雙語教學主題(國中八年級教材):介紹因式分解

Topic: Factoring quadratic polynomials

The vocabulary we use in this topic

quadratic, squared 平方, square root 開根號, polynomial, polynomial identity 恆等式, GCF=greatest common factor, factoring polynomial 因式分解, factoring out 提公因式(數), grouping 分組, expand, expression, binomial 二項多項式, trinomial 三項多項式, leading coefficient, constant term

各位老師，因式分解法這個單元，在內容上主要是<u>因式分解的方法</u>，國內外稍有不同，但是目的是一樣的。所以在資料的整理上，主要是配合國內的教學習慣，並以部分篇幅介紹國外的常用的因式分解方法。老師們可以根據自己的興趣，時間以及進度考量，參考使用部分內容。老師們辛苦了!

We are going to talk about factoring quadratic polynomials in this class. So what is factoring quadratic polynomials? Generally speaking, factoring a polynomial is expressing the polynomial as a product of two or more factors; it is basically the reverse process of multiplying.

因式分解就是將一個多項式表示成為若干個質因式的連乘積。

一般來說，就是多項式相乘的逆向過程。

Let's look at some examples	
Ex1:	2 times the quantity of x plus 5
$2(x+5)$	We expand the expression by using the distributive property of multiplication and get
$=2x+10$	2 times x + 10 These two expressions are equivalent. They mean the same thing.
$2(x+5)$ expanding ↓ factoring ↑ $=2x+10$	When we work from top to bottom, we say we are expanding this polynomial. And when we work from bottom to top, we say we are factoring this polynomial.
Ex2:	

Remember we learned polynomial identities? For instance: *The difference of two squares* factoring → $a^2-b^2=(a+b)(a-b)$ ← expanding	Please review all the polynomial identities we learned before. We are going to use some of them when we factor polynomials. a squared minus b squared is equal to parentheses a plus b times parentheses a minus b But normally we skip the word parentheses. For when we teach, we talk and write on the chalkboard at the same time. Students will listen to us and watch us to write. It's easy for them to understand what we mean. But if you want to play safe, you can write down all the details when you work on your teaching plan. (通常我們會省略像”括號”這類的字，因為我們上課一定是邊說邊寫，學生會很清楚我們的意思)

We now introduce factoring binomials and trinomials respectively in this lesson. Let's start with factoring binomials.

Ex3: Factor the following binomials:	
1. 4x-8	4 times x minus 8
The GCF of 4 and 8 is 4, factor out 4	
We get	
4x-8	4 times x minus 8 is equal to 4 times whole x minus 2
=4(x-2)# $\frac{4x}{4}$ $\frac{8}{4}$	And after factoring out 4, we write down the remaining term. In parentheses, the first term would be 4x divided by 4, so we have x as the first term. And the second term is 8 divided by 4, which is 2. The operation sign "minus" remains.
Then factoring this binomial 4x-8 is completed.	
We can check the reverse process by using the distributive property	
4(x-2) =4x-8	Please keep it in mind that from now on, before you do any polynomial factoring, you need to look into the common factors of the coefficients and variables in the polynomials you work on first.
This is called the GCF method.	

2. x^2-3x x^2-3x $=x(x-3)$ #	We see there is a common variable x from this binomial. So we factor out x from both terms x squared minus 3 times x We can see the common variable in both terms is x. We factor out x and get X times x minus 3
3. x^2-25 x^2-25 $= x^2- 5^2$ $=(x+5)(x-5)$ #	x squared minus 25 This binomial has 2 squared terms. It reminds us of the polynomial identity *DIFFERENCE OF TWO SQUARES* $a^2-b^2=(a+b)(a-b)$ So we transform it to x squared minus 5 squared According to the polynomial identity mentioned above, we get x plus 5 times x minus 5

Now we are going to introduce factoring trinomials. Before we do that, let us review another polynomial identity we always use when factoring trinomials-

PERFECT SQUARE TRINOMIAL

$$(a\pm b)^2=a^2\pm 2ab+b^2$$

Let's see some examples here.

ATTENTION: no matter what methods we use to factor polynomials, we always, always check the GCF method first.

Ex 4: factoring the following trinomials	
1. x^2+6x+9	x squared plus 6 times x plus 9
	The trinomial we have here has no GCF from all terms other than 1. So we don't use the GCF method here.
x^2+6x+9 ↑ ↑ x^2 3^2	It has two perfect squares x squared and 9 9 equals 3 squared
Corresponding to the identity $(a+b)^2=a^2+2ab+b^2$ We have a=x and b=3.	
$2ab$ $=2(x)(3)$ $=6x$	Replace a with x and b with 3, we get 2 times a times b is equal to 2 times x times 3 and equals to 6 times x
That is x^2+6x+9 $= x^2+2(x)(3)+3^2$ $=(x+3)^2$#	

2. $49x^2-70x+25$ This trinomial also has two perfect squares $49x^2$ and 25 $49x^2-70x+25$ $(7x)^2$ 5^2 Corresponding to the identity $(a-b)^2=a^2-2ab+b^2$ We have a=7x and b=5. 2ab =2 (7x)(5) =70x So $49x^2-70x+25$ $= (7x)^2-2\ (7x)(5)+5^2$ $=(7x-5)^2$ #	49 times x squared minus 70 times x plus 25 49 times x squared equals 7x whole squared 25 equals 5 squared Replace a with 7x and b with 5 we get 2 times a times b is equal to 2 times 7x times 5 and equal to 70 times x

The second method to factor trinomials is *GROUPING*. The general form of a quadratic trinomial is like ax^2+bx+c, we can set a, b, and c are whole numbers. We will show you how to factor trinomials by grouping them in terms of a=1 and a≠1 respectively.

Ex5: Factoring the trinomials ax^2+bx+c when a=1 1. x^2+3x+2 x^2+3x+2 We copy the process of x^2+bx+c $=(x+m)(x+m)$ $= x^2+(m+n)x+mn$ $= x^2+mx+nx+mn$	（下面這一段的解釋介紹，老師們可以自行斟酌是否要演示給學生） When a=1, the general form of a trinomial is like x^2+bx+c We assume that after factoring, the general form would be like x^2+bx+c $=(x+m)(x+m)$ (m,and n are whole numbers) Expand the final expression, we get x^2+bx+c $=(x+m)(x+m)$ $= x^2+(m+n)x+mn$ $= x^2+mx+nx+mn$ That means we simply split the x term bx to mx plus nx, and grouping will be done easily. X squared plus b times x plus c equals x plus m times x plus n and equals x squared plus m times x plus n times x plus m times n

We first need to find out m and n by factoring the constant term 2. $2=1\cdot 2=(-1)(-2)$ Then m=1, n=2 or m=-1, n=-2	2 equals 1 times 2 Equals negative 1 times negative 2
But m+n needs to be 3 $m+n=1+2$ or $=(-1)+(-2)$ $=3$ or -3 So m=1 and n=2 meet the requirement of the x term 3x.	
Rewrite the trinomial x^2+3x+2 $= x^2+x+2x+2$	Split the middle term 3x to x plus 2x
Then *GROUPING* and factoring out the GCF in each group $=(x^2+x)+(2x+2)$ $=x(x+1)+2(x+1)$ $\frac{x^2}{x}$ $\frac{x}{x}$ $\frac{2x}{2}$ $\frac{2}{2}$	After grouping, factor out the common factor x from the first parentheses and 2 from the second parentheses(using the GCF method) After factoring out the common factor x from the first parentheses, the remains in parentheses are x squared divided by x is x and x divided by x is 1 2x divided by 2 is x and 2 divided by 2 is 1
The last step would be x^2+3x+2 $= x^2+x+2x+2$ $=(x^2+x)+(2x+2)$ $=x\underline{(x+1)}+2\underline{(x+1)}$ $=(x+1)(x+2)$ #	These two terms have a common factor (x+1). Factor out the common factor (x+1), the remains in the parentheses are (x+2)

2. $x^2-5x-24$	
$-24=mn$ and need $m+n=-5$	Factor the constant term -24 first.

m	1	2	3	4	6	8	12	24
n	-24	-12	-8	-6	-4	-3	-2	-1

So we choose $m=3$ and $n=-8$	We need to have m and n which satisfy m plus n equals negative 5
Then the trinomial can be written as	
$x^2-5x-24$	
$= x^2+3x-8x-24$	
$=(x^2+3x)+(-8x-24)$	Grouping
$=(x\cdot x+3\cdot x)+[(-8)\cdot x+(-8)\cdot 3]$	Factor out the GCF
$=x(x+3)+(-8)(x+3)$	Factor out the GCF (x+3)
$=(x+3)(x-8)$ #	
3. $3x^2+9x-84$	3 x squared plus 3x minus 84
$3x^2+9x-84$	There is a GCF 3 in this trinomial, we will factor out the GCF first.
$=3(x^2+3x-28)$	Then we factor the constant term negative 28
$-28=1(-28)=2(-14)=4(--7)=7(-4)$ $=14(-1)=28(-1)$	
	We need m,n that $mn=-28$, $m+n=3$
$=3(x^2+7x-4x-28)$	So m equals 7 and n equals negatived 4
$=3[(x^2+7x)+(-4x-28)]$	Grouping and factoring GCF x from the first parentheses and negative 4 from the second parentheses.
$=3[x(x+7)-4(x+7)]$	Factoring the common term x plus 7,
$=3(x+7)(x-4)$ #	

Now we are going to factor the trinomials ax^2+bx+c when $a\neq1$. As we said before, we can factor out the negative GCF when a is negative. So we simply consider a>1.

這一段是寫給老師們參考的。利用 ax^2+bx+c 因式分解的結果 $mrx^2+(ms+nr)x+rs$ 反推如何拆開一次項係數進行分組因式分解。國外稱 ac 因式分解法。

We first introduce *ac METHOD.*

For any trinomial ax^2+bx+c, a>0

If ax^2+bx+c

$=(mx+n)(rx+s)$

Expand the expression

ax^2+bx+c

$=(mx+n)(rx+s)$

$=mrx^2+(ms+nr)x+rs$

We have

$ac=mnrs$ and $b=ms+nr$

we need to find out two numbers in which the product of these two numbers equals ac and the sum of these two numbers equals b

由結果可以看到，我們要找兩個數，相乘等於首項係數跟常數項的乘積，相加等於一次項係數。

Ex6:: Factoring the trinomials ax^2+bx+c when a>1	
1. $2x^2+5x+3$	2 times x squared plus 5 times x plus 3
$2\cdot3=6$	
$=1\cdot6=2\cdot3=(-1)(-6)=(-2)(-3)$	a times c is 2 times 3 is 6
And $2x+3x=5x$	and 2x plus 3x equals the middle term 5x
So $2x^2+5x+3$	So we split 5x to 2x+3x
$=2x^2+2x+3x+3$	Grouping and factoring the GCF
$=[(2x^2+2x)+(3x+3)]$	Factor the GCF (x+1)
$=[2x(x+)+3(x+1)]$	
$=(x+1)(2x+3)$ #	

2. $20x^2-6x-2$ $20x^2-6x-2$ $=2(10x^2-3x-1)$ $10(-1)=-10$ $=1(-10)=2(-5)=5(-2)=10(-1)$ $-3x=2x+(-5x)$ Then $20x^2-6x-2$ $=2(10x^2-3x-1)$ $=2(10x^2+2x-5x-1)$ $=2[(10x^2+2x)+(-5x-1)]$ $=2[2x(5x+1)+(-1)(5x+1)]$ $=2(5x+1)(2x-1)$ #	20 x squared minus 6x minus 1 We see all the coefficients are even numbers, we can first factor out the GCF 2. The middle term is negative 3x So we choose 2 and negative 5 such that negative 3x equals 2x plus negative 5x Grouping and factoring out each GCF Factor out (5x+1)

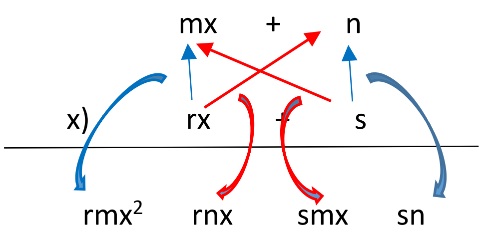
We are going to introduce the CROSS method we generally use in our class. Somewhat it's not quite the same as they use in some other countries.
最後，我們介紹常用的"十字交乘法"。不過，我們的十字交乘跟某些國外所說的 CROSS method 不同。
Let's explain our CROSS method in a general way.
Let

$ax^2+bx+c \qquad a>0$
$=(mx+n)(rx+s)$
$=mrx^2+(ms+nr)x+rs$

We have

$a=mn, b=ms+nr, c=rs$

The process of multiplying (mx+n) and (rx+s) is



When we do the polynomial factoring, we reverse the process of the multipling above. That is:

We need two numbers n and s of which their product is c

two numbers m and r of which their product is a

the sum of the two cross products sm and rn is b

For instance, if we want to factor the polynomial

$2x^2+5x+3$

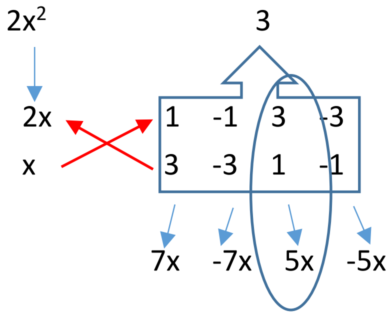
We first factor the leading coefficient $2=1\cdot 2=(-1)(-2)$. However, we don't consider negative leading coefficients, so we always have positive factors for the leading coefficient.

Then we factor the constant term $3=1\cdot 3=(-1)(-3)= 3\cdot 1=(-3)(-1)$

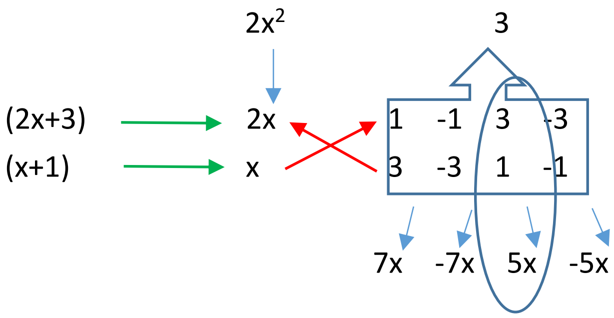
The reverse process would be like this:

Fix the x^2 terms first

The constant term would have many choices



When $3=3\cdot 1$, the result of the cross product is 5x. Please notify students that when we write down the answer, we need to do it from left to right, not from top to bottom.



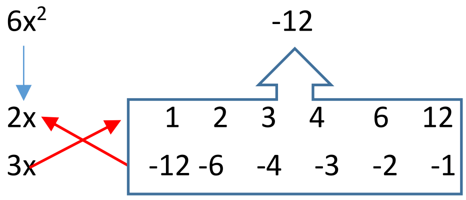
We get

$2x^2+5x+3=(2x+3)\ (x+1)$ #

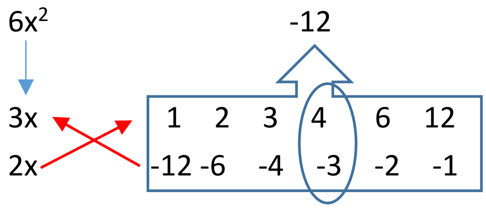
One more example:

Factoring the polynomial $6x^2-x-12$

We factor the leading term $6x^2$ and the constant term -12 together.



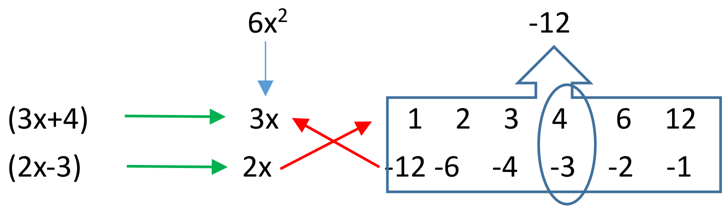
Or



$-9x+8x=-x$

We found that when $-12=3(-4)$, the cross product $-9x+8x=-x$ is what we need.

So we can write down the factoring result right away.



$6x^2-x-12=(3x+4)(2x-3)$ #

Check:

Expand the expression $(3x+4)(2x-3)$, we get

$$(3x+4)(2x-3)=6x^2-9x+8x-12$$
$$=6x^2-x-12$$

Doing more practice will make you an expert!

Reference:

1.
https://davenport.libguides.com/math-skills-overview/polynomials/factoring-polynomials (DAVENPORT UNIVERSITY)

2.
https://terms.naer.edu.tw/search/?q=DEFINITION+OF+FACTORING+POLYN0MIALS&field=ti&op=AND&group=&num=10 (國家教育研究院雙語詞彙)

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