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Parabolic Second-Order Directional Differentiability in the Hadamard Sense of the Vector-Valued Functions Associated with Circular Cones

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Abstract In this paper, we study the parabolic second-order directional derivative in the Hadamard sense of a vector-valued function associated with circular cone. The vector-valued function comes from applying a given real-valued function to the spectral decomposition associated with circular cone. In particular, we present the exact formula of second-order tangent set of circular cone by using the parabolic second-order directional derivative of projection operator. In addition, we also deal with the relationship of second-order differentiability between the vector-valued function and the given real-valued function. The results in this paper build fundamental bricks to the characterizations of second-order necessary and sufficient conditions for circular cone optimization problems.

Keywords Parabolic second-order derivative · Circular cone · Second-order tangent set

Mathematics Subject Classification 90C30 · 49J52 · 46G05

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1 Introduction

The parabolic second-order derivatives were originally introduced by Ben-Tal and Zowe in [1,2]; please refer to [3] for more details about properties of parabolic second-order derivatives. Usually the parabolic second-order derivatives can be employed to characterize the optimality conditions for various optimization problems; see [1,4–7] and references therein. The so-called generalized parabolic second-order derivatives are studied in [4,5,8], whereas the parabolic second-order derivatives for certain types of functions are investigated in [5,8–10]. In this paper, we mainly focus on the parabolic second-order directional derivative in the Hadamard sense for the vector-valued functions associated with circular cones. This vector-valued function, called circular cone function, comes from applying a given real-valued function to the spectral decomposition associated with circular cone.

For the circular cone function, by using the basic tools of nonsmooth analysis, various properties such as directional derivative, differentiability, B-subdifferentiability, semismoothness, and positive homogeneity have been studied in [11,12]. The aforementioned results can be regarded as the first-order type of differentiability analysis. Here, we further discuss the second-order type of differentiability analysis for the circular cone function. As mentioned above, the concept of parabolic second-order directional differentiability plays an important role in second-order necessary and sufficient conditions. Recently, there was an investigation on the parabolic second-order directional derivative of singular values of matrices and symmetric matrix-valued functions in [10]. Inspired by this work, we study the parabolic second-order directional derivative for the vector-valued circular cone function. The relationship of parabolic second-order directional derivative between the vector-valued circular cone function and the given real-valued function is established, in which we do not require that the real-valued function is second-order differentiable. This allows us to apply our result to more general nonsmooth functions. For example, we obtain the exact formula of second-order tangent set by using the parabolic second-order directional differentiability of projection operator associated with circular cone, which is corresponding to the nonsmooth max-type function. In addition, we study the relationship of second-order differentiability between circular cone function and the given real-valued function. It is surprising that, not like the first-order differentiability, the relationship in the second-order differentiability case really depends on the angle. This further shows the essential role played by the angle in the circular cone setting.

2 Preliminaries

The n-dimensional circular cone is defined as

$$\mathcal{L}_\theta := \left\{ x = (x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \,:\, \cos\theta \|x\| \le x_1 \right\},$$

which is a nonsymmetric cone in the standard inner product. In our previous works [12–15], we have explored some important features about circular cone, such as characterizing its tangent cone, normal cone, and second-order regularity. In particular, the spectral decomposition associated with $\mathcal{L}_\theta$ was discovered, i.e., for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, one has

$$x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}, \tag{1}$$

where

$$\lambda_1(x) := x_1 - \|x_2\| \cot\theta, \quad \lambda_2(x) := x_1 + \|x_2\| \tan\theta$$

and

$$u_x^{(1)} := \frac{1}{1+\cot^2\theta}\begin{bmatrix}1 & 0 \\ 0 & \cot\theta \cdot I\end{bmatrix}\begin{bmatrix}1 \\ -\bar{x}_2\end{bmatrix}, \quad u_x^{(2)} := \frac{1}{1+\tan^2\theta}\begin{bmatrix}1 & 0 \\ 0 & \tan\theta \cdot I\end{bmatrix}\begin{bmatrix}1 \\ \bar{x}_2\end{bmatrix}$$

with $\bar{x}_2 := x_2/\|x_2\|$ if $x_2 \neq 0$, and $\bar{x}_2$ being any vector $w \in \mathbb{R}^{n-1}$ satisfying $\|w\| = 1$ if $x_2 = 0$. With this spectral decomposition (1), we can define a vector-valued function associated with circular cone as below. More specifically, for a given real-valued function $f : \mathbb{R} \to \mathbb{R}$, the circular cone function $f^{\mathcal{L}_\theta} : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$f^{\mathcal{L}_\theta}(x) := f\left(\lambda_1(x)\right) u_x^{(1)} + f\left(\lambda_2(x)\right) u_x^{(2)}.$$

Let X, Y be normed vector spaces and consider $x, d, w \in X$. Assume that $\psi : X \to Y$ is directionally differentiable. The function ψ is said to be parabolical second-order directionally differentiable in the Hadamard sense at x, if ψ is directionally differentiable at x and for any $d, w \in X$ the following limit exists:

$$\psi^{''}(x; d, w) := \lim_{\substack{t \downarrow 0 \\ w' \to w}} \frac{\psi\left(x + td + \frac{1}{2}t^2 w'\right) - \psi(x) - t\psi^{'}(x; d)}{\frac{1}{2}t^2}. \tag{2}$$

To the contrary, the function ψ is said to be parabolical second-order directionally differentiable at x, if w' is fixed to be w in (2). Generally speaking, the concept of parabolical second-order directional differentiability in the Hadamard sense is stronger than that of parabolical second-order directional differentiability. However, when ψ is locally Lipschitz at x, these two concepts coincide. It is known that if ψ is parabolical second-order directional differentiability in the Hadamard sense at x along d, w, then

$$\psi\left(x + td + \frac{1}{2}t^2 w + o\left(t^2\right)\right) = \psi(x) + t\psi'(x; d) + \frac{1}{2}t^2\psi^{''}(x; d, w) + o\left(t^2\right). \tag{3}$$

At the first glance on (3), the concept of parabolical second-order directional differentiability in the Hadamard sense is likely to say that ψ has a second-order Taylor expansion along some directions. In fact, for the expression (3), the main difference lies on the appearance of w. Why do we need such expansion (3), We say a few words about it. For standard nonlinear programming, corresponding to the nonnegative orthant, a polyhedral is targeted. Hence, considering the way $x + td$, a radial line, is enough. However, for optimization problems involved the circular cones, second-order cones, or semidefinite matrices cones, they are all nonpolyhedral cones. Thus, we need to describe the curves thereon. To this end, the curved approach $x + td + \frac{1}{2}t^2 w$ is needed, which, to some extent, reflects the nonpolyhedral properties of nonpolyhedral cones. This point can be seen in Sect. 3, where the parabolic second-order directional derivative is used to study the second-order tangent sets of circular cones. The exact expression of second-order tangent set is important for describing the second-order necessary and sufficient conditions for conic programming, since its support function is appeared in the second-order necessary and sufficient conditions for conic programming; see [16] for more information.

3 Second-Order Directional Derivative

For subsequent analysis, we will frequently use the second-order derivative of $\bar{x} := \frac{x}{\|x\|}$ at $x \neq 0$. To this end, we present the second-order derivative of $\bar{x}$ in below theorem. For convenience of notation, we also denote $\Phi(x) := \bar{x}$ for $x \neq 0$, which does not cause any confusion from the context.

Theorem 3.1 *Let a function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be given as $\Phi(x) := \frac{x}{\|x\|}$ for $x \neq 0$. Then, the function Φ is second-order continuous differentiable at $x \neq 0$ with*

$$\mathcal{J}\Phi(x) = \frac{I - \bar{x}\bar{x}^T}{\|x\|}$$

and

$$\mathcal{J}^2\Phi(x)(w, w) = -2\left(\frac{\bar{x}^T w}{\|x\|^2}\right) w + w^T \left(\frac{3\bar{x}\bar{x}^T - I}{\|x\|^3}\right) wx, \quad \forall\, w \in \mathbb{R}^n.$$

Proof It is clear that Φ is second-order continuous differentiable because of $x \neq 0$. The Jacobian of Φ at $x \neq 0$ is obtained from direct calculation. To obtain the second-order derivative, for any given $a \in \mathbb{R}^n$, we define $\psi : \mathbb{R}^n \to \mathbb{R}$ as

$$\psi(x) := \Phi(x)^T a = \frac{x^T a}{\|x\|}.$$

We also denote $h(x) := a^T x$ and $g(x) := 1/\|x\|$ so that $\psi(x) = h(x)g(x)$. Since $x \neq 0$, it is clear that g and h are twice continuously differentiable at x with $\mathcal{J}h(x) = a$, $\mathcal{J}^2 h(x) = O$, and

$$\mathcal{J}g(x) = -\frac{\bar{x}}{\|x\|^2}, \quad \mathcal{J}^2 g(x) = -\frac{\left(I - \bar{x}\bar{x}^T\right) - 2\bar{x}\bar{x}^T}{\|x\|^3} = \frac{3\bar{x}\bar{x}^T - I}{\|x\|^3}.$$

Hence, from the chain rule, we have $\mathcal{J}\psi(x) = g(x)\mathcal{J}h(x) + h(x)\mathcal{J}g(x)$ and

$$\mathcal{J}^2\psi(x) = \mathcal{J}g(x)^T \mathcal{J}h(x) + h(x)\mathcal{J}^2 g(x) + g(x)\mathcal{J}^2 h(x) + \mathcal{J}h(x)^T \mathcal{J}g(x),$$

which implies

$$\begin{aligned}\mathcal{J}^2\psi(x)(w,w) &= 2\mathcal{J}g(x)(w)\mathcal{J}h(x)(w) + h(x)\mathcal{J}^2 g(x)(w,w)\\ &\quad + g(x)\mathcal{J}^2 h(x)(w,w)\\ &= 2\mathcal{J}g(x)(w)\mathcal{J}h(x)(w) + h(x)\mathcal{J}^2 g(x)(w,w)\\ &= a^T\left[-2\frac{\bar{x}^T w}{\|x\|^2}w + w^T\left(\frac{3\bar{x}\bar{x}^T - I}{\|x\|^3}\right)wx\right]. \end{aligned} \tag{4}$$

On the other hand, we see that $\mathcal{J}^2\psi(x)(w,w) = a^T\mathcal{J}^2\Phi(x)(w,w)$. Since $a \in \mathbb{R}^n$ is arbitrary, this together with (4) yields

$$\mathcal{J}^2\Phi(x)(w,w) = -2\frac{\bar{x}^T w}{\|x\|^2}w + w^T\left(\frac{3\bar{x}\bar{x}^T - I}{\|x\|^3}\right)wx,$$

which is the desired result. □

Next, we characterize the parabolic second-order directional derivative of the spectral values $\lambda_i(x)$ for $i = 1, 2$.

Theorem 3.2 *Let $x \in \mathbb{R}^n$ with spectral decomposition $x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}$ given as in (1). Then, the parabolic second-order directional differentiability in the Hadamard sense of $\lambda_i(x)$ for $i = 1, 2$ reduces to the parabolic second-order directional differentiability. Moreover, given $d, w \in \mathbb{R}^n$, we have*

$$\lambda_1''(x;d,w) = \begin{cases} w_1 - \left(\bar{x}_2^T w_2 + \frac{\|d_2\|^2 - (\bar{x}_2^T d_2)^2}{\|x_2\|}\right)\cot\theta, & \text{if } x_2 \neq 0,\\ w_1 - \bar{d}_2^T w_2 \cot\theta, & \text{if } x_2 = 0,\ d_2 \neq 0,\\ w_1 - \|w_2\|\cot\theta, & \text{if } x_2 = 0,\ d_2 = 0, \end{cases}$$

and

$$\lambda_2''(x;d,w) = \begin{cases} w_1 + \left(\bar{x}_2^T w_2 + \frac{\|d_2\|^2 - (\bar{x}_2^T d_2)^2}{\|x_2\|}\right)\tan\theta, & \text{if } x_2 \neq 0,\\ w_1 + \bar{d}_2^T w_2 \tan\theta, & \text{if } x_2 = 0,\ d_2 \neq 0,\\ w_1 + \|w_2\|\tan\theta, & \text{if } x_2 = 0,\ d_2 = 0. \end{cases}$$

Proof Note that $\lambda_i(x)$ for $i = 1, 2$ is Lipschitz continuous [12]; hence, the parabolic second-order directional differentiability in the Hadamard sense of $\lambda_i(x)$ for $i = 1, 2$ reduces to the parabolic second-order directional differentiability.

To compute the parabolic second-order directional derivative, we consider the following three cases.

(i) If $x_2 \neq 0$, then $x + td + \frac{1}{2}t^2 w = (x_1 + td_1 + \frac{1}{2}t^2 w_1, x_2 + td_2 + \frac{1}{2}t^2 w_2)$. Note that $\lambda_1'(x;d) = d_1 - \bar{x}_2^T d_2 \cot\theta$ and

$$\|x_2 + td_2 + \frac{1}{2}t^2 w_2\| = \|x_2\| + t\bar{x}_2^T d_2 + \frac{1}{2}t^2\left(\bar{x}_2^T w_2 + \frac{\|d_2\|^2 - \left(\bar{x}_2^T d_2\right)^2}{\|x_2\|}\right) + o\left(t^2\right).$$

Thus, we obtain

$$\frac{\lambda_1\left(x + td + \frac{1}{2}t^2 w\right) - \lambda_1(x) - t\lambda_1'(x; d)}{\frac{1}{2}t^2} \to w_1 - \left(\bar{x}_2^T w_2 + \frac{\|d_2\|^2 - \left(\bar{x}_2^T d_2\right)^2}{\|x_2\|}\right)\cot\theta.$$

(ii) If $x_2 = 0$ and $d_2 \neq 0$, then $x + td + \frac{1}{2}t^2 w = (x_1 + td_1 + \frac{1}{2}t^2 w_1, td_2 + \frac{1}{2}t^2 w_2)$ and $\lambda_1'(x; d) = d_1 - \|d_2\|\cot\theta$. Hence,

$$\frac{\lambda_1\left(x + td + \frac{1}{2}t^2 w\right) - \lambda_1(x) - t\lambda_1'(x; d)}{\frac{1}{2}t^2} \to w_1 - \bar{d}_2^T w_2 \cot\theta.$$

(iii) If $x_2 = 0$ and $d_2 = 0$, then $x + td + \frac{1}{2}t^2 w = (x_1 + td_1 + \frac{1}{2}t^2 w_1, \frac{1}{2}t^2 w_2)$. Thus, $\lambda_1'(x; d) = d_1$ and

$$\frac{\lambda_1\left(x + td + \frac{1}{2}t^2 w\right) - \lambda_1(x) - t\lambda_1'(x; d)}{\frac{1}{2}t^2} \to w_1 - \|w_2\|\cot\theta.$$

From all the above, the formula of $\lambda_1''(x; d, w)$ is proved. Similar arguments can be applied to obtain the formula of $\lambda_2''(x; d, w)$. □

The relationship of parabolic second-order directional differentiability in the Hadamard sense between $f^{\mathcal{L}_\theta}$ and f is given below.

Theorem 3.3 *Suppose that $f : \mathbb{R} \to \mathbb{R}$. Then, $f^{\mathcal{L}_\theta}$ is parabolic second-order directionally differentiable at x in the Hadamard sense if and only if f is parabolic second-order directionally differentiable at $\lambda_i(x)$ in the Hadamard sense for $i = 1, 2$. Moreover,*

(a) if $x_2 = 0$ and $d_2 = 0$, then

$$\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) = f''(x_1; d_1, w_1 - \|w_2\|\cot\theta)\, u_w^{(1)} + f''(x_1; d_1, w_1 + \|w_2\|\tan\theta)\, u_w^{(2)};$$

(b) if $x_2 = 0$ and $d_2 \neq 0$, then

$$\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) = f''\left(x_1; d_1 - \|d_2\|\cot\theta, w_1 - \bar{d}_2^T w_2 \cot\theta\right) u_d^{(1)}$$

$$+f^{''}\left(x_1; d_1 + \|d_2\| \tan\theta, w_1 + \bar{d}_2^T w_2 \tan\theta\right) u_d^{(2)}$$
$$+\frac{1}{\tan\theta + \cot\theta}\left(f'(x_1; d_1 + \|d_2\| \tan\theta) - f'(x_1; d_1 - \|d_2\| \cot\theta)\right)\mathcal{J}\widetilde{\Phi}(d)w;$$

(c) if $x_2 \neq 0$, then

$$\left(f^{\mathcal{L}_\theta}\right)^{''}(x; d, w)$$
$$= f^{''}\left(x_1 - \|x_2\| \cot\theta; d_1 - \bar{x}_2^T d_2 \cot\theta, w_1 - \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2)d_2\right]\cot\theta\right) u_x^{(1)}$$
$$+f^{''}\left(x_1 + \|x_2\| \tan\theta; d_1 + \bar{x}_2^T d_2 \tan\theta, w_1 + \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2)d_2\right]\tan\theta\right) u_x^{(2)}$$
$$+\frac{2}{\cot\theta + \tan\theta}\Gamma_1 \mathcal{J}\widetilde{\Phi}(x)d + \frac{1}{\cot\theta + \tan\theta}\Gamma_2\left(\mathcal{J}\widetilde{\Phi}(x)w + \mathcal{J}^2\widetilde{\Phi}(x)(d, d)\right),$$

where

$$\Gamma_1 := f'\left(x_1 + \|x_2\| \tan\theta; d_1 + \bar{x}_2^T d_2 \tan\theta\right)$$
$$-f'\left(x_1 - \|x_2\| \cot\theta; d_1 - \bar{x}_2^T d_2 \cot\theta\right)$$
$$\Gamma_2 := f\left(x_1 + \|x_2\| \tan\theta\right) - f\left(x_1 - \|x_2\| \cot\theta\right)$$

and $\widetilde{\Phi}(x) := (1, \Phi(x_2))^T$ for all $x \in \mathbb{R}^n$ with $x_2 \neq 0$.

Proof "$\Leftarrow$" Suppose that f is parabolic second-order directionally differentiable at $\lambda_i(x)$ for $i = 1, 2$ in the Hadamard sense. Given $d, w \in \mathbb{R}^n$ and $w' \to w$, we consider the following four cases. First we denote $z := x + td + \frac{1}{2}t^2 w'$.
Case 1: For $x_2 = 0, d_2 = 0$, and $w_2 = 0$, we have $f^{\mathcal{L}_\theta}(x) = \left(f(x_1), 0\right) = f(x_1)u_z^{(1)} + f(x_1)u_z^{(2)}$ and

$$\left(f^{\mathcal{L}_\theta}\right)'(x; d) = \left(f'(x_1; d_1), 0\right) = f'(x_1; d_1)\, u_z^{(1)} + f'(x_1; d_1)\, u_z^{(2)}.$$

Note that $u_z^{(i)} \to u_\xi^{(i)}$ as $i = 1, 2$ for some $\xi \in \{(1, w) : \|w\| = 1\}$. Thus, we conclude that

$$\frac{f^{\mathcal{L}_\theta}\left(x + td + \frac{1}{2}t^2 w'\right) - f^{\mathcal{L}_\theta}(x) - t\left(f^{\mathcal{L}_\theta}\right)'(x; d)}{\frac{1}{2}t^2}$$
$$\to f^{''}(x_1; d_1, w_1)\, u_\xi^{(1)} + f^{''}(x_1; d_1, w_1)\, u_\xi^{(2)}$$
$$= \left(f^{''}(x_1; d_1, w_1), 0\right).$$

Case 2: For $x_2 = 0, d_2 = 0$, and $w_2 \neq 0$, since f is parabolic second-order directionally differentiable, we have

$$\frac{f\left(\lambda_1(z)\right) - f(x_1) - tf'(x_1; d_1)}{\frac{1}{2}t^2} \to f^{''}\left(x_1; d_1, w_1 - \|w_2\| \cot\theta\right)$$

and

$$\frac{f(\lambda_2(z)) - f(x_1) - tf'(x_1; d_1)}{\frac{1}{2}t^2} \to f''(x_1; d_1, w_1 + \|w_2\| \tan\theta).$$

Note that $u_z^{(i)} \to u_w^{(i)}$ for $i = 1, 2$. Therefore, we also conclude that

$$\frac{f^{\mathcal{L}_\theta}\left(x + td + \frac{1}{2}t^2 w'\right) - f^{\mathcal{L}_\theta}(x) - t\left(f^{\mathcal{L}_\theta}\right)'(x; d)}{\frac{1}{2}t^2}$$
$$\to f''(x_1; d_1, w_1 - \|w_2\| \cot\theta)\, u_w^{(1)} + f''(x_1; d_1, w_1 + \|w_2\| \tan\theta)\, u_w^{(2)}.$$

In summary, from Cases 1 and 2, we see that under $x_2 = 0$ and $d_2 = 0$

$$\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) = f''(x_1; d_1, w_1 - \|w_2\| \cot\theta)\, u_w^{(1)} + f''(x_1; d_1, w_1 + \|w_2\| \tan\theta)\, u_w^{(2)}.$$

Case 3: For $x_2 = 0$, $d_2 \neq 0$, we have

$$(f^{\mathcal{L}_\theta})'(x; d) = f'(x_1; d_1 - \|d_2\| \cot\theta) u_d^{(1)} + f'(x_1; d_1 + \|d_2\| \tan\theta) u_d^{(2)}.$$

Note that

$$\begin{aligned}
& f\left(x_1 + td_1 + \frac{1}{2}t^2 w_1' - t\|d_2 + \frac{1}{2}tw_2'\| \cot\theta\right) \\
&= f\left(x_1 + td_1 + \frac{1}{2}t^2 w_1' - t\Big[\|d_2\| \cot\theta + \frac{1}{2}t\bar{d}_2^T w_2' \cot\theta + o(t)\Big]\right) \\
&= f\left(x_1 + td_1 + \frac{1}{2}t^2 w_1 - t\Big[\|d_2\| \cot\theta + \frac{1}{2}t\bar{d}_2^T w_2 \cot\theta\Big] + o\left(t^2\right)\right) \\
&= f(x_1) + tf'(x_1; d_1 - \|d_2\| \cot\theta) \\
&\quad + \frac{1}{2}t^2 f''\left(x_1; d_1 - \|d_2\| \cot\theta, w_1 - \bar{d}_2^T w_2 \cot\theta\right) + o\left(t^2\right),
\end{aligned} \tag{5}$$

where we use the facts that $w' \to w$ and f is parabolic second-order directionally differentiable at $\lambda_1(x)$ in the Hadamard sense. Similarly, we obtain

$$\begin{aligned}
& f\left(x_1 + td_1 + \frac{1}{2}t^2 w_1' + t\|d_2 + \frac{1}{2}tw_2'\| \tan\theta\right) \\
&= f(x_1) + tf'(x_1; d_1 + \|d_2\| \tan\theta) \\
&\quad + \frac{1}{2}t^2 f''\left(x_1; d_1 + \|d_2\| \tan\theta, w_1 + \bar{d}_2^T w_2 \tan\theta\right) + o\left(t^2\right).
\end{aligned} \tag{6}$$

Thus, the first component of $\frac{f^{\mathcal{L}_\theta}(x+td+\frac{1}{2}t^2w')-f^{\mathcal{L}_\theta}(x)-t(f^{\mathcal{L}_\theta})'(x;d)}{\frac{1}{2}t^2}$ converges to

$$\frac{1}{1+\cot^2\theta}f''\left(x_1;d_1-\|d_2\|\cot\theta,w_1-\bar{d}_2^Tw_2\cot\theta\right)$$
$$+\frac{1}{1+\tan^2\theta}f''\left(x_1;d_1+\|d_2\|\tan\theta,w_1+\bar{d}_2^Tw_2\tan\theta\right).$$

In addition, according to Theorem 3.1, we know

$$\begin{aligned}\frac{d_2+\frac{1}{2}tw_2'}{\|d_2+\frac{1}{2}tw_2'\|}&=\Phi\left(d_2+\frac{1}{2}tw_2'\right)\\&=\Phi(d_2)+\frac{1}{2}t\mathcal{J}\Phi(d_2)w_2'+\frac{1}{8}t^2\mathcal{J}^2\Phi(d_2)\left(w_2',w_2'\right)+o\left(t^2\right)\\&=\Phi(d_2)+\frac{1}{2}t\mathcal{J}\Phi(d_2)w_2'+\frac{1}{8}t^2\mathcal{J}^2\Phi(d_2)\left(w_2,w_2\right)+o\left(t^2\right).\end{aligned}\tag{7}$$

Hence, it follows from (5) to (7) that

$$\begin{aligned}&-f\left(\lambda_1(z)\right)\Phi\left(d_2+\frac{1}{2}tw_2'\right)+f(x_1)\Phi(d_2)+tf'\left(x_1;d_1-\|d_2\|\cot\theta\right)\Phi(d_2)\\&=-\frac{1}{2}tf(x_1)\mathcal{J}\Phi(d_2)w_2'-\frac{1}{2}t^2\Bigg[f''\left(x_1;d_1-\|d_2\|\cot\theta,w_1-\bar{d}_2^Tw_2\cot\theta\right)\Phi(d_2)\\&\quad+f'\left(x_1;d_1-\|d_2\|\cot\theta\right)\mathcal{J}\Phi(d_2)w_2'+\frac{1}{4}f(x_1)\mathcal{J}^2\Phi(d_2)\left(w_2,w_2\right)\Bigg]+o\left(t^2\right)\end{aligned}$$

and

$$\begin{aligned}&f(\lambda_2(z))\Phi\left(d_2+\frac{1}{2}tw_2'\right)-f(x_1)\Phi(d_2)-tf'(x_1;d_1+\|d_2\|\tan\theta)\Phi(d_2)\\&=\frac{1}{2}tf(x_1)\mathcal{J}\Phi(d_2)w_2'+\frac{1}{2}t^2\Bigg[f''\left(x_1;d_1+\|d_2\|\tan\theta,w_1+\bar{d}_2^Tw_2\tan\theta\right)\Phi(d_2)\\&\quad+f'\left(x_1;d_1+\|d_2\|\tan\theta\right)\mathcal{J}\Phi(d_2)w_2'+\frac{1}{4}f(x_1)\mathcal{J}^2\Phi(d_2)\left(w_2,w_2\right)\Bigg]+o\left(t^2\right).\end{aligned}$$

Thus, the second component of $\frac{f^{\mathcal{L}_\theta}(x+td+\frac{1}{2}t^2w')-f^{\mathcal{L}_\theta}(x)-t(f^{\mathcal{L}_\theta})'(x;d)}{\frac{1}{2}t^2}$ converges to

$$\frac{1}{\tan\theta+\mathrm{ctan}\theta}\left(\kappa_1\mathcal{J}\Phi(d_2)w_2+\kappa_2\Phi(d_2)\right),$$

where

$$\begin{aligned}\kappa_1&:=f'\left(x_1;d_1+\|d_2\|\tan\theta\right)-f'\left(x_1;d_1-\|d_2\|\cot\theta\right)\\\kappa_2&:=f''\left(x_1;d_1+\|d_2\|\tan\theta,w_1+\bar{d}_2^Tw_2\tan\theta\right)\\&\quad-f''\left(x_1;d_1-\|d_2\|\cot\theta,w_1-\bar{d}_2^Tw_2\cot\theta\right).\end{aligned}$$

To sum up, we can conclude that

$$\begin{aligned}
&\left(f^{\mathcal{L}_\theta}\right)^{''}(x; d, w)\\
&= f^{''}\left(x_1; d_1 - \|d_2\|, w_1 - \bar{d}_2^T w_2 \cot\theta\right) u_d^{(1)}\\
&\quad + f^{''}\left(x_1; d_1 + \|d_2\| \tan\theta, w_1 + \bar{d}_2^T w_2 \tan\theta\right) u_d^{(2)}\\
&\quad + \frac{1}{\tan\theta + \cot\theta}\left(f'(x_1; d_1 + \|d_2\| \tan\theta) - f'(x_1; d_1 - \|d_2\| \cot\theta)\right)\mathcal{J}\widetilde{\Phi}(d)w.
\end{aligned}$$

Case 4: For $x_2 \neq 0$, under this case, we know

$$\begin{aligned}
(f^{\mathcal{L}_\theta})'(x; d) = f'\left(\lambda_1(x); d_1 - \bar{x}_2^T d_2 \cot\theta\right) u_x^{(1)} + f'\left(\lambda_2(x); d_1 + \bar{x}_2^T d_2 \tan\theta\right) u_x^{(2)}\\
+ \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)} \begin{bmatrix} 0 & 0 \\ 0 & I - \bar{x}_2 \bar{x}_2^T \end{bmatrix} d.
\end{aligned}$$

Note that

$$\begin{aligned}
\|x_2 + td_2 + \frac{1}{2}t^2 w_2'\| &= \|x_2\| + t\bar{x}_2^T d_2 + \frac{1}{2}t^2\left[\bar{x}_2^T w_2' + d_2^T \mathcal{J}\Phi(x_2)d_2\right] + o\left(t^2\right)\\
&= \|x_2\| + t\bar{x}_2^T d_2 + \frac{1}{2}t^2\left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2)d_2\right] + o\left(t^2\right).
\end{aligned}$$

Since f is parabolic second-order directionally differentiable at $\lambda_1(x)$ in the Hadamard sense, we have

$$\begin{aligned}
&f\left(x_1 + td_1 + \frac{1}{2}t^2 w_1' - \|x_2 + td_2 + \frac{1}{2}t^2 w_2'\| \cot\theta\right)\\
&= f(x_1 - \|x_2\| \cot\theta) + tf'\left(x_1 - \|x_2\| \cot\theta; d_1 - \bar{x}_2^T d_2 \cot\theta\right)\\
&\quad + \frac{1}{2}t^2 f^{''}\left(x_1 - \|x_2\| \cot\theta; d_1 - \bar{x}_2^T d_2 \cot\theta, w_1\right.\\
&\quad\quad \left. - \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2)d_2\right] \cot\theta\right) + o\left(t^2\right).
\end{aligned}$$

Besides, we know that

$$\begin{aligned}
&\Phi\left(x_2 + td_2 + \frac{1}{2}t^2 w_2'\right)\\
&= \Phi(x_2) + t\mathcal{J}\Phi(x_2)d_2 + \frac{1}{2}t^2\left(\mathcal{J}\Phi(x_2)w_2' + \mathcal{J}^2\Phi(x_2)(d_2, d_2)\right) + o\left(t^2\right)\\
&= \Phi(x_2) + t\mathcal{J}\Phi(x_2)d_2 + \frac{1}{2}t^2\left(\mathcal{J}\Phi(x_2)w_2 + \mathcal{J}^2\Phi(x_2)(d_2, d_2)\right) + o\left(t^2\right).
\end{aligned}$$

Thus, the first component of $\frac{f^{\mathcal{L}_\theta}(x+td+\frac{1}{2}t^2w')-f^{\mathcal{L}_\theta}(x)-t(f^{\mathcal{L}_\theta})'(x;d)}{\frac{1}{2}t^2}$ converges to

$$\begin{aligned}
&\frac{1}{1+\cot^2\theta} f''\Big(x_1-\|x_2\|\cot\theta; d_1-\bar{x}_2^T d_2\cot\theta, w_1\\
&\quad -\Big[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2)d_2\Big]\cot\theta\Big)\\
&+\frac{1}{1+\tan^2\theta} f''\Big(x_1+\|x_2\|\tan\theta; d_1+\bar{x}_2^T d_2\tan\theta, w_1\\
&\quad +\Big[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2)d_2\Big]\tan\theta\Big).
\end{aligned}$$

Moreover, the second component of $\frac{f^{\mathcal{L}_\theta}(x+td+\frac{1}{2}t^2w')-f^{\mathcal{L}_\theta}(x)-t(f^{\mathcal{L}_\theta})'(x;d)}{\frac{1}{2}t^2}$ converges to

$$\begin{aligned}
&-\frac{\cot\theta}{1+\cot^2\theta}\Big(f\,(x_1-\|x_2\|\cot\theta)\left[\mathcal{J}\Phi(x_2)w_2+\mathcal{J}^2\Phi(x_2)\,(d_2,d_2)\right]\\
&+2f'\Big(x_1-\|x_2\|\cot\theta; d_1-\bar{x}_2^T d_2\cot\theta\Big)\mathcal{J}\Phi(x_2)d_2\\
&+f''\big(x_1-\|x_2\|\cot\theta; d_1-\bar{x}_2^T d_2\cot\theta, w_1\\
&\quad -\big[\bar{x}_2^T w_2+d_2^T\mathcal{J}\Phi(x_2)d_2\big]\cot\theta\big)\Phi(x_2)\Big)\\
&+\frac{\tan\theta}{1+\tan^2\theta}\Big(f\,(x_1+\|x_2\|\tan\theta)\left[\mathcal{J}\Phi(x_2)w_2+\mathcal{J}^2\Phi(x_2)\,(d_2,d_2)\right]\\
&+2f'\Big(x_1+\|x_2\|\tan\theta; d_1+\bar{x}_2^T d_2\tan\theta\Big)\mathcal{J}\Phi(x_2)d_2\\
&+f''\Big(x_1+\|x_2\|\tan\theta; d_1+\bar{x}_2^T d_2\tan\theta, w_1\\
&\quad +\big[\bar{x}_2^T w_2+d_2^T\mathcal{J}\Phi(x_2)d_2\big]\tan\theta\Big)\,\Phi(x_2)\Big).
\end{aligned}$$

To sum up, we can conclude that

$$\begin{aligned}
&\left(f^{\mathcal{L}_\theta}\right)''(x;d,w)\\
&=f''\Big(x_1-\|x_2\|\cot\theta; d_1-\bar{x}_2^T d_2\cot\theta, w_1-\Big[\bar{x}_2^T w_2+d_2^T\mathcal{J}\Phi(x_2)d_2\Big]\cot\theta\Big)u_x^1\\
&\quad+f''\Big(x_1+\|x_2\|\tan\theta; d_1+\bar{x}_2^T d_2\tan\theta, w_1+\Big[\bar{x}_2^T w_2+d_2^T\mathcal{J}\Phi(x_2)d_2\Big]\tan\theta\Big)u_x^2\\
&\quad+\frac{2}{\cot\theta+\tan\theta}\Gamma_1\mathcal{J}\widetilde{\Phi}(x)d+\frac{1}{\cot\theta+\tan\theta}\Gamma_2\Big(\mathcal{J}\widetilde{\Phi}(x)w+\mathcal{J}^2\widetilde{\Phi}(x)(d,d)\Big),
\end{aligned}$$

where we use the facts that $\mathcal{J}\widetilde{\Phi}(x)w = (0, \mathcal{J}\Phi(x_2)w_2)$ and $\mathcal{J}^2\widetilde{\Phi}(x)(d,d) = (0, \mathcal{J}^2\Phi(x_2)(d_2,d_2))$.

"⇒" Suppose that $f^{\mathcal{L}_\theta}$ is parabolic second-order directionally differentiable at x in the Hadamard sense. Given $\tilde{d}, \tilde{w} \in \mathbb{R}$ and $\tilde{w}' \to \tilde{w}$. To proceed, we also discuss the following two cases.
Case 1: For $x_2 = 0$, let $d = \tilde{d}e$, $w' = \tilde{w}'e$, and $w = \tilde{w}e$. Denote $z := x + td + \frac{1}{2}t^2 w'$. Then

$$\begin{aligned} &\frac{f\left(x_1 + t\tilde{d} + \frac{1}{2}t^2\tilde{w}'\right) - f(x_1) - tf'(x_1, \tilde{d})}{\frac{1}{2}t^2} \\ &= \left\langle \frac{f^{\mathcal{L}_\theta}(z) - f^{\mathcal{L}_\theta}(x) - t\left(f^{\mathcal{L}_\theta}\right)'(x; d)}{\frac{1}{2}t^2}, e \right\rangle. \end{aligned}$$

Thus, we obtain $f''(x_1; \tilde{d}, \tilde{w}) = \langle (f^{\mathcal{L}_\theta})''(x; d, w), e\rangle$.
Case 2: For $x_2 \neq 0$, let $d = \tilde{d}u_x^{(1)}$, $w' = \tilde{w}'u_x^{(1)}$, and $w = \tilde{w}u_x^{(1)}$. Then, we have

$$x + td + \frac{1}{2}t^2 w' = \left(\lambda_1(x) + t\tilde{d} + \frac{1}{2}t^2\tilde{w}'\right) u_x^{(1)} + \lambda_2(x) u_x^{(2)}$$

with $t > 0$ satisfying $t\tilde{d} + \frac{1}{2}t^2\tilde{w}' < \lambda_2(x) - \lambda_1(x)$. This implies

$$f^{\mathcal{L}_\theta}\left(x + td + \frac{1}{2}t^2 w'\right) = f\left(\lambda_1(x) + t\tilde{d} + \frac{1}{2}t^2\tilde{w}'\right) u_x^{(1)} + f\left(\lambda_2(x)\right) u_x^{(2)}$$

and $(f^{\mathcal{L}_\theta})'(x; d) = f'(\lambda_1(x); \tilde{d})u_x^{(1)}$. Thus,

$$\begin{aligned} &\frac{f\left(\lambda_1(x) + t\tilde{d} + \frac{1}{2}t^2\tilde{w}'\right) - f\left(\lambda_1(x)\right) - tf'\left(\lambda_1(x); \tilde{d}\right)}{\frac{1}{2}t^2} \\ &= \left(1 + \cot^2\theta\right)\left\langle \frac{f^{\mathcal{L}_\theta}\left(x + td + \frac{1}{2}t^2 w'\right) - f^{\mathcal{L}_\theta}(x) - t\left(f^{\mathcal{L}_\theta}\right)'(x; d)}{\frac{1}{2}t^2}, u_x^1 \right\rangle, \end{aligned}$$

which says

$$f''\left(\lambda_1(x); \tilde{d}, \tilde{w}\right) = \left(1 + \cot^2\theta\right)\left\langle \left(f^{\mathcal{L}_\theta}\right)''(x; d, w), u_x^{(1)} \right\rangle.$$

The similar arguments can be used for f at $\lambda_2(x)$. From all the above, the proof is complete. □

4 Second-Order Tangent Sets

In this section, we turn our attention to f being the special function $f(t) = \max\{t, 0\}$. In this case, the corresponding $f^{\mathcal{L}_\theta}$ is just the projection operator associated with circular cone. For $x \in \mathcal{L}_\theta$, from [16], we know the tangent cone is given by

$$\begin{aligned}T_{\mathcal{L}_\theta}(x) &:= \{d : \text{dist}(x+td, \mathcal{L}_\theta) = o(t),\ t \geq 0\} \\ &= \left\{d : \Pi_{\mathcal{L}_\theta}(x+td) - (x+td) = o(t),\ t \geq 0\right\} \\ &= \left\{d : \Pi'_{\mathcal{L}_\theta}(x; d) = d\right\},\end{aligned} \tag{8}$$

which, together with the formula of $\Pi'_{\mathcal{L}_\theta}$, yields

$$T_{\mathcal{L}_\theta}(x) = \begin{cases} \mathbb{R}^n, & \text{if } x \in \text{int}\mathcal{L}_\theta, \\ \mathcal{L}_\theta, & \text{if } x = 0, \\ \left\{d : d_2^T x_2 - d_1 x_1 \tan^2\theta \leq 0\right\}, & \text{if } x \in \text{bd}\mathcal{L}_\theta/\{0\}. \end{cases}$$

Definition 4.1 [16, Definition 3.28] The set limits

$$T_S^{i,2}(x, d) := \left\{w \in \mathbb{R}^n : \text{dist}\left(x + td + \frac{1}{2}t^2 w, S\right) = o\left(t^2\right),\ t \geq 0\right\}$$

and

$$T_S^2(x, d) := \left\{w \in \mathbb{R}^n : \exists\, t_n \downarrow 0 \text{ such that dist}\left(x + t_n d + \frac{1}{2}t_n^2 w, S\right) = o\left(t_n^2\right)\right\}$$

are called the inner and outer second-order tangent sets, respectively, to the set S at x in the direction d.

In [13], we have shown that the circular cone is second-order regular, which means $T^{i,2}_{\mathcal{L}_\theta}(x; d)$ is equal to $T^2_{\mathcal{L}_\theta}(x; d)$ for all $d \in T_{\mathcal{L}_\theta}(x)$. Since the inner and outer second-order tangent sets are equal, we simply say that $T^2_{\mathcal{L}_\theta}(x; d)$ is the second-order tangent set. Next, we provide two different approaches to establish the exact formula of second-order tangent set of circular cone. One is following from the parabolic second-order directional derivative of the spectral value $\lambda_1(x)$, and the other is using the parabolic second-order directional derivative of projection operator $\Pi_{\mathcal{L}_\theta}$.

Theorem 4.1 *Given $x \in \mathcal{L}_\theta$ and $d \in T_{\mathcal{L}_\theta}(x)$, then*

$$T^2_{\mathcal{L}_\theta}(x, d) = \begin{cases} \mathbb{R}^n, & \text{if } d \in \text{int}T_{\mathcal{L}_\theta}(x), \\ T_{\mathcal{L}_\theta}(d), & \text{if } x = 0, \\ \left\{w : w_2^T x_2 \cot\theta - w_1 x_1 \tan\theta \leq d_1^2 \tan\theta - \|d_2\|^2 \cot\theta\right\}, & \text{otherwise.} \end{cases}$$

Proof First, we note that $\mathcal{L}_\theta = \{x : -\lambda_1(x) \leq 0\}$. With this, we have

$$\begin{aligned} w \in T^2_{\mathcal{L}_\theta}(x; d) &\Longleftrightarrow -\lambda_1\left(x + td + \frac{1}{2}t^2 w + o\left(t^2\right)\right) \leq 0 \\ &\Longleftrightarrow -\lambda_1(x) - t\lambda_1'(x; d) - \frac{1}{2}t^2\lambda_1''(x; d, w) + o\left(t^2\right) \leq 0. \end{aligned} \tag{9}$$

The case of $x \in \text{int}\mathcal{L}_\theta$ (corresponding to $-\lambda_1(x) < 0$) or $x \in \text{bd}\mathcal{L}_\theta$ and $d \in \text{int}T_{\mathcal{L}_\theta}(x)$ (corresponding to $\lambda_1(x) = 0$ and $-\lambda_1'(x; d) < 0$) ensures that (9) holds for all $w \in \mathbb{R}^n$.

For the case $x = 0$ and $d = 0$, it follows from Theorem 3.2 and (9) that

$$w \in T^2_{\mathcal{L}_\theta}(x; d) \Longrightarrow -w_1 + \|w_2\| \cot\theta \le 0 \Longleftrightarrow w \in \mathcal{L}_\theta.$$

Conversely, if $w \in \mathcal{L}_\theta$, then $\mathrm{dist}(\frac{1}{2}t^2 w, \mathcal{L}_\theta) = 0$ due to $\mathcal{L}_\theta$ is a cone, which implies $w \in T^2_{\mathcal{L}_\theta}(x; d)$. Hence, $T^2_{\mathcal{L}_\theta}(x; d) = T_{\mathcal{L}_\theta}(x)$.
For the case $x = 0$ and $d \in \mathrm{bd}T_{\mathcal{L}_\theta}(x)\backslash\{0\} = \mathrm{bd}\mathcal{L}_\theta\backslash\{0\}$, it follows from Theorem 3.2 and (9) that

$$w \in T^2_{\mathcal{L}_\theta}(x; d) \Longrightarrow -w_1 d_1 \tan^2\theta + d_2^T w_2 \le 0 \Longleftrightarrow w \in T_{\mathcal{L}_\theta}(d).$$

Conversely, if $w \in T_{\mathcal{L}_\theta}(d)$, then $\mathrm{dist}(d + tw, \mathcal{L}_\theta) = o(t)$, and hence, $\mathrm{dist}(d + \frac{1}{2}tw, \mathcal{L}_\theta) = o(\frac{1}{2}t) = o(t)$. Thus, we obtain $\mathrm{dist}(x + td + \frac{1}{2}t^2 w, \mathcal{L}_\theta) = \mathrm{dist}(td + \frac{1}{2}t^2 w, \mathcal{L}_\theta) = o(t^2)$, which means $w \in T^2_{\mathcal{L}_\theta}(x; d)$.
The case remained is $x \in \mathrm{bd}\mathcal{L}_\theta/\{0\}$ and $d \in \mathrm{bd}T_{\mathcal{L}_\theta}(x)$, i.e., $x_1 = \|x_2\| \cot\theta$ and $d_2^T x_2 = d_1 x_1 \tan^2\theta$. Since $x_2 \ne 0$, $-\lambda_1$ is second-order differentiable at x. Hence, it follows from Theorem 3.2 that

$$\begin{aligned} T^2_{\mathcal{L}_\theta}(x; d) &= \left\{ w \,:\, -\lambda_1''(x; d, w) \le 0 \right\} \\ &= \left\{ w \,:\, -x_1 w_1 \tan\theta + x_2^T w_2 \cot\theta + \|d_2\|^2 \cot\theta - d_1^2 \tan\theta \le 0 \right\}, \end{aligned}$$

where the last step is due to $\bar{x}_2^T d_2 = d_1 \tan\theta$. □

As below, we provide the second approach to establish the formula of second-order tangent set by using the parabolic second-order directional derivative of projection operator associated with circular cone. To this end, we need a technical lemma.

Lemma 4.1 *For $x \in \mathcal{L}_\theta$ and $d \in T_{\mathcal{L}_\theta}(x)$, we have*

$$T^2_{\mathcal{L}_\theta}(x, d) = \left\{ w \,:\, \Pi''_{\mathcal{L}_\theta}(x; d, w) = w \right\}.$$

Proof The desired result follows from

$$\begin{aligned} T^2_{\mathcal{L}_\theta}(x, d) &= \left\{ w \,:\, \mathrm{dist}\left(x + td + \frac{1}{2}t^2 w, \mathcal{L}_\theta \right) = o\left(t^2\right),\ t \ge 0 \right\} \\ &= \left\{ w \,:\, \Pi_{\mathcal{L}_\theta}\left(x + td + \frac{1}{2}t^2 w \right) - \left(x + td + \frac{1}{2}t^2 w \right) = o\left(t^2\right),\ t \ge 0 \right\} \\ &= \left\{ w \,:\, \Pi_{\mathcal{L}_\theta}\left(x + td + \frac{1}{2}t^2 w \right) - \Pi_{\mathcal{L}_\theta}(x) - t\Pi'_{\mathcal{L}_\theta}(x; d) - \frac{1}{2}t^2 w \right. \\ &\qquad \left. = o\left(t^2\right),\ t \ge 0 \right\} \\ &= \left\{ w \,:\, \Pi''_{\mathcal{L}_\theta}(x; d, w) = w \right\}, \end{aligned}$$

where the third step uses the fact that $d = \Pi'_{\mathcal{L}_\theta}(x; d)$ since $d \in T_{\mathcal{L}_\theta}(x)$ by (8). □

Recall first from [15] that $\Pi_{\mathcal{L}_\theta}$, the projection operator, is the vector-valued function corresponding to $f(t) = \max\{t, 0\}$. To present the second approach, we will also use the parabolic second-order directional derivative of the $f(t) = \max\{t, 0\}$, which can be found in [10]. Now the second approach to prove Theorem 4.1 is given below.

Proof Notice first that as $x_1 > \|x_2\| \cot\theta$ or $x_1 = \|x_2\| \cot\theta \neq 0$ and $d_1 \geq \bar{x}_2^T d_2 \cot\theta$, then

$$\frac{2}{\tan\theta + \cot\theta} \Gamma_1 \mathcal{J}\widetilde{\Phi}(x)d + \frac{1}{\tan\theta + \cot\theta} \Gamma_2 \big(\mathcal{J}\widetilde{\Phi}(x)w + \mathcal{J}^2\widetilde{\Phi}(x)(d, d)\big)$$
$$= \left(0,\, w_2 - \left[\bar{x}_2^T w_2 - \frac{\left(\bar{x}_2^T d_2\right)^2}{\|x_2\|} + \frac{\|d_2\|^2}{\|x_2\|}\right] \frac{x_2}{\|x_2\|}\right)^T. \tag{10}$$

As $x_1 \geq 0$ and $d_1 \geq \|d_2\| \cot\theta$, we know that

$$\frac{1}{\tan\theta + \cot\theta} \Big(f'(x_1; d_1 + \|d_2\| \tan\theta) - f'(x_1; d_1 - \|d_2\| \cot\theta) \Big) \mathcal{J}\widetilde{\Phi}(d)w$$
$$= \left(0,\, w_2 - \bar{d}_2^T w_2 \bar{d}_2\right)^T. \tag{11}$$

We point it out that, in the above formulas (10) and (11), we have applied the parabolic second-order directional derivative of the max-type function $f(t) = \max\{t, 0\}$. To proceed, we discuss the following three cases.

Case 1: For $d \in \mathrm{int}T_{\mathcal{L}_\theta}(x)$, we keep going to discuss three subcases.

Subcase (1): $x = 0$. Under this subcase, we see $d \in \mathrm{int}\mathcal{L}_\theta$, i.e., $d_1 > \|d_2\| \cot\theta$. If $d_2 = 0$, then $d_1 > 0$ which yields

$$f''(x_1; d_1, w_1 - \|w_2\| \cot\theta)\, u_w^1 + f''(x_1; d_1, w_1 + \|w_2\| \tan\theta)\, u_w^2 = w, \quad \forall w \in \mathbb{R}^n.$$

If $d_2 \neq 0$, it then follows from (11) that

$$\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) = \left(w_1 - \bar{d}_2^T w_2 \cot\theta\right) u_d^{(1)} + \left(w_1 + \bar{d}_2^T w_2 \tan\theta\right) u_d^{(2)}$$
$$+ \left(0,\, w_2 - \bar{d}_2^T w_2 \bar{d}_2\right)^T = w.$$

Subcase (2): $x \in \mathrm{int}\mathcal{L}_\theta$. Under this subcase, it is clear that $T_{\mathcal{L}_\theta}(x) = \mathbb{R}^n$. If $x_2 = 0$, it follows from Theorem 3.3 that $(f^{\mathcal{L}_\theta})''(x; d, w) = w$ whenever $d_2 = 0$ or $d_2 \neq 0$ due to $x_1 > 0$ in this case. If $x_2 \neq 0$, from (10), we know that

$$\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) = \begin{pmatrix} w_1 \\ \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2) d_2\right] \frac{x_2}{\|x_2\|} \end{pmatrix}$$
$$+ \begin{pmatrix} 0 \\ w_2 - \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2) d_2\right] \frac{x_2}{\|x_2\|} \end{pmatrix} = w.$$

Subcase (3): $x \in \text{bd}\mathcal{L}_\theta/\{0\}$. Then $d \in \text{int}T_{\mathcal{L}_\theta}(x)$ means $d_2^T x_2 < d_1 x_1 \tan^2\theta = d_1\|x_2\|\tan\theta$, i.e., $\bar{x}_2^T d_2 \cot\theta < d_1$. Thus, $(f^{\mathcal{L}_\theta})''(x; d, w) = w$ for all $w \in \mathbb{R}^n$ by the similar argument as above.

In summary, we have $T^2_{\mathcal{L}_\theta}(x, d) = \mathbb{R}^n$ in this case.

Case 2: For $x = 0$, since $d \in T_{\mathcal{L}_\theta}(x) = \mathcal{L}_\theta$, we see that $d_1 \geq \|d_2\|\cot\theta$. It only remains to show the case of $d_1 = \|d_2\|\cot\theta$. If $d_2 = 0$, then $d_1 = 0$, and hence,

$$\begin{aligned}\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) &= f''(x_1; d_1, w_1 - \|w_2\|\cot\theta)\, u_w^1 \\ &\quad + f''(x_1; d_1, w_1 + \|w_2\|\tan\theta)\, u_w^2 \\ &= (w_1 - \|w_2\|\cot\theta)_+\, u_w^1 + (w_1 + \|w_2\|\tan\theta)_+\, u_w^2 \\ &= \Pi_{\mathcal{L}_\theta}(w).\end{aligned}$$

This, together with Lemma 4.1, yields $w \in T^2_{\mathcal{L}_\theta}(x; d) \iff \Pi_{\mathcal{L}_\theta}(w) = w$, i.e., $w \in \mathcal{L}_\theta = T_{\mathcal{L}_\theta}(d)$. If $d_2 \neq 0$, then $d_1 = \|d_2\|\cot\theta > 0$. Hence,

$$\begin{aligned}\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) = \left(w_1 - \bar{d}_2^T w_2 \cot\theta\right)_+ u_d^{(1)} + \left(w_1 + \bar{d}_2^T w_2 \tan\theta\right) u_d^{(2)} \\ + \left(0, w_2 - \bar{d}_2^T w_2 \bar{d}_2\right)^T.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) &= w \iff \frac{1}{1+\cot^2\theta}\left(w_1 - \bar{d}_2^T w_2 \cot\theta\right)_+ \\ &= w_1 - \frac{\cot^2\theta}{1+\cot^2\theta}\left(w_1 + \bar{d}_2^T w_2 \tan\theta\right) \\ &\iff \left(w_1 - \bar{d}_2^T w_2\cot\theta\right)_+ = w_1 - \bar{d}_2^T w_2 \cot\theta \\ &\iff w_1 d_1 \tan^2\theta \geq d_2^T w_2 \\ &\iff w \in T_{\mathcal{L}_\theta}(d),\end{aligned}$$

where we have used the fact that $d_1 = \|d_2\|\cot\theta$.

Case 3: For $x \in \text{bd}\mathcal{L}_\theta/\{0\}$ and $d \in \text{bd}T_{\mathcal{L}_\theta}(x)$, we have $d_1 = \bar{x}_2^T d_2 \cot\theta$. This says that

$$\begin{aligned}\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) &= \left(w_1 - \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2) d_2\right]\cot\theta\right)_+ u_x^{(1)} \\ &\quad + \left(w_1 + \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2) d_2\right]\tan\theta\right) u_x^{(2)} \\ &\quad + \left(0, w_2 - \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2) d_2\right]\frac{x_2}{\|x_2\|}\right)^T.\end{aligned}$$

Hence,

$$\left(f^{\mathcal{L}_\theta}\right)''(x; d, w) = w \iff \left(w_1 - \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2) d_2\right]\cot\theta\right)_+$$

$$
\begin{aligned}
&= \quad w_1 - \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2) d_2 \right] \cot\theta \\
&\Longleftrightarrow w_1 - \left[\bar{x}_2^T w_2 + d_2^T \mathcal{J}\Phi(x_2) d_2 \right] \cot\theta \geq 0 \\
&\Longleftrightarrow w_1 x_1 \tan\theta - x_2^T w_2 \cot\theta \geq \|d_2\|^2 \cot\theta - d_1^2 \tan\theta \\
&\Longleftrightarrow w \in T^2_{\mathcal{L}_\theta}(x; d),
\end{aligned}
$$

where the third equivalence is due to the fact $d_1 = \bar{x}_2^T d_2 \cot\theta$ in this case. □

5 Second-Order Differentiability

The relationship for the first-order differentiability between $f^{\mathcal{L}_\theta}$ and f has been studied in [11,12]. More specifically, $f^{\mathcal{L}_\theta}$ is first-order differentiable at x if and only if f is first-order differentiable at $\lambda_i(x)$ for $i = 1, 2$. It is natural to ask whether analogous relationship for the second-order differentiability (in the Fréchet sense) between $f^{\mathcal{L}_\theta}$ and f exists or not. In this section, we provide an answer for this question.

Theorem 5.1 *Let $x \in \mathbb{R}^n$ with spectral decomposition $x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}$ given as in (1). Suppose that f is second-order differentiable at $\lambda_i(x)$ for $i = 1, 2$. Then,*

(a) for $x_2 \neq 0$, $f^{\mathcal{L}_\theta}$ is second-order differentiable at x with

$$
\mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(d,d) = \Big(d^T A_1(x) d, d^T A_2(x) d, \ldots, d^T A_n(x) d \Big)^T,
$$

where

$$
\begin{aligned}
A_1(x) &:= \begin{bmatrix} \tilde{\xi} & \tilde{\varrho}\bar{x}_2^T \\ \tilde{\varrho}\bar{x}_2 & \tilde{a} I + (\tilde{\eta} - \tilde{a})\bar{x}_2\bar{x}_2^T \end{bmatrix}, \\
A_i(x) &:= C(x) \frac{(x_2)_i}{\|x_2\|} + B_i(x), \quad i = 2, \ldots, n.
\end{aligned}
$$

Here

$$
\begin{aligned}
C(x) &:= \begin{bmatrix} \tilde{\varrho} & (\tilde{\eta} - \tilde{a})\bar{x}_2^T \\ (\tilde{\eta} - \tilde{a})\bar{x}_2 & \tilde{\tau} I + (\varpi - 3\tilde{\tau})\bar{x}_2\bar{x}_2^T \end{bmatrix}, \\
B_i(x) &:= v e_i^T + e_i v^T, \quad v := \left(\tilde{a}, \tilde{\tau}\bar{x}_2^T \right)^T,
\end{aligned}
$$

and

$$
\begin{aligned}
a &:= \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}, \quad \tilde{\xi} := \frac{f''(\lambda_1(x))}{1 + \cot^2\theta} + \frac{f''(\lambda_2(x))}{1 + \tan^2\theta}, \quad \tilde{\tau} := \frac{\eta - a}{\|x_2\|}, \\
\tilde{a} &:= \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)},
\end{aligned}
$$

$$\tilde{\varrho} := -\frac{\cot\theta}{1+\cot^2\theta} f^{''}(\lambda_1(x)) + \frac{\tan\theta}{1+\tan^2\theta} f^{''}(\lambda_2(x)),$$
$$\eta := \frac{\cot^2\theta}{1+\cot^2\theta} f^{'}(\lambda_1(x)) + \frac{\tan^2\theta}{1+\tan^2\theta} f^{'}(\lambda_2(x)),$$
$$\tilde{\eta} := \frac{\cot^2\theta}{1+\cot^2\theta} f^{''}(\lambda_1(x)) + \frac{\tan^2\theta}{1+\tan^2\theta} f^{''}(\lambda_2(x)),$$
$$\varpi := -\frac{\cot^3\theta}{1+\cot^2\theta} f^{''}(\lambda_1(x)) + \frac{\tan^3\theta}{1+\tan^2\theta} f^{''}(\lambda_2(x)).$$

(b) for $x_2 = 0$ and $\theta = 45°$, $f^{\mathcal{L}_\theta}$ is second-order differentiable at x with

$$\mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(d,d) = \left(d^T A_1(x)d, d^T A_2(x)d, \ldots, d^T A_n(x)d\right)^T,$$

where

$$A_1(x) := f^{''}(x_1)I, \quad A_i(x) := f^{''}(x_1)\begin{bmatrix} 0 & e_{i-1}^T \\ e_{i-1} & 0 \end{bmatrix}, \quad i = 2, 3, \ldots, n.$$

Proof (a) Note that $\|x_2\|$ and $\bar{x}_2$ are second-order differentiable at $x_2 \neq 0$, which together with that f is second-order differentiable, ensures that $f^{\mathcal{L}_\theta}$ is also second-order differentiable at x with $x_2 \neq 0$. Since $f^{\mathcal{L}_\theta}$ is second-order differentiable, according to the definition of the parabolic second-order differentiability, we have $\mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(d,d) = (f^{\mathcal{L}_\theta})^{''}(x; d, 0)$. Note also that $f^{''}(\lambda_i(x); \tilde{d}, \tilde{w}) = f'(\lambda_i(x))\tilde{w} + f^{''}(\lambda_i(x))(\tilde{d}, \tilde{d})$ for $i = 1, 2$ whenever f is second-order differentiable. Hence, taking $w = 0$ in Theorem 3.3 yields

$$\begin{aligned}
&\mathcal{J}^2\left(f^{\mathcal{L}_\theta}\right)(x)(d,d) \\
&= \begin{pmatrix}
\tilde{\xi} d_1^2 + 2\tilde{\varrho} d_1 \bar{x}_2^T d_2 + \left[\tilde{\eta} - \tilde{a}\right]\left(\bar{x}_2^T d_2\right)^2 + \tilde{a}\|d_2\|^2 \\
\left[\tilde{\varrho} d_1^2 + 2\tilde{\eta} d_1 \bar{x}_2^T d_2 + \varpi\left(\bar{x}_2^T d_2\right)^2 + \frac{\eta}{\|x_2\|}\|d_2\|^2 - \frac{\eta}{\|x_2\|}\left(\bar{x}_2^T d_2\right)^2\right]\bar{x}_2 \\
+2\tilde{a} d_1 d_2 - 2\tilde{a} d_1\left(\bar{x}_2^T d_2\right)\bar{x}_2 + 2\frac{\eta}{\|x_2\|}\left(\bar{x}_2^T d_2\right) d_2 - 2\frac{\eta}{\|x_2\|}\left(\bar{x}_2^T d_2\right)^2\bar{x}_2 \\
-2\frac{a}{\|x_2\|}\bar{x}_2^T d_2 d_2 + 3\frac{a}{\|x_2\|}\left(\bar{x}_2^T d_2\right)^2\bar{x}_2 - a\frac{\|d_2\|^2}{\|x_2\|}\bar{x}_2
\end{pmatrix} \\
&= \begin{pmatrix}
\tilde{\xi} d_1^2 + 2\tilde{\varrho} d_1 \bar{x}_2^T d_2 + \left[\tilde{\eta} - \tilde{a}\right]\left(\bar{x}_2^T d_2\right)^2 + \tilde{a}\|d_2\|^2 \\
\left[\tilde{\varrho} d_1^2 + 2(\tilde{\eta} - \tilde{a}) d_1 \bar{x}_2^T d_2 + (\varpi - 3\tilde{\tau})(\bar{x}_2^T d_2)^2 + \tilde{\tau}\|d_2\|^2\right]\bar{x}_2 \\
2\left(\tilde{a} d_1 + \tilde{\tau}\bar{x}_2^T d_2\right) d_2
\end{pmatrix}.
\end{aligned}$$

(b) When $\theta = 45°$, then circular cone reduces to the second-order cone and the circular cone function $f^{\mathcal{L}_\theta}$ is the SOC function f^{soc}. The result follows from [17,18]. □

Theorem 5.2 *Let $x \in \mathbb{R}^n$ with spectral decomposition $x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}$ given as in (1). For $x_2 \neq 0$, if $f^{\mathcal{L}_\theta}$ is second-order differentiable at x, then f is second-order differentiable at $\lambda_i(x)$ for $i = 1, 2$. For $x_2 = 0$, if $f^{\mathcal{L}_\theta}$ is second-order differentiable at x, then f is second-order differentiable at x_1 and $\theta = 45°$. In particular,*

(a) when $x_2 = 0$ and $\theta = 45°$, $f''(x_1) = \left\langle \mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(e, e), e \right\rangle$;
(b) when $x_2 \neq 0$, $f''(\lambda_i(x)) = \frac{1}{\|u_x^{(i)}\|^2} \left\langle \mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(u_x^{(i)}, u_x^{(i)}), u_x^{(i)} \right\rangle$, $i = 1, 2$.

Proof To proceed, we consider the following two cases.
Case 1: For $x_2 = 0$, from the second-order differentiability of $f^{\mathcal{L}_\theta}$, we know that

$$f^{\mathcal{L}_\theta}(x + d) = f^{\mathcal{L}_\theta}(x) + \mathcal{J} f^{\mathcal{L}_\theta}(x)d + \frac{1}{2}\mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(d, d) + o\left(\|d\|^2\right). \tag{12}$$

For $t \in \mathbb{R}$, taking $d = te$ in (12) yields

$$\begin{bmatrix} f(x_1 + t) \\ 0 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ 0 \end{bmatrix} + \begin{bmatrix} f'(x_1)t \\ 0 \end{bmatrix} + \frac{1}{2}t^2 \mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(e, e) + o\left(t^2\right),$$

which in turn implies

$$f(x_1 + t) = f(x_1) + f'(x_1)t + \frac{1}{2}t^2 \left\langle \mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(e, e), e \right\rangle + o\left(t^2\right).$$

This is equivalent to saying that f is second-order differentiable with $f''(x_1) = \langle \mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(e, e), e \rangle$. This together with the fact $f''(x_1; \tilde{d}, \tilde{w}) = f'(x_1)\tilde{w} + f''(x_1)(\tilde{d}, \tilde{d})$ and Theorem 3.3 yields

$$\begin{aligned} \mathcal{J}^2\left(f^{\mathcal{L}_\theta}\right)(x)(d, d) &= \left(d^T A_1(x)d, d^T A_2(x)d, \ldots, d^T A_n(x)d\right)^T \\ &\quad + \left(0, d^T E_2(x)d, \ldots, d^T E_n(x)d\right)^T, \end{aligned}$$

where

$$A_1(x) := f''(x_1)I, \quad A_i(x) := f''(x_1)\begin{bmatrix} 0 & e_{i-1}^T \\ e_{i-1} & 0 \end{bmatrix}, \quad i = 2, 3, \ldots, n,$$

and

$$E_i(x) := f''(x_1)(\tan\theta - \cot\theta)\begin{pmatrix} 0 \\ \bar{d}_2 \end{pmatrix}\left(0, e_{i-1}^T\right), \quad i = 2, \ldots, n.$$

Because $f^{\mathcal{L}_\theta}$ is second-order differentiable at x, then $\mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(d, d)$ is a bilinear mapping. Since

$$\mathcal{J}^2 f^{\mathcal{L}_\theta}(x)(d, d) - \left(d^T A_1(x)d, d^T A_2(x)d, \ldots, d^T A_n(x)d\right)^T$$

is a bilinear mapping, this requires that for $i = 2, \ldots, n$,

$$d^T E_i(x)d = f^{''}(x_1)(\tan\theta - \cot\theta)d^T \begin{pmatrix} 0 \\ \bar{d}_2 \end{pmatrix} \left(0, e_{i-1}^T\right) d$$
$$= f^{''}(x_1)(\tan\theta - \cot\theta)\|d_2\|(d_2)_i$$

is also a bilinear mapping with respect to d, which holds if and only if $\tan\theta = \cot\theta$, i.e., $\theta = 45°$.
Case 2: For $x_2 \neq 0$, taking $d = tu_x^{(1)}$ in (12), we have

$$\begin{aligned} f\left(\lambda_1(x) + t\right) u_x^{(1)} &= f(\lambda_1(x))u_x^{(1)} + t\mathcal{J} f^{\mathcal{L}_\theta}(x)u_x^{(1)} \\ &\quad + \frac{1}{2}t^2\mathcal{J}^2 f^{\mathcal{L}_\theta}(x)\left(u_x^1, u_x^1\right) + o\left(t^2\right) \\ &= f\left(\lambda_1(x)\right) u_x^{(1)} + tf'\left(\lambda_1(x)\right) u_x^{(1)} \\ &\quad + \frac{1}{2}t^2\mathcal{J}^2 f^{\mathcal{L}_\theta}(x)\left(u_x^{(1)}, u_x^{(1)}\right) + o\left(t^2\right). \end{aligned}$$

This leads to

$$\begin{aligned} f(\lambda_1(x) + t) &= f(\lambda_1(x)) + tf'(\lambda_1(x)) \\ &\quad + \frac{1}{2}t^2\frac{1}{\|u_x^{(1)}\|^2}\left\langle \mathcal{J}^2 f^{\mathcal{L}_\theta}(x)\left(u_x^{(1)}, u_x^{(1)}\right), u_x^{(1)}\right\rangle + o\left(t^2\right), \end{aligned}$$

which implies

$$f^{''}\left(\lambda_1(x)\right) = \frac{1}{\|u_x^{(1)}\|^2}\left\langle \mathcal{J}^2 f^{\mathcal{L}_\theta}(x)\left(u_x^{(1)}, u_x^{(1)}\right), u_x^{(1)}\right\rangle.$$

The similar arguments can be used to obtain the formula of $f^{''}(\lambda_2(x))$. □

Putting Theorem 5.2 and Theorem 5.3 together, we immediately obtain the following result.

Theorem 5.3 *Let $x \in \mathbb{R}^n$ with spectral decomposition $x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}$ given as in (1). Then, the following statements hold.*

(a) For $x_2 \neq 0$, $f^{\mathcal{L}_\theta}$ is second-order differentiable at x if and only if f is second-order differentiable at $\lambda_i(x)$ for $i = 1, 2$.
(b) For $x_2 = 0$, $f^{\mathcal{L}_\theta}$ is second-order differentiable at x if and only if f is second-order differentiable at x_1 and $\theta = 45°$.

The below example illustrates that the converse statement in Theorem 5.3(b) is false when $\theta \neq 45°$.

Example 5.1 Consider $n = 2$ and $f(t) = t^2$. Then, by a simple calculation, we have

$$f^{\mathcal{L}_\theta}(x) = \begin{bmatrix} x_1^2 + x_2^2 \\ 2x_1x_2 + (\tan\theta - \cot\theta)|x_2|x_2 \end{bmatrix}.$$

Note that the function

$$|x_2|x_2 = \begin{cases} x_2^2, & \text{if } x_2 > 0, \\ 0, & \text{if } x_2 = 0, \\ -x_2^2, & \text{if } x_2 < 0. \end{cases}$$

is differentiable at $x_2 = 0$, but not second-order differentiable at $x_2 = 0$. Hence, $f^{\mathcal{L}_\theta}$ is not second-order differentiable at x with $x_2 = 0$ unless θ satisfies $\tan\theta = \cot\theta$, i.e., $\theta = 45°$.

To sum up, from Theorem 5.3 and Example 5.1, we conclude that

"$f^{\mathcal{L}_\theta}$ is second-order differentiable at $x \Longleftrightarrow f$ is second-order differentiable at $\lambda_i(x)$" is not always true.

This phenomenon differs from what occurs in the first-order differentiability case. Precisely, the relationship for the first-order differentiability is independent of the angle, while the relationship for second-order differentiability really depends on the angle.

6 Conclusions

The parabolic second-order directional differentiability and second-order differentiability of the circular cone function were discussed in this paper. These results belong to the second-order type of differentiability analysis and help us to understand the relationship between the vector-valued circular cone function and the given real-valued function more clearly. In particular, the parabolic second-order directional differentiability of projection operator was used to establish the expression of second-order tangent sets, which plays an important role to develop the second-order optimality conditions for circular programming problems. The second-order differentiability of the given real-valued function cannot ensure the second-order differentiability of circular cone function unless some additional assumption is given on the angle. This is a very interesting and surprising fact. It further indicates that some results holding in second-order cone setting, such as second-order cone monotonicity and convexity, cannot be extended to circular cone setting, because in the latter case the angle plays an important role [14,19]. Thus, the further study to discover the difference between second-order cone programming and circular cone programming is necessary.

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