雙語教學主題(國中八年級教材):介紹一元二次方程式的公式解
Topic: introducing the quadratic formula

這個單元常用到的一些用語
The vocabulary we use for this topic
add, subtract, multiply, divide, plus(+), minus(-), times(x), divided(÷), isolate, squared 平方, square root 開根號, ± plus or minus 正負, complete the square 配完全平方, solution or root 解或根, perfect square 完全平方

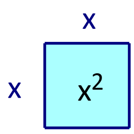
We are going to introduce the super formula for solving quadratic equations in this class. The reason we say it's super is that no matter what method you have learned to solve quadratic equations like factorizing, grouping, or using the cross method, mostly, you can only solve certain kinds of quadratic equations with any one of the methods mentioned above. However, with this super formula, you can solve any quadratic equations you see. It's super powerful.
這個單元我們要介紹二元一次方程式的公式解。你們前面所學過的各種解二元一次方程式的方法，都只能夠解部分的方程式，而今天我們要學的公式解，是一個所向無敵的超級方法，大家要好好學喔!

Before we learn this formula, I want you to look into the intuitive understanding of completing the square you learned from your algebra class by a simple example.
但是在我們介紹之前，我要帶你們用直觀的方法看你們曾經在前面的代數課中學過的配完全平方法

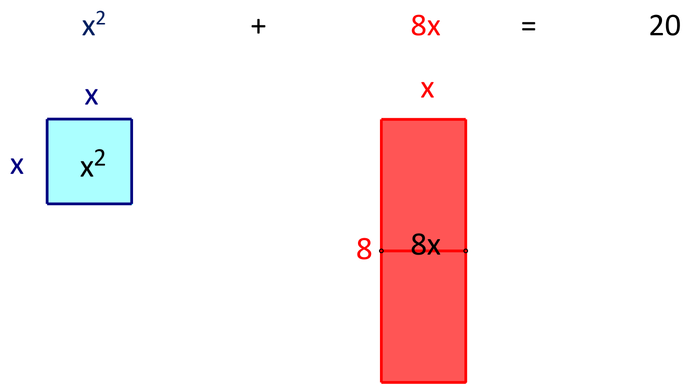
Example: Solve the quadratic equation by completing the square
$x^2 + 8x = 20$ (x squared plus 8 times x is equal to 20)
Let's visualize the polynomial on the left-hand side of the equation. Each term is going to represent the area of the shape.

We first look at x^2, we represent it by a square that has a side length of x. The area of the square is equal to x^2.

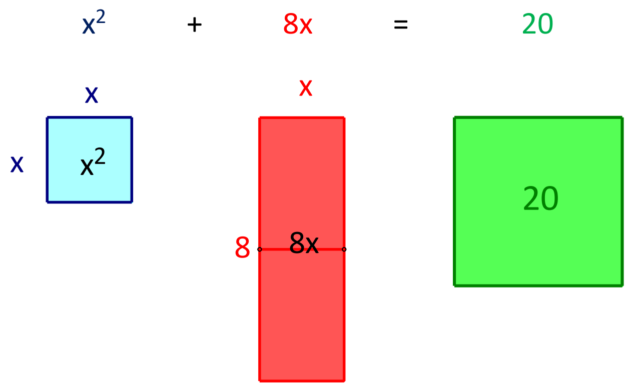
x^2 + $8x$ = 20



Then the second term 8x, we represent it by a rectangle that has a side length of x and another side length of 8. The area of the rectangle is equal to 8x.



Now comes to the last term, which is the constant term 20, we represent it by a rectangle that has the area of 20. We don't write down the side length of it yet.

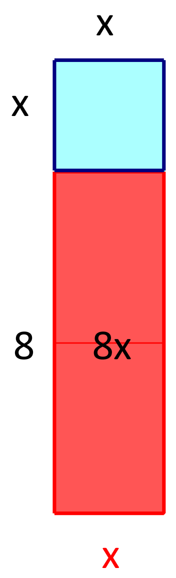


What we want to do is to combine the area of the two shapes on the left-hand side and let the area be equal to 20 which is the area of the shape on the right-hand side.

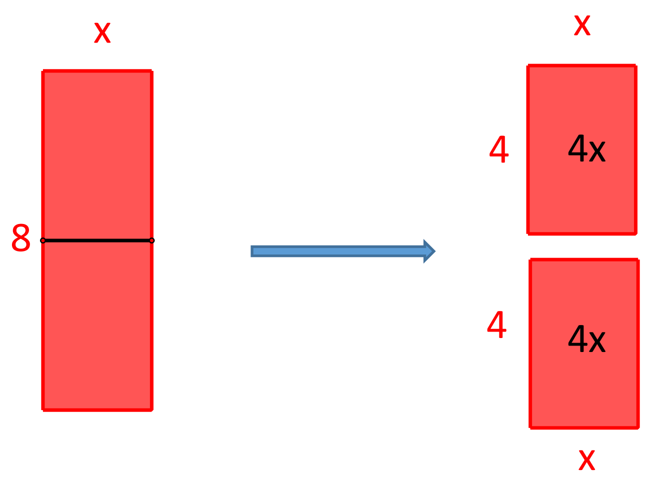
But how do we combine the two shapes on the left-hand side?

Let's look at the shapes on the left-hand side, they do have a common side x. we can then combine these two shapes on their common side x.

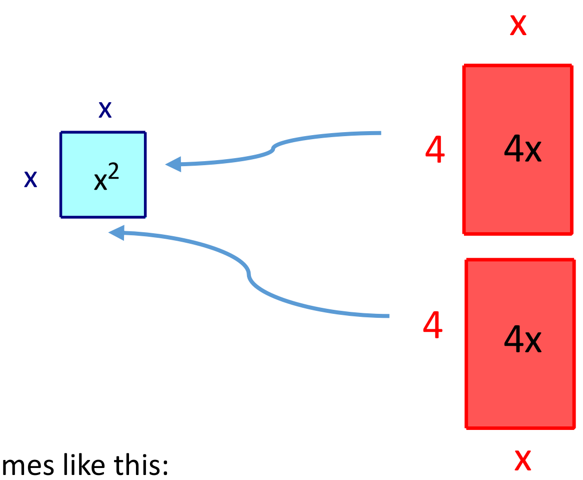
Before we do this, we split the red rectangle in half from the side length of 8. Why do we split the rectangle in half? Think about it! We have only 8x in the equation, if we just join the blue and red shapes together and don't do anything else, it'll be like the figure below:



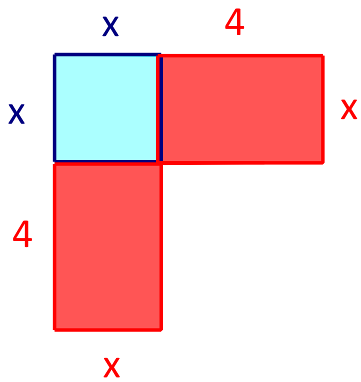
It's harder this way to get what we want, which is "completing the SQUARE". Therefore, we split the red rectangle evenly in half to try to reach our goal. And of course, it works at the end.



Here we have two half red rectangles which have the side length of x and 4. The area of these two half red rectangles is the same, 4x
We rotate the upper half rectangle 90° and connect it to the side x of the blue square. Then we move the lower half red rectangle right under the blue square.



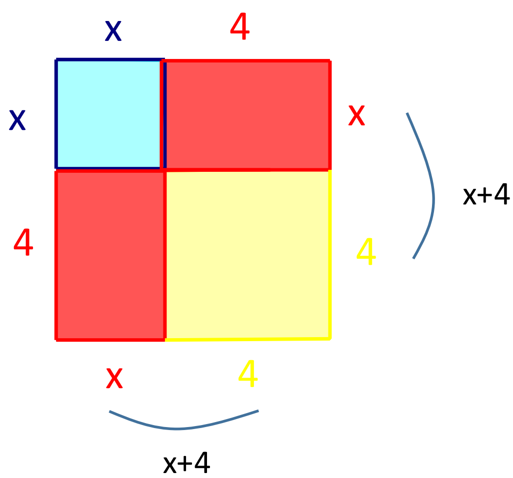
And it becomes like this:



Look at the yellow area shown below, it's a square with a side length of 4, and its area is 16

If we add this yellow square to the above figure, we get a bigger square with a side length of (x+4), and the area of this bigger square is $(x+4)^2$.

(the quantity of x plus 4 squared)



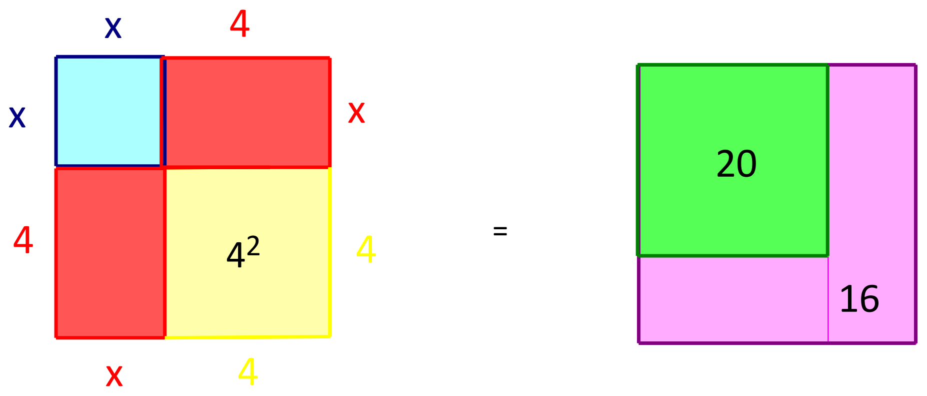
Now look back to our equation

$x^2+8x=20$

Add $4^2=16$ on both sides of the equation

$x^2+8x+4^2=20+16$

This is what we learned about "completing the square" before. It helps us to solve the equation.



It shows that the area of the square with the side length of x+4 on the left-hand side should be equal to the area of 36 on the right hand side, which is also a square with a side length of 6

$(x+4)^2=36$

(the quantity of x+4 squared is equal to 36)

Square root both sides of the equation

$$\sqrt{(x+4)^2}=\pm\sqrt{36}$$

(the reason we add the sign $\pm$ (plus or minus)here is because $6^2=36$, also $(-6)^2=36$)

Simplify the equation

$X+4=\pm 6$ (x plus 4 is equal to plus or minus 6)

We solve two linear equations:

$X+4=6$

(x plus 4 equals 6)

$X+4-4=6-4$

(Subtract 4 on both sides)

$X=2$

$x+4=-6$

(x plus 4 equals negative 6)

$x+4-4=-6-4$

(Subtract 4 on both sides)

$x=-10$

The solutions to the equation are x=2 or -10. Completing the square helps us to solve quadratic equations.

Since it's so useful and kind of complicated, we will now introduce the formula form to you by using the same method: completing the square

Quadratic equation formula

The general form of a quadratic equation is like

$ax^2+bx+c=0$

a is the leading coefficient of the equation, so a can not be 0.

And when a is negative, we can multiply both sides with (-1), then the leading coefficient of this equation turns to be positive. It's always simpler to discuss polynomials or equations which have positive leading coefficients. So we only need to discuss when a is positive.

$ax^2+bx+c=0 \qquad a>0$

It's easier to introduce the formula when the leading coefficient is 1

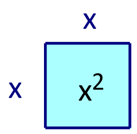
We divide both sides by a

$x^2+\frac{b}{a}x+\frac{c}{a}=0$ we only look at the x^2 term and the x term when we complete the square on the left hand side of the equation, we move the constant term $\frac{c}{a}$ to the right hand side.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

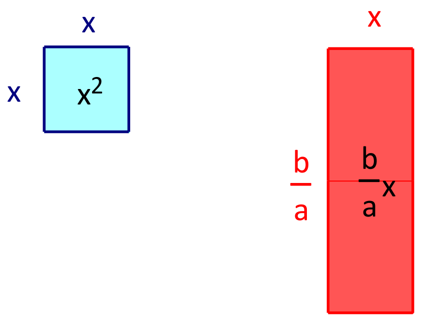
Then we use the same way by representing x^2 as the area of a square with a side length of x.

$$x^2 \quad + \quad \frac{b}{a}x \quad = \quad -\frac{c}{a}$$



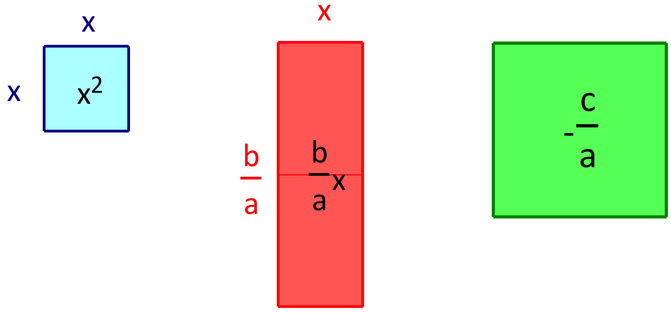
Then the second term $\frac{b}{a}x$, we represent it by a rectangle that has a side length of x and another side length of $\frac{b}{a}$. The area of the rectangle is equal to $\frac{b}{a}x$.

$$x^2 \quad + \quad \frac{b}{a}x \quad = \quad -\frac{c}{a}$$



Now comes to the last term which is the constant term $-\frac{c}{a}$, we represent it by a rectangle that has the area of $-\frac{c}{a}$. We don't write down the side length of it yet.

$$x^2 \quad + \quad \frac{b}{a}x \quad = \quad -\frac{c}{a}$$

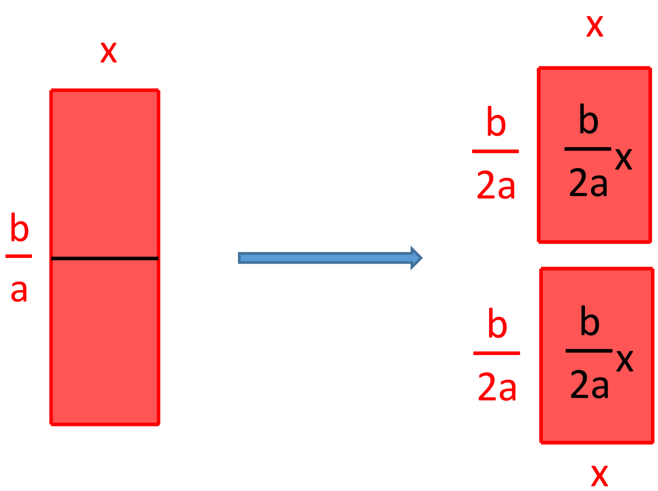


What we want to do is to combine the area of two shapes on the left hand side and let the area be equal to $-\frac{c}{a}$ which is the area of the shape on the right hand side.

But how do we combine the two shapes on the left hand side?

Let's look at the shapes on the left hand side, they do have a common side x. we can then combine these two shapes on their common side x.

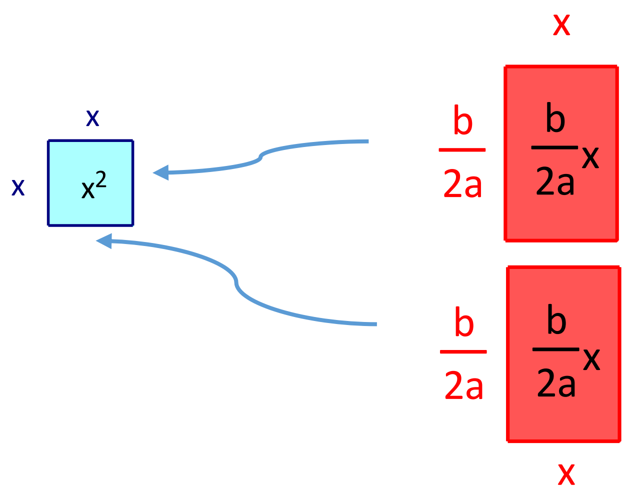
Before we do this, we split the red rectangle in half from the side length of $\frac{b}{a}$



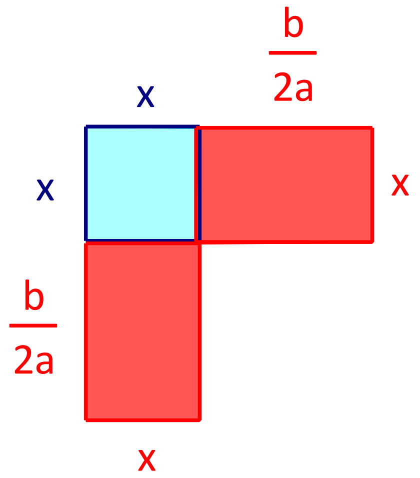
Here we have two half red rectangles which have the side length of x and $\frac{b}{2a}$.

The area of these two half red rectangles is the same, $\frac{b}{2a}x$.

We rotate the upper half rectangle 90° and connect it to the side x of the blue square. Then we move the lower half red rectangle right under the blue square.



And it becomes like this:

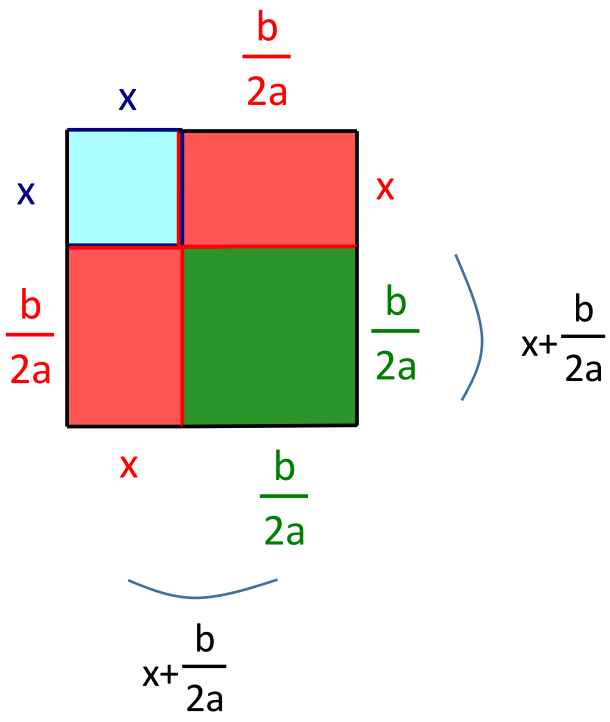


Look at the dark green area shown below, it's a square with a side length of $\frac{b}{2a}$,

and its area is $\left(\frac{b}{2a}\right)^2$. If we add this dark green square to the above figure, we

get a bigger square with a side length of $(x+\frac{b}{2a})$, and the area of this bigger

square is $\left(x+\frac{b}{2a}\right)^2$. (the quantity of x plus $\frac{b}{2a}$ squared)



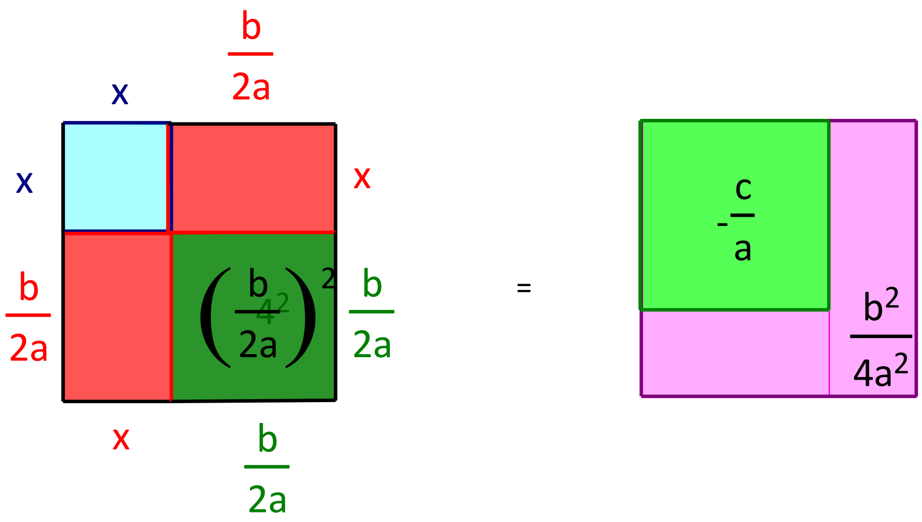
Now look back to our equation

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

According to the result we have above, add $\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$ on both sides of the equation

(the quantity of b over 2a squared is equal to b squared over 4 times a squared)

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$



This is what we learned "completing the square" before.

It shows that the area $(x+\frac{b}{2a})^2$ of the square with the side length of $(x+\frac{b}{2a})$ on the left hand side should be equal to the area $(-\frac{c}{a}+\frac{b^2}{4a^2})$ on the right hand side

Combine like terms of the constant term

$$-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}$$

$$=\frac{b^2-4ac}{4a^2}$$

The equation of the equal area is

$$(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2} \qquad (1)$$

(the quantity of $x+\frac{b}{2a}$ squared equals b squared minus 4ac all over 4 times a)

Square root on both sides of the equation (1)

$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$ (the same reason we add the sign $\pm$ here

Simplify the equation

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a} \qquad a>0,\ \sqrt{4a^2}=2a$$

Move $\frac{b}{2a}$ to the right hand side, we get

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ *Da la…*

(x equals to negative b plus or minus the square root of b squared minus 4ac all over 2a)

This is the super quadratic formula.

The two solutions to the quadratic equation are

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a},\ x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Now when we want to solve a quadratic equation like $ax^2+bx+c=0$, we only need to identify what the corresponding values of a, b and c are, simply plug in the numbers and we will get the answers right away. Isn't it fantastic!

Let's do some practice

Solve the following quadratic equations:

1. $x^2-1=0$ we need to find out the corresponding a, b and c by comparing these two equations $x^2-1=0$ and $ax^2+bx+c=0$ rewrite the first equation as $x^2+0x-1=0$ Then we get a=1, b=,0, c=-1 Plug in the values of a, b and c to the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-0\pm\sqrt{0^2-4(1)(-1)}}{2(1)}$ $x=\frac{\pm\sqrt{4}}{2}$ $x=\pm1$ The solutions of the equation $x^2-1=0$ are 1,-1 verify: Replace x in the equation $x^2-1=0$ with1 $1^2-1=0$ true and Replace x in the equation $x^2-1=0$ With-1 $(-1)^2-1=0$ true So both 1 and -1 are the solutions	Of course you can solve this equation by factorizing it, but we want to practice the quadratic formula we just learned x squared minus 1 equals 0 a times x squared plus b times x plus c equals 9 x equals to negative b plus or minus square root of b squared minus 4 times a times c all over 2 times a x equals plus or minus the square root of 4 over 2 x equals plus or minus 1 Normally verifying the answers we get will help us get higher scores when taking a test. Please do it whenever you have time.

2. $x^2+3x+2=0$ Comparing this equation with the formula $ax^2+bx+c=0$ We get a=1, b=3, c=2 Plug into the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-3\pm\sqrt{3^2-4(1)(2)}}{2(1)}$ $x=\frac{-3\pm\sqrt{1}}{2}$ $x=\frac{-3+1}{2}$ or $x=\frac{-3-1}{2}$ x=-1 or x=-2 The solutions of the equation are 1,-1 verify: Replace x in the equation $x^2+3x+2=0$ with-1 $(-1)^2+3(-1)+2$ =1-3+2 =0 true and Replace x in the equation $x^2+3x+2=0$ with-2 $(-2)^2+3(-2)+2$ =4-6+2 =0 true So both -1 and -2 are the solutions	We know we can solve this equation by using the cross method, but we practice using the formula to solve it. x square plus 3x plus 2 equals 0

3. $3x^2+5x+1=0$ Comparing the formula, we get a=3, b=5, c=1 Plug into the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-5\pm\sqrt{5^2-4(3)(1)}}{2(3)}$ $x=\frac{-5\pm\sqrt{13}}{6}$ $x=\frac{-5+\sqrt{13}}{6}$ or $x=\frac{-5-\sqrt{13}}{6}$ The solutions to the equation are $\frac{-5+\sqrt{13}}{6}, \frac{-5-\sqrt{13}}{6}$ verify: Replace x in the equation $3x^2+5x+1=0$ with $\frac{-5+\sqrt{13}}{6}$ $3(\frac{-5+\sqrt{13}}{6})^2+5(\frac{-5+\sqrt{13}}{6})+1$ $=\frac{25-10\sqrt{13}+13}{12}+\frac{-50+10\sqrt{13}}{12}+\frac{12}{12}$ =0 true and Replace x in the equation $3x^2+5x+1=0$ With $\frac{-5-\sqrt{13}}{6}$ (skip the process here, it's about the same) true So both $\frac{-5+\sqrt{13}}{6}$ and $\frac{-5-\sqrt{13}}{6}$ - are the solutions	3 times X squared plus 5 times x plus 1 equals 0 x equals negative 5 plus or minus the square root of 15 all over 6 Simplify the expression and combine like terms

Listen and relax. Enjoy!

https://www.youtube.com/results?search_query=quadratic+formula+song

Reference:

https://www.youtube.com/watch?v=EBbtoFMJvFc –youtube

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