多項式的除法原理

Division Algorithm for Polynomials

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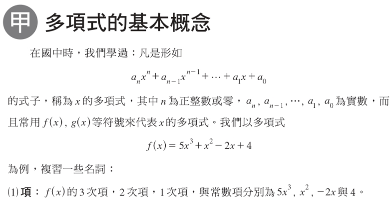

甲 多項式的基本概念

在國中時，我們學過：凡是形如

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

的式子，稱為 x 的多項式，其中 n 為正整數或零，$a_n, a_{n-1}, \cdots, a_1, a_0$ 為實數，而且常用 $f(x), g(x)$ 等符號來代表 x 的多項式。我們以多項式

$$f(x)=5x^3+x^2-2x+4$$

為例，複習一些名詞：

(1) **項**：$f(x)$ 的 3 次項，2 次項，1 次項，與常數項分別為 $5x^3, x^2, -2x$ 與 4。

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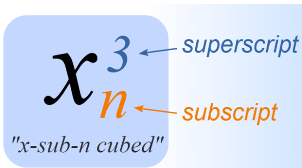
Vocabulary: Polynomial (多項式), Nonnegative Integer (非負整數), Subscript (下標), Superscript (上標), Ellipsis (刪節號 informally as dot dot dot).

Translations:

In junior high school, we have learned that the expressions like $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ (a sub n times x to the power of n plus a sub n minus 1 times x to the power of n minus 1 plus dot dot dot a sub 1 times x plus a sub 0.) are called polynomials in x. Where n is a positive integer or a zero (nonnegative integer) and $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers. A polynomial in x is generally represented as f(x) or g(x).



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Illustrations

Vocabulary: Term (項), Namely (亦即), Coefficient (係數), Constant Polynomial (常數多項式), Zero Polynomial (零多項式) , Exponent (指數), Variable (變數), Degree of the Polynomial (多項式的次數).

In the polynomial x^2+2x, the expressions x^2 and $2x$ are called the terms of the polynomials. Similarly, the polynomial $3y^2+5y+7$ has three terms, namely, $3y^2$, $5y$ and 7.

在多項式 x^2+2x 中，x^2 及 $2x$ 稱為多項式的項。同樣的，多項式 $3y^2+5y+7$ 有三項，亦即 $3y^2$ 、$5y$ 及 7。

Each term of a polynomial has a coefficient. So, in $-x^3+4x^2+7x-2$, the coefficient of x^3 is

-1, the coefficient of x^2 is 4, the coefficient of x is 7 and -2 is the coefficient of x^0.

多項式的每項都有係數。在多項式$-x^3+4x^2+7x-2$裡，三次項x^3的係數是-1，二次項x^2的係數是 4，一次項 x 的係數是 2，最後常數項x^0的係數是-2。

2 is also a polynomial. 2, -5, 7, etc. are examples of constant polynomials. The constant polynomial 0 is called the zero polynomial.

2 也是多項式，2、-5、7 都稱為常數多項式。而 0 則稱為零多項式。

In the polynomial $p(x)=3x^7-4x^6+x-9$, the term with highest power of x is $3x^7$. The exponent of x in this term is 7. We call the highest power of the variable in a polynomial as the degree of the polynomial. Particualrly, **the degree of a non-zero constant polynomial is zero**.

多項式$p(x)=3x^7-4x^6+x-9$中，x 的最高次項為$3x^7$，其指數為 7。我們稱最高次數為多項式的次數。特別地，非零的常數多項式稱為零次多項式。

Vocabulary: Operation (運算), Monomial (單項式), Classify (分類), According to (根據), Binomial (二項式), Trinomial (三項式)

A polynomial is defined as an expression which is composed of variables, constants and exponents that are combined using mathematical operations such as addition, subtraction, multiplication and division (No division operation by a variable).

多項式由變數、常數、指數組成，以四則運算如加減乘除（惟除法不能使用在變數相除）結合。

The expressions with one term are called monomials and the expressions with more than one term are called polynomials.

只有一項的式子稱為單項式，超過一項的式子則稱多項式。

Polynomials are classified according to the number of terms; for instance, a binomial has two terms and a trinomial has three terms.

多項式可以依據項數分類，舉例來說，二項式只有二項而三項式只有三項。

Material	Note
(二)乘法 兩個單項的多項式之乘積為係數相乘而次方相加，例如 $(2x^5)(3x^4)=(2\times 3)x^{5+4}=6x^9$。 而兩多項式相乘，可以利用乘法對加法的分配律來計算。	**Vocabulary:** Numerical (數值), Distributive Property (分配律), Descending Order (降冪). **Translations:** 1. To multiply two monomials, multiply their numerical coefficients and find the product of the variable factors according to the laws of exponents. (兩個單項的多項式之乘積為係數相乘而次方相加。) 2. For example $(2x^5)(3x^4)=(2\times 3)x^{5+4}=6x^9$ (2 times x to the fifth, times 3 times x to the forth equals 2 times 3 times x to the power of 5 plus 4, which equals 6 times x to the ninth.) (例如，$(2x^5)(3x^4)=(2\times 3)x^{5+4}=6x^9$ 。) 3. To multiply two polynomials, use the distributive property of multiplication over addition. (兩多項式相乘，可以利用乘法對加法的分配律來計算。) **Note:** The answer is often written in descending order of the exponents, ending with the constant term.

Summaries

Vocabulary: Standard Form (標準式), Parentheses (圓括號：()).

1. A polynomial in x is said to be in standard form if :

    I. All parentheses are removed.
    II. Like terms are combined.
    III. The terms are arranged in order of descending powers of x.

多項式的一般式應符合：

    I. 所有括號都應該被消除。
    II. 相同項應合併。

III. 以降冪排列。

2. The degree of polynomial in x is the greatest power of x.

多項式的次數為 x 最高次方。

Material	Note
(三)除法 我們仿照整數的除法，進行多項式的除法運算，以 $2x^3+3x^2+8x+5$ 除以 x^2+x+3 為例（前者稱為**被除式**，後者稱為**除式**），計算如下： $\begin{array}{r\|rrrr} & 2x & +1 & & \\ \hline x^2+x+3 & 2x^3 & +3x^2 & +8x & +5 \\ & 2x^3 & +2x^2 & +6x & \\ \hline & & x^2 & +2x & +5 \\ & & x^2 & +x & +3 \\ \hline & & & x & +2 \end{array}$ 因為 $2x^3 \div x^2 = 2x$，所以放 $2x$。 因為 $x^2 \div x^2 = 1$，所以放 1。 次數小於除式的次數，停止計算。 得**商式**為 $2x+1$，**餘式**為 $x+2$。這種演算法稱為多項式的**長除法**。	**Vocabulary:** Dividend (被除式), Divisor (除式), Quotient (商式), Remainder (餘式). **Translations:** The process of dividing a polynomial is essentially the same as dividing an integer. To show how it works, let's divide $2x^3+3x^2+8x+5$ by x^2+x+3. (The former is called the dividend. The latter is called the divisor.) Put $2x$ on the top, because $2x^3 \div x^2 = 2x$. Put $+1$ on the top, because $x^2 \div x^2 = 1$. Stop when the degree of remainder is less than divisor. We say the quotient is $2x+1$, remainder is $x+2$. This result is called the Division Algorithm for polynomials.

Illustrations

Because $x^2 \times 2x = 2x^3$. The goal is to match $2x^3$ and we have $\left(x^2+x+3\right)\times 2x = 2x^3+2x^3+6x$. We put this part below the dividend.

$$\begin{array}{r|rrrr} & 2x & & & \\ \hline x^2+x+3 & 2x^3 & 3x^2 & +8x & +5 \\ & 2x^3 & 2x^2 & +6x & \end{array}$$

Finally, we subtract each element. We bring down the 5.

$$\begin{array}{r|rrrr} & 2x & & & \\ \hline x^2+x+3 & 2x^3 & 3x^2 & +8x & +5 \\ & 2x^3 & 2x^2 & +6x & \\ \hline & & x^2 & +2x & +5 \end{array}$$

How many times dose x^2+x+3 go into x^2? 1 times. Our goal at each step is to get the same exponent and the same coefficient as the term with the highest power. We put the $+1$ up top and $(+1)\times(x^2+x+3)=x^2+x+3$ on the bottom.

	$2x$	$+1$		
x^2+x+3	$2x^3$	$3x^2$	$+8x$	$+5$
	$2x^3$	$2x^2$	$+6x$	
		x^2	$+2x$	$+5$
		x^2	$+x$	$+3$

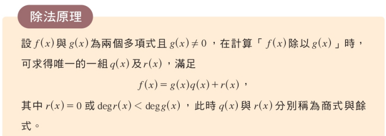
As the second step that we subtract each term, we get $x+2$ at the end.

	$2x$	$+1$		
x^2+x+3	$2x^3$	$3x^2$	$+8x$	$+5$
	$2x^3$	$2x^2$	$+6x$	
		x^2	$+2x$	$+5$
		x^2	$+x$	$+3$
			x	$+2$

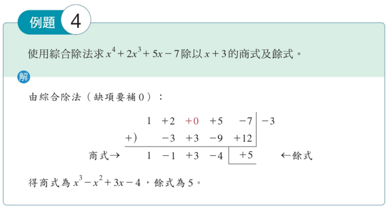
The result is as follows.

$$2x^3+3x^2+8x+5=(x^2+x+3)(2x+1)+(x+2)$$

$$\text{Dividend}=\text{Divisor}\times\text{Quotient}+\text{Remainder}$$

Material	Note
 除法原理 設 $f(x)$ 與 $g(x)$ 為兩個多項式且 $g(x)\neq 0$，在計算「$f(x)$ 除以 $g(x)$」時，可求得唯一的一組 $q(x)$ 及 $r(x)$，滿足 $f(x)=g(x)q(x)+r(x)$， 其中 $r(x)=0$ 或 $\deg r(x)<\deg g(x)$，此時 $q(x)$ 與 $r(x)$ 分別稱為商式與餘式。	**Vocabulary**: Divide Evenly Into/Be Divisible By (整除), Factor (因數), Multiple (倍數). **Translations**: **The Division Algorithm** If $f(x)$ and $g(x)$ are polynomials such that $g(x)\neq 0$, when doing "divide $f(x)$ by $g(x)$" there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x)=g(x)q(x)+r(x)$ Dividend = Divisor × Quotient + Remainder Where $r(x)=0$ or the degree of $r(x)$ is less

	than the degree of $g(x)$. **Note:** 1. If the remainder $r(x)$ is zero, $g(x)$ divides evenly into $f(x)$. (如果餘式 $r(x)$ 等於 0，$g(x)$ 被 $f(x)$ 整除。) 2. $r(x)$ is a factor of $f(x)$; $f(x)$ is a multiple of $r(x)$. ($r(x)$ 是 $f(x)$ 的因式，$f(x)$ 是 $r(x)$ 的倍式。)

Material	Note
 例題 4 使用綜合除法求 x^4+2x^3+5x-7 除以 $x+3$ 的商式及餘式。 解 由綜合除法（缺項要補0）： $\begin{array}{r r r r r r \| r} & 1 & +2 & +0 & +5 & -7 & -3 \\ +) & & -3 & +3 & -9 & +12 & \\ \hline \text{商式}\rightarrow & 1 & -1 & +3 & -4 & +5 & \leftarrow\text{餘式} \end{array}$ 得商式為 x^3-x^2+3x-4，餘式為 5。	**Vocabulary**: Synthetic Division (綜合除法), Procedure (程序).

Illustrations

Let's look at the procedure for dividing polynomials by synthetic divisions.

Consider this problem: Divide x^4+2x^3+5x-7 by $x+3$

Step 1: List the coefficients of the dividend in a horizontal row. Note that the dividend does not contain an x^2-term. Remember to insert the coefficient $+0$ into the dividend.

$$\begin{array}{r r r r r r|} & 1 & +2 & +0 & +5 & -7 \\ +) & & & & & \\ \hline \end{array}$$

Step 2: To the right of the number, list the opposite of the constant in the divisor. The constant in the divisor $x+3$ is 3. So, we take the opposite of that number which is -3.

$$\begin{array}{r r r r r r|r} & 1 & +2 & +0 & +5 & -7 & -3 \\ +) & & & & & & \\ \hline \end{array}$$

Step 3: Drop the leading coefficient, 1, below the horizontal line.

$$\begin{array}{r r r r r r|r} & 1 & +2 & +0 & +5 & -7 & -3 \\ +) & & & & & & \\ \hline & 1 & & & & & \end{array}$$

Step 4: Multiple the number -3 by 1 and write the result $\left(1\cdot(-3)=-3\right)$ beneath the coefficient $(+2)$ and above the horizontal line.

$$\begin{array}{rrrrrr|r} & 1 & +2 & +0 & +5 & -7 & -3 \\ +) & & -3 & & & & \\ \hline & 1 & & & & & \end{array}$$

Step 5: Add -3 and $+2$ $(-3+2=-1)$ and write the sum in the same column as those values, below the horizontal line.

$$\begin{array}{rrrrrr|r} & 1 & +2 & +0 & +5 & -7 & -3 \\ +) & & -3 & & & & \\ \hline & 1 & -1 & & & & \end{array}$$

Repeat the **step 4** and **step 5**.

$$\begin{array}{rrrrrr|r} & 1 & +2 & +0 & +5 & -7 & -3 \\ +) & & -3 & +3 & -9 & +12 & \\ \hline & 1 & -1 & +3 & -4 & & \end{array}$$

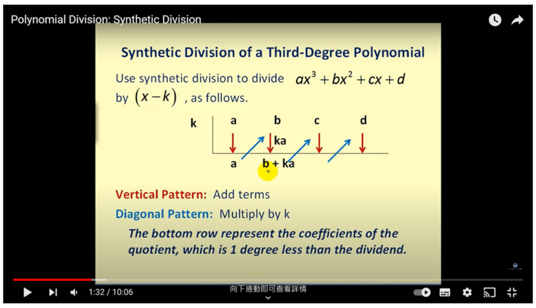
Combine the numbers in the rightmost column $(-7+12=5)$ and write the result in the same column, blow the horizontal line.

$$\begin{array}{rrrrrr|r} & 1 & +2 & +0 & +5 & -7 & -3 \\ +) & & -3 & +3 & -9 & +12 & \\ \hline & 1 & -1 & +3 & -4 & +5 & \end{array}$$

The number below the horizontal line represents the coefficients of the quotient and the remainder. Note that the degree of the quotient is always one less than the degree of the dividend, so this quotient has degree three: x^3-x^2+3x-4. The rightmost number the horizontal line represents the remainder, which is 5 for this problem.

$$\begin{array}{rrrrrr|r} & 1 & +2 & +0 & +5 & -7 & -3 \\ +) & & -3 & +3 & -9 & +12 & \\ \hline & \boxed{1} & \boxed{-1} & \boxed{+3} & \boxed{-4} & +5 & \end{array}$$

Therefore, $x^4+2x^3+5x-7=(x+3)\left(x^3-x^2+3x-4\right)+5$.



Vocabulary: Column (直行), Row (橫列), Pattern (模式), Diagonal (對角線), Preference (偏愛).

Video: Mathispower4U - *Polynomial Division: Synthetic Division.*

https://www.youtube.com/watch?v=5dBAdzl2Mns

丙 餘式定理

想求多項式 $f(x)=x^4-5x+3$ 除以 $x-2$ 的商式與餘式，可以使用綜合除法：

$$\begin{array}{rrrrr|l} 1 & +0 & +0 & -5 & +3 & 2 \\ +) & +2 & +4 & +8 & +6 & \\ \hline 1 & +2 & +4 & +3 & +9 & \end{array}$$

得商式為 x^3+2x^2+4x+3，餘式為 9。但是如果只要求餘式，而不求商式時，還有另一種較簡捷的方法：

Translations:

Finding the quotient and the remainder, we can use synthetic division. We will have the quotient is x^3+2x^2+4x+3, and the remainder is 9. However, if it is only the remainder which is needed, there is another easier and quicker method to use.

根據除法原理，設 $f(x)$ 除以 $x-2$ 的商式為 $q(x)$，餘式為常數 r，且滿足

$$f(x)=(x-2)q(x)+r \text{。}$$

將 $x=2$ 代入上式，得

$$f(2)=(2-2)q(2)+r=r \text{，}$$

因此，餘式 $r=f(2)=2^4-5\times2+3=9$。

一般而言，仿照這簡捷的方法，可以推得**餘式定理**。

餘式定理

多項式 $f(x)$ 除以一次式 $ax-b$ 的餘式等於 $f\left(\frac{b}{a}\right)$。

By the division algorithm,

Dividend = (Divisor × Quotient) + Remainder

Set $f(x)$ is divided by $x-2$, the quotient is $q(x)$ and the remainder is r, which satisfy the equation:

$$f(x)=(x-2)q(x)+r.$$

Substitute $x=2$ into the former equation, yielding

$$f(2)=(2-2)q(2)+r=r$$

Hence, the remainder is $r=f(2)=2^4-5\times2+3=9$

Remainder Theorem

If a polynomial $f(x)$ is divided by the binomial $ax-b$, the remainder is $f\left(\frac{b}{a}\right)$.

Supplementary Materials I

If $x-1$ and $x+1$ are both factors of the polynomial $ax^4+bx^3-3x^2+5x$ and "a" and "b" are constants, what is the value of a?

Solution

Using the remainder theorem, we can set up a system of equations. When the polynomial is divided by $x-1$ or $x+1$, the remainder is 0, which means that if we let $p(x)$ denote the polynomial, $p(1)=0$ and $p(-1)=0$.

$$\begin{cases} a(1)^4+b(1)^3-3(1)^2+5(1)=0 \\ a(-1)^4+b(-1)^3-3(-1)^2+5(-1)=0 \end{cases} \Rightarrow \begin{cases} a+b-3+5=0 \\ a-b-3-5=0 \end{cases}$$

Adding the equations together, $2a-6=0 \Rightarrow a=3$

Supplementary Materials II

If the expression $\dfrac{5x^3-2x^2-14x+1}{x-2}$ is written in the form $5x^2-2x^2-14x+1+\dfrac{B}{x-2}$, where B is a constant. Find the value of B.

Solution

Based on where it is, B represents the remainder of the division. Using the remainder theorem, we can find B by plugging $x=2$ into the polynomial $5x^3-2x^2-14x+1$.

$$5(2)^3-2(2)^2-14(2)+1=5$$

We can write the result of this division as $5x^2-2x^2-14x+1+\dfrac{5}{x-2}$, from which $B=5$.

References

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