two coterminal angles differ by an integer multiple of 360°. (任意兩同界角的度數相差 360° 的整數倍。) 2. An angle is in standard position if its vertex is located at the origin, and its initial side extends along the positive x-axis. (當廣義角置於坐標平面上，角的頂點與原點重合，角的始邊在 x 軸正向上。) <p>Other definitions:</p> <p>Coterminal angles are two angles in a standard position that have the same terminal side. (同界角是兩個在標準位置上的兩個角有相同的終邊。)</p>
<p>將廣義角置於坐標平面上，角的頂點與原點重合，角的始邊在x軸正向上，這樣的角稱為標準位置角。當角的終邊落在第一、二、三或四象限時，分別稱這個角為第一、二、三或四象限角。舉例如下。</p> 	<p>Vocabulary: Quadrantal Angle (象限角).</p> <p>Translations:</p> <ol style="list-style-type: none"> 1. An angle is in standard position if its vertex is located at the origin, and its initial side extends along the positive x-axis. (將廣義角置於標準位置，角的頂點與原點重合，角的始邊在 x 軸正向上，這樣的角稱為標準位置角) 2. An angle in standard position is said to lie in a

quadrant if its terminal side lies in that quadrant.
 (將廣義角置於標準位置，當角的終邊落在某象限時，稱這個角在某象限。)

3. In the figures below, α lies in **the 1st quadrant**, β lies in **the 2nd quadrant**, γ lies in **the 3rd quadrant** and δ lies in **the 4th quadrant**. (如下圖所示， α 角的終邊落在第一象限， β 角的終邊落在第二象限， γ 角的終邊落在第三象限， δ 角的終邊落在第四象限。)

此外，當角的終邊落在 x 軸或 y 軸上時，稱作**象限角**，例如 90° 與 180° 都是象限角。



Vocabulary: Quadrantal Angle (象限角), etc. (The abbreviation of et cetera) (... 等等).

Translations:

An angle in standard position is called a **quadrantal angle** if its terminal side lies on the x-axis or y-axis. (當角置於標準位置，若角的終邊落在 x 軸或 y 軸上時，稱作**象限角**。)

Sentences:

Quadrantal angles include 0° , $\pm 90^\circ$, $\pm 180^\circ$, $\pm 270^\circ$, $\pm 360^\circ$ etc.

(象限角包括 0° , $\pm 90^\circ$, $\pm 180^\circ$, $\pm 270^\circ$, $\pm 360^\circ$... 等等。)

乙 廣義角三角比

將銳角 θ 放在坐標平面上，並在 θ 角的終邊上任取異於原點的一點 $P(x, y)$ ，令 $r = \overline{OP} = \sqrt{x^2 + y^2}$ ，如圖 6 所示，根據銳角三角比的定義，可得

$$\sin \theta = \frac{y}{r}, \cos \theta = \frac{x}{r}, \tan \theta = \frac{y}{x}.$$

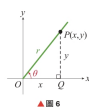
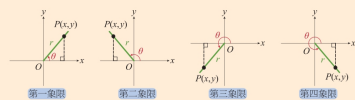
而且三角比不受 P 點在終邊上的位置所影響，事實上，廣義角的三角比可利用上述的方法來定義。

廣義角三角比的定義

當廣義角 θ 是一個標準位置角時，在 θ 的終邊上任取異於原點的一點

$P(x, y)$ ，令 $r = \overline{OP} = \sqrt{x^2 + y^2}$ ，定義 θ 的三角比為

$$\sin \theta = \frac{y}{r}, \cos \theta = \frac{x}{r}, \tan \theta = \frac{y}{x} \quad (x \neq 0).$$



▲圖 6

Vocabulary: General Angle (廣義角), Acute Angle (銳角).

Translations:

Let θ be an acute angle in standard position and suppose that $P(x, y)$ is any point other than $O(0, 0)$ on the terminal side of θ (Figure 6). If

$r = \overline{OP} = \sqrt{x^2 + y^2}$ is the distance between $P(x, y)$ and $O(0, 0)$, then the three trigonometric functions of θ are defined by

$$\sin \theta = \frac{y}{r}, \cos \theta = \frac{x}{r}, \tan \theta = \frac{y}{x}.$$

Defining Sine and Cosine Functions from the Unit Circle

The sine function relates a real number t to the y -coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y -value of the endpoint on the unit circle of an arc of length t . In Figure 2, the sine is equal to y . Like all functions, the sine function has an input and an output; its input is the measure of the angle; its output is the y -coordinate of the corresponding point on the unit circle. The cosine function of an angle t equals the x -value of the endpoint on the unit circle of an arc of length t . In Figure 3, the cosine is equal to x .

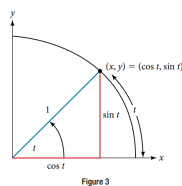


Figure 3

sine and cosine functions

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t , then

$$\cos t = x$$

$$\sin t = y$$

Vocabulary: Precisely (精確地), Parentheses (圓括號), Shorthand (速記的).

Sentences:

1. Defining sine and cosine functions from the unit circle. (在單位圓上正弦與餘弦函數的定義。)
2. The sine function relates a real number t to the y -coordinate of the point where the corresponding angle intercepts the unit circle. (正弦函數為圓心角 t 與單位圓交點的 y 坐標。)
3. The cosine function of an angle t equals the x -value of the endpoint on the unit circle of an arc of length t . (餘弦函數為圓心角 t 與單位圓交點的 x 坐標。)

Note:

(三角函數的平方符號位置說明)

Because it is understood that sine and cosine are functions, we do not always need to write them with **parentheses**: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used **shorthand** notation for $(\cos(t))^2$.

丙 三角比的換算公式

在例題 7 中，我們利用「同界角有相同的三角比」這個觀念先將角度化到 0° 到 360° 之間再做計算；事實上，透過一些換算公式，我們可以更進一步將角度化成銳角 (0° 到 90° 之間)，說明如下。

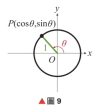
在坐標平面上，設廣義角 θ 的終邊與單位圓 (以原點 O 為圓心，半徑為 1 的圓) 交於一點 $P(x, y)$ ，如圖 8 所示，根據廣義角三角比的定義，得

$$\cos \theta = \frac{x}{1} = x, \sin \theta = \frac{y}{1} = y.$$

因此 P 點的坐標為 $(\cos \theta, \sin \theta)$ ，即

廣義角 θ 的終邊與單位圓的交點為 $(\cos \theta, \sin \theta)$ ，如圖 9 所示。

接下來，我們推導三角比的換算公式。



Vocabulary: Intersection Point (交點).

Sentences:

1. By the concept of “Coterminal Angles have the same ratios”, we simplify angles from 0° to 360° in order to make the calculation easier. (我們利用「同界角有相同的三角比」這個觀念先將角度化到 0° 到 360° 之間再做計算。)
2. $P(x, y)$ is the point where the general angle θ intercepts the unit circle which the center is origin $O(0, 0)$ and the radius is 1. (設廣義角 θ 的終邊與單位圓交於一點 $P(x, y)$ 。)
3. According to the definition of trigonometric ratios of the general angles, we have... (根據廣義角三角比的定義，得...)
4. The point of intersection with the terminal side of the general angle and unit circle is $(\cos \theta, \sin \theta)$. (廣義角的終邊與單位圓的交點為 $(\cos \theta, \sin \theta)$)

(一) $(180^\circ - \theta)$ 換算公式

設 θ 與 $(180^\circ - \theta)$ 的終邊與單位圓的交點分別為 P 與 Q 。根據上面的結論，得

$$P(\cos\theta, \sin\theta), Q(\cos(180^\circ - \theta), \sin(180^\circ - \theta))$$

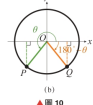
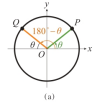
因為 P 與 Q 對稱於 y 軸，如圖 10 所示，所以 P 與 Q 的 x 坐標互為相反數，且 y 坐標相等，即

$$x \text{ 坐標: } \cos\theta = -\cos(180^\circ - \theta),$$

$$y \text{ 坐標: } \sin\theta = \sin(180^\circ - \theta).$$

($180^\circ - \theta$) 換算公式

$$\sin(180^\circ - \theta) = \sin\theta, \cos(180^\circ - \theta) = -\cos\theta.$$



▲圖 10

Vocabulary: Conversion Equation (換算公式), Symmetric with Respect (對稱於), Opposite Number (相反數)

Sentences:

1. Point P and Point Q are symmetric with respect to the Y-Axis. (P 與 Q 對稱於 Y 軸。)
2. The x-coordinate of P and Q are opposite to each other, and the y-coordinate of these two points are the same. (P 與 Q 的 x 坐標互為相反數，且 y 坐標相等。)

GeoGebra Resources:

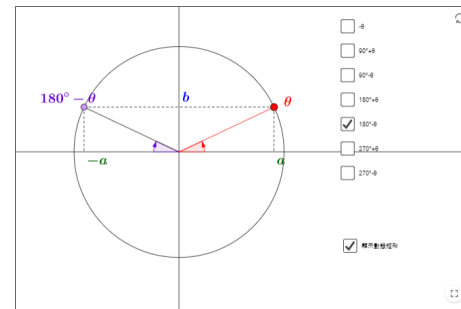
羅驥韡 (Pegasus Roe) – 餘弦定理

<https://reurl.cc/O4EM39>

0, 90, 180, 270 ± θ

Author: 羅驥韡 (Pegasus Roe)

Topic: Trigonometry



丁 直角坐標與極坐標的轉換

生活中亦常利用距離和方向描述一個點的位置，例如「東北方，20 公尺遠」，這種描述位置的方法，就是下面所要介紹的極坐標之概念。

如圖 15，在平面上選定一點 O (稱為原點或極點)，以 O 為端點向右作一水平射線 (稱為極軸)，對於平面上異於 O 的任一點 P ，令 $r = OP$ 且 θ 代表以極軸為始邊，射線 OP 為終邊的廣義角；此時，我們以 $[r, \theta]$ 來表示 P 點的位置，並稱 $[r, \theta]$ 為 P 點的極坐標，記作 $P[r, \theta]$ 。



▲圖 15

Vocabulary: Polar Axis (極軸), Pole (極點), Polar Coordinate (極坐標)。

Sentences:

1. We often use distances and directions to describe the position of a point, such as “East-North, 20 meters”. (我們常利用距離和方向描述一個點的位置，例如「東北方，20 公尺」。)

2. Figure 15 shows a horizontal ray extending to the right, which is called the **polar axis**. The endpoint O of the ray is called the **pole**. (如圖 15，以 O 為端點(稱為極點)，向右作一水平射線(稱為極軸)。
3. Set any P besides from O , and r is a direct distance from the pole O to P . (異於 O 的任意一點 P ，令 r 為有向線段 OP 。)
4. θ is a general angle where polar axis is the initial side; the OP ray is the terminal side. (θ 為廣義角，其中始邊為極軸；終邊為射線 OP 。)
5. We refer to the ordered pair as the $[r, \theta]$ **polar coordinates** of P . (我們稱 $[r, \theta]$ 為 P 點的極坐標。)

補充題

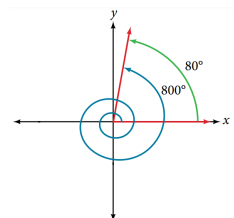
Materials

Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the **least positive angle** θ that is coterminal with an angle measuring 800° , where $0^\circ \leq \theta < 360^\circ$.

Solution:

An angle with measure 800° is coterminal with an angle with measure $800 - 360 = 440^\circ$, but 440° is still greater than 360° , so we subtract 360° again to find another coterminal angle: $440 - 360 = 80^\circ$. The angle $\theta = 80^\circ$ is coterminal with 800° .



To put it another way, 800° equals 80° plus **two full rotations**, as shown in the Figure on the

right side.

Note

Vocabulary: the Least Positive Angle (最小正同界角), Two Full Rotations (兩整圈).

Sentences:

1. An angle with measure 800° is coterminal with an angle with measure 440° . (800° 為 440° 的同界角。)
2. 440° is greater than 360° . (440° 大於 360° 。)
3. The angle $\theta = 80^\circ$ is coterminal with 800° . (80° 為 800° 的同界角)

參考資料

References

1. 許志農、黃森山、陳清風、廖森游、董涵冬 (2019)。數學 2：單元 11 廣義角三角比與極坐標。龍騰文化。
2. Openstax. (2022, May 24). *AlgebraAndTrigonometry*. <https://reurl.cc/x9pKbe>.
3. Miami Beach Senior High School. (2022, July 4). *Chapter 6 Additional Topics in Trigonometry*. <https://reurl.cc/zNrWdN>.

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