實數

The Real Numbers

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甲 有理數

圖 2 的麵包食譜中有許多常見的數字形式，例如整數 320、小數 25.6、分數 $\frac{1}{3}$ 等，這些數都可以寫成分數的形式。數學上，凡是可以寫成分數的形式的數都稱為**有理數**。

有理數的定義

可以表示成 $\frac{q}{p}$ 的數，稱為有理數（其中 p, q 為整數，且 $p \neq 0$）。

▲圖2

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Vocabulary: Rational number (有理數), Integer (整數), Decimal (小數), Fraction (分數), Irreducible/Reduced Fractions (最簡分數).

Sentences:

1. There are many forms of numbers, such as integers, decimals, fractions, etc. (數字有很多形式，例如整數、小數、分數等。)
2. A rational number is a number that can be expressed as a quotient or fraction $\frac{q}{p}$. Where p and q are integers, and p is not equal to zero. (有理數是一個可以被表示成分數 $\frac{q}{p}$ 形式的數，其中 p、q 是整數，p 不能為 0。)

Extra Vocabulary: Flour (麵粉), Bread Flour (高筋麵粉), All-Purpose Flour (中筋麵粉), Cake Flour (低筋麵粉), Yeast (酵母), Table Spoon (一大匙), Tea Spoon (一茶匙), Milliliter (毫升).

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<tr><td>Improper Fractions</td><td>Fractions in which the numerator is more than or equal to the denominator.</td><td>$\frac{5}{3}$</td></tr>
<tr><td>Mixed Fractions</td><td>Mixed fractions consist of a whole number along with a proper fraction.</td><td>$8\frac{2}{3}$</td></tr>
<tr><td>Like Fractions</td><td>Fractions with the same denominators.</td><td>$\frac{1}{4}$ and $\frac{3}{4}$</td></tr>
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Vocabulary: Irreducible/Reduced Fraction (最簡分數), Improper Fraction (假分數), Mixed Fraction (帶分數), Denominator (分母), Numerator (分子).

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	Note: 1. Expand a fraction. (擴分。) 2. Reduce a fraction. (約分。) 3. Reduce to a common denominator. (通分。)
由例題 1 可發現，將 $\frac{2}{11}$ 化成小數時，小數點以後的數字 1 與 8 依序不斷的循環出現，這種小數稱為**循環小數**，記為 $\frac{2}{11}=0.181818\cdots=0.\overline{18}$， 讀作「零點一八，一八循環」。在循環小數中，小數點之後重複出現的那一段數字稱為**循環節**，如上述例子的循環節為 18。將 $\frac{2}{11}$ 化成小數的過程中，因為除數為 11，所以餘數只可能是 0, 1, 2, ⋯, 10 之中的一數。這表示最多經過 11 次運算，餘數就會重複地出現。像這樣利用除法將有理數化成小數，如果不能除盡成為有限小數，就一定可以化成循環小數。	**Vocabulary:** Repeating/Recurring Decimal, Period (循環節), Periodicity (循環節長度), Terminating Decimal (有限小數), Non-terminating and Recurring decimal (無限循環小數), Repetend (or Reptend) (循環小數). **Sentences:** 1. $\frac{2}{11}$ is also represented as $0.18181818\cdots=0.\overline{18}$ zero point one eight, one eight repeating. ($\frac{2}{11}$ 可以被化成 $0.18181818\cdots=0.\overline{18}$ 。) 2. We notice that after the decimal point, there is a group of one eight is repeating itself. (我們可以發現小數點後數字 18 一直重複。) 3. One eight is the repetend (or reptend) in the decimal representation of $\frac{2}{11}$, with 2 repeating digits. ($\frac{2}{11}$ 的循環節是 18，循環節的長度是 2。) **Note:** $1.2\overline{34}$ may be read "one point two repeating three four", "one point two recurring three four", "one point two repetend three four" or "one point two into infinity three four".

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<td>有理數的稠密性

任兩個相異有理數之間，至少存在一個有理數。</td>
<td>Vocabulary: Density (稠密性), Distinct (不同的).

Translations:

The density of the rational numbers

There is a rational number that exists between any two distinct rational numbers</td>
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<td>無理數

不能表示成 $\frac{q}{p}$ 的數（p, q 為整數，$p \neq 0$），稱為無理數。

之前提過，任意有理數都可化成整數、有限小數或循環小數，而無理數不能化成以上三種形式。事實上，所有的無理數都是不循環的無限小數。

那麼，怎麼知道 $\sqrt{2}$ 不是有理數呢？如果 $\sqrt{2}$ 是有理數，那麼 $\sqrt{2}$ 可表為 $\sqrt{2} = \frac{q}{p}$（最簡分數），其中 p, q 為正整數。將上式整理可得 $2p^2 = q^2$，可推得 $2p^2 - pq = q^2 - pq$，即 $p(2p-q) = q(q-p)$，移項可得

$$\frac{2p-q}{q-p} = \frac{q}{p} = \sqrt{2} 。 \quad ①$$

又因為 $1 < \frac{q}{p} = \sqrt{2} < 2$，即 $p < q < 2p$，所以

$$0 < q - p < p 。$$

也就是說，式子 ① 中的分數 $\frac{2p-q}{q-p}$ 是由分數 $\frac{q}{p}$ 約分得來，但這與 $\frac{q}{p}$ 為 $\sqrt{2}$ 的最簡分數矛盾。因此 $\sqrt{2}$ 是無理數。

像上述的證明方法稱為反證法，其原理為：從「假設結論不成立」出發，通過一系列的推理，導出矛盾的結果；因此，「假設結論不成立」是錯誤的，也就是說，「結論成立」才是正確的。</td>
<td>Vocabulary: Non-terminating and Non-recurring Decimal (無限不循環小數), Proof by Contradiction/Indirect Proof (反證法), Conclusion (結論).

Sentences:

1. Any number which cannot be written in the form $\frac{q}{p}$, where p and q are integers and p is unequal to 0, is called an irrational number. (不能表示成 $\frac{q}{p}$ 的數，其中 p、q 是整數且 p 不能為 0，稱為無理數。)

2. By squaring both sides. (兩邊平方。)

3. Start by suppose that the opposite is true. (從「假設結論不成立」出發。)

4. The contradiction confirms that the original supposition must be false. (得到「假設結論不成立」是錯的，也就是「結論成立」才是對的。)</td>
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任意兩個實數作加、減、乘、除（除數不可以是 0）運算後仍然是實數，且實數在運算上有下列性質。

1. **實數的運算性質**：設 a, b, c 是任意實數。
   (1) 交換律：$a+b=b+a$, $ab=ba$。
   (2) 結合律：$(a+b)+c=a+(b+c)$, $(ab)c=a(bc)$。
   (3) 分配律：$a(b+c)=ab+ac$。
   (4) 消去律：若 $a+c=b+c$，則 $a=b$。
   若 $ac=bc$ 且 $c\neq 0$，則 $a=b$。

任意實數都可以在數線上找到對應的點，而且愈往右邊的點所對應的實數愈大。實數的大小關係有下列性質。

2. **實數的次序關係**：設 a, b, c 是任意實數。
   (1) 三一律：「$a<b$, $a=b$, $a>b$」三式中恰有一個成立。
   (2) 遞移律：若 $a<b$ 且 $b<c$，則 $a<c$。
   (3) 不等量加法：若 $a<b$，則 $a+c<b+c$。
   (4) 不等量乘法：若 $a<b$ 且 $c>0$，則 $ac<bc$；
   若 $a<b$ 且 $c<0$，則 $ac>bc$。
   (5) 對任一實數 a, $a^2\geq 0$ 恆成立。（$a^2=0$ 僅在 $a=0$ 時成立）

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Vocabulary: Law/Property (律), Commutative (交換的), Associative (結合的), Distributive (分配的), Cancellation (消去), Trichotomy Law(三一律), Transitive (遞移的), Addition Property of Inequality (不等量加法), Multiplicative Property of Inequality (不等量乘法), Arbitrary (隨意的).

Sentences:

1. We can swap numbers over and still get the same result. (我們可以交換數字位置，且得到相同結果。)
2. For any/**arbitrary** real numbers a and b, exactly one of the relations a<b, a=b, or a>b holds. (對任意實數 a、b，必符合大於、等於或小於其中一項。)
3. If the same quantity is added to both sides of an inequality, then the inequality is still true (將不等式兩邊加上一樣的數，不等式依然成立。)
4. Both sides of an inequality can be multiplied or divided by the same positive number and an equivalent inequality can be formed. (將不等式兩邊乘或除一樣的數，不等式依然成立。)

Notes:

ℝ : Real Number ℚ : Quotient

ℤ : Zahlen (“Numbers” in German)

ℕ : Natural Number

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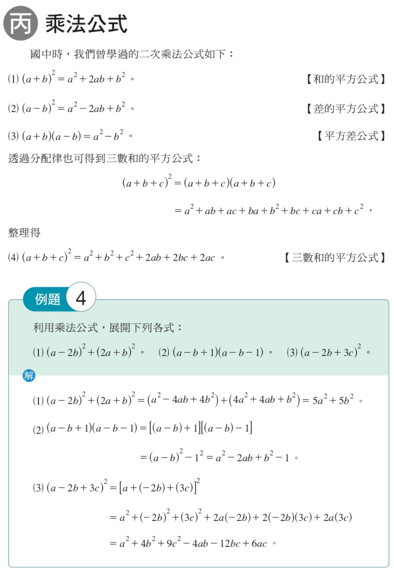

丙 乘法公式

國中時，我們曾學過的二次乘法公式如下：

(1) $(a+b)^2=a^2+2ab+b^2$。 【和的平方公式】

(2) $(a-b)^2=a^2-2ab+b^2$。 【差的平方公式】

(3) $(a+b)(a-b)=a^2-b^2$。 【平方差公式】

透過分配律也可得到三數和的平方公式：

$$(a+b+c)^2=(a+b+c)(a+b+c)$$
$$=a^2+ab+ac+ba+b^2+bc+ca+cb+c^2，$$

整理得

(4) $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac$。 【三數和的平方公式】

例題 4

利用乘法公式，展開下列各式：

(1) $(a-2b)^2+(2a+b)^2$。 (2) $(a-b+1)(a-b-1)$。 (3) $(a-2b+3c)^2$。

解

(1) $(a-2b)^2+(2a+b)^2=(a^2-4ab+4b^2)+(4a^2+4ab+b^2)=5a^2+5b^2$。

(2) $(a-b+1)(a-b-1)=[(a-b)+1][(a-b)-1]$
$=(a-b)^2-1^2=a^2-2ab+b^2-1$。

(3) $(a-2b+3c)^2=[a+(-2b)+(3c)]^2$
$=a^2+(-2b)^2+(3c)^2+2a(-2b)+2(-2b)(3c)+2a(3c)$
$=a^2+4b^2+9c^2-4ab-12bc+6ac$。
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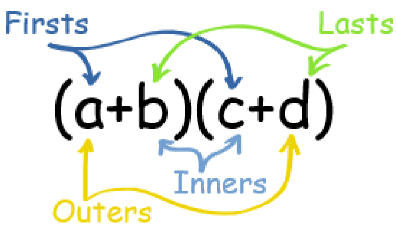
Vocabulary: Algebra Formula(乘法公式), Perfect Square Trinomial (完全平方三項), Square of a Sum (和的平方), Square of a binomial (二項的平方). Square of a Difference (差的平方), Difference of Square (平方差), Square of a Trinomial (三項平方).

Sentences:

1. Identify and factor perfect square trinomials. (請將完全平方三項化成完全平方。)
2. FOIL the expression. We do the FIRST terms “a” times “a”. Do the OUTER terms “a” times “b”, and do the INNER terms “b” times “c”. Last we do the LAST terms “b” times “d”. (利用 FOIL 的方式展開式子：首先將第一項相乘，再乘外面那項，再將內項相乘，最後再乘後面兩項。)

Notes:

FOIL: The FOIL method stands for First, Outer, Inner, and Last.



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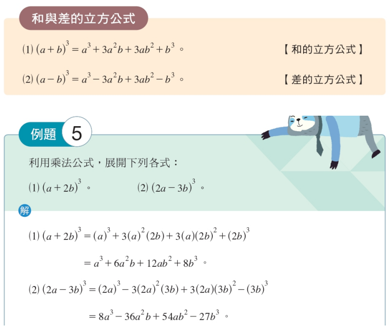

和與差的立方公式

(1) $(a+b)^3=a^3+3a^2b+3ab^2+b^3$。 【和的立方公式】

(2) $(a-b)^3=a^3-3a^2b+3ab^2-b^3$。 【差的立方公式】

例題 5

利用乘法公式，展開下列各式：

(1) $(a+2b)^3$。 (2) $(2a-3b)^3$。

解

(1) $(a+2b)^3=(a)^3+3(a)^2(2b)+3(a)(2b)^2+(2b)^3$
$=a^3+6a^2b+12ab^2+8b^3$。

(2) $(2a-3b)^3=(2a)^3-3(2a)^2(3b)+3(2a)(3b)^2-(3b)^3$
$=8a^3-36a^2b+54ab^2-27b^3$。
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Vocabulary: Cube of the Sum (和的立方), Cube of the Difference (差的立方), Cube of a Binomial (二項的立方), Sum of Cubes (立方和), Difference of Cubes (立方差).
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<td>立方和與立方差公式
(1) $a^3+b^3=(a+b)(a^2-ab+b^2)$。 【立方和公式】
(2) $a^3-b^3=(a-b)(a^2+ab+b^2)$。 【立方差公式】</td>
<td>Sentences:
1. Expand/Simplify the following expressions by using algebraic formulas. (利用乘法公式展開/化簡下列各式。)
2. Perform the indicated operations and write the result in descending power of x. (寫出下列算式的展開過程，並將答案以 x 的降冪呈現。)</td>
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<td>(二)雙重根式
形如$\sqrt{5-2\sqrt{6}}$的根式稱為雙重根式。有些雙重根式可以利用平方公式來化簡：因為
$$5-2\sqrt{6}=5-2\times\sqrt{2}\times\sqrt{3}=\left(\sqrt{3}-\sqrt{2}\right)^2，$$
所以$\sqrt{5-2\sqrt{6}}=\sqrt{3}-\sqrt{2}$。
一般而言，當$a\geq b\geq 0$時，因為
$$\left(\sqrt{a}-\sqrt{b}\right)^2=(a+b)-2\sqrt{ab}，$$
所以
$$\sqrt{(a+b)-2\sqrt{ab}}=\sqrt{a}-\sqrt{b}。$$
同理可得$\sqrt{(a+b)+2\sqrt{ab}}=\sqrt{a}+\sqrt{b}$。</td>
<td>Vocabulary: Nested Radical (雙重根式), Simplest Radical Form (最簡根式), Square Root/Root (平方根), Systems of Equations (聯立方程組), Substitute A for B/ Replace A with B (以 A 代替 B).
Sentences:
1. The nested radical is a radical expression that contains another radical expression, like the square root of 5 minus 2 square root of 6. (雙重根式就是根式裡還有根式，形如$\sqrt{5-2\sqrt{6}}$。)
2. Rewrite it in a form that it’s not nested. (將其改寫成不是雙重的形式。)
3. Rewrite the expression under the radical as a perfect square. (將第一個根號底下的式子寫成完全平方。)
4. The square root of a square must be nonnegative. (平方再開根號，其值為正。)
5. Express the answer in the simplest radical form. (將答案以最簡根式呈現。)</td>
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<td>6. Rationalize the denominator of ... (將分母...有理化。)</td>
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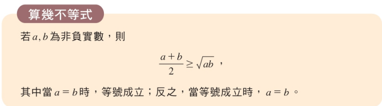
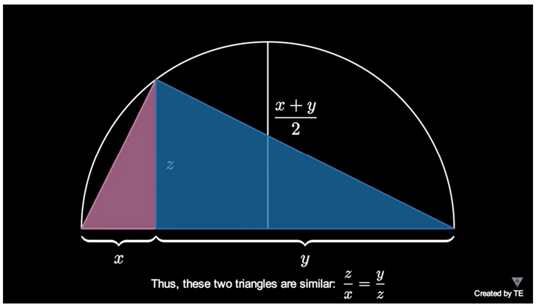


Youtube Materials:



Mathematical Visual Proofs - AM-GM Inequality II (visual proof).

https://reurl.cc/m3b13Y
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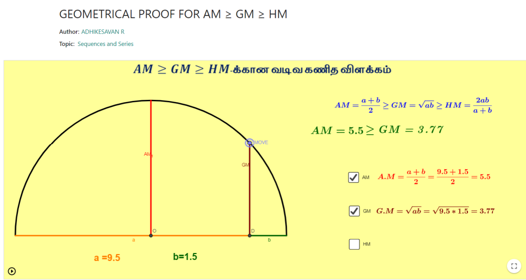
Vocabulary: Inequality of arithmetic and geometric means/AM-GM Inequality (算術-幾何平均值不等式/算幾不等式), Non-negative Real Numbers (非負實數).

Translations:

If a and b are non-negative real numbers, then “a” plus “b” divided by 2 is greater than or equal to the square root of “ab” with equality it and only if “a” equals “b”.

GeoGebra Materials:



Adhikesavan R - Sequences and Series.

https://www.geogebra.org/m/mfxhdpue
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<td colspan="2">Materials</td>
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<td colspan="2">A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs $2 per foot, while the fence for the other three sides costs $1 per foot. How much of each type of fencing will he have to buy in order to keep expenses to a minimum? What is the minimum expense?</td>
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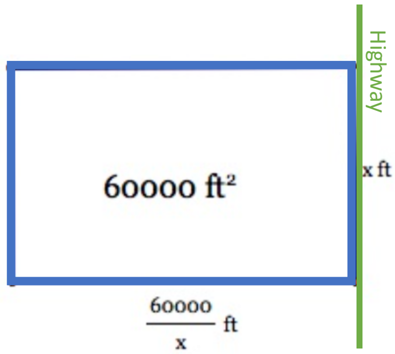
Solution:

Let x equal the distance along the highway. The area equals 60,000 square feet, which means $\frac{60000}{x}$ is the distance along one side of the field that is not parallel to the highway. The cost function is



$$\text{Cost }(x) = 2x + x + \frac{60000}{x} + \frac{60000}{x}$$

$$\text{Cost }(x) = 3x + \frac{120000}{x}$$

By the **AM-GM Inequality**,

$$\text{Cost }(x) \geq 2\sqrt{(3x)\left(\frac{120000}{x}\right)} = 2\sqrt{360000} = 2(600) = \$1{,}200$$

The cost of fencing for any such rectangular plot is always greater than or equal to $1,200. Thus, $1,200 is the minimum cost. It reaches this amount when

$$3x = \frac{120000}{x}$$

, which means x = 200 feet. So the plot of land is 200 feet along the highway and 300 feet along the side perpendicular to the highway.

Note

Vocabulary: Fence (柵欄), Plot (一個區域的土地), Highway (高速公路), Minimum (最小值), Parallel (平行), Reach (達到), Perpendicular(垂直).

Sentences:

1. A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. (一位農夫欲沿著一高速公路，用柵欄圍出一塊 60,000 平方英尺的地。)
2. How much of each type of fencing will he have to buy in order to keep expenses to a minimum? (他至少須花費多少錢購買柵欄？)

參考資料 **References**
1. 許志農、黃森山、陳清風、廖森游、董涵冬（2019）。*數學 1：單元 1 實數*。龍騰文化。 2. SplashLearn. *Fraction – Definition with Examples.* https://www.splashlearn.com/math-vocabulary/fractions/fraction. 3. Mathsisfun. *Common Number Sets.* https://www.mathsisfun.com/sets/number-types.html. 4. Mathsisfun. *FOIL Method.* https://www.mathsisfun.com/definitions/foil-method.html 5. Jim Wilson. *Using the Arithmetic Mean-Geometric Mean Inequality in Problem Solving.* http://jwilson.coe.uga.edu/EMT725/AMGM/SSMA.Bnghm.html

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