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A continuation approach for solving binary quadratic program based on a class of NCP-functions

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ABSTRACT

In the paper, we consider a continuation approach for the binary quadratic program (BQP) based on a class of NCP-functions. More specifically, we recast the BQP as an equivalent minimization and then seeks its global minimizer via a global continuation method. Such approach had been considered in [11] which is based on the Fischer–Burmeister function. We investigate this continuation approach again by using a more general function, called the generalized Fischer–Burmeister function. However, the theoretical background for such extension can not be easily carried over. Indeed, it needs some subtle analysis.

1. Introduction

In this paper, we consider the following binary quadratic program (BQP)

$$\min x^T Qx + c^T x \quad \text{over} \quad x \in S, \tag{1}$$

where Q is an $n \times n$ symmetric matrix, c is a vector in $\mathbb{R}^n$ and S is the binary discrete set $\{0,1\}^n$. It is known that BQP is NP-hard and has a variety of applications in computer science, operations research and engineering, see [1,3,8,13,14] and references therein. There have been proposed several continuous approaches for solving BQP [9,12,15] which often need to cooperate with branch and bound algorithms or some heuristic strategies to generate an exact or approximate solution. In [10], another type of continuous approach was proposed which is to reformulate BQP as an equivalent mathematical programming problem with equilibrium constraints (MPEC) and then consider an effective algorithm to find its global solution. In this approach, many NCP-functions are employed to convert equilibrium constraints into a collection of quasi-linear equality constraints. Among others, the Fischer–Burmeister function $\phi_{\rm FB} : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\phi_{\rm FB}(a,b) = \sqrt{a^2 + b^2} - (a+b) \tag{2}$$

is a popular one. In this paper, we investigate this continuation approach again by using a more general function $\phi_p : \mathbb{R}^2 \to \mathbb{R}$, called the generalized Fischer–Burmeister function and defined by

$$\phi_p(a,b) := \|(a,b)\|_p - (a+b), \tag{3}$$

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where $p > 1$ is an arbitrary fixed real number and $\|(a,b)\|_p$ denotes the p-norm of (a,b), i.e., $\|(a,b)\|_p = \sqrt[p]{|a|^p + |b|^p}$. In other words, in the generalized FB function ϕ_p, we replace the 2-norm of (a,b) appeared in the FB function by a more general p-norm. The function ϕ_p is still an NCP-function, which naturally induces another NCP-function $\psi_p : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$\psi_p(a,b) := \frac{1}{2}|\phi_p(a,b)|^2. \tag{4}$$

For any given $p > 1$, the function ψ_p is shown to possess all favorable properties of the FB function $\psi_{\rm FB}$.

Traditionally, in the continuation approach for BQP, one needs to utilize the fact that

$$x \in \{0,1\}^n \Longleftrightarrow x_i = x_i^2, \quad i = 1,2,\ldots,n. \tag{5}$$

To the contrast, our proposed continuous optimization approach arises from the complementarity condition formulation of $0-1$ vector $x \in \{0,1\}^n$, which includes the equivalence (5) with redundant constraints

$$0 \leqslant x_i \leqslant 1, \quad i = 1,2,\ldots,n.$$

so that it can generate an integer feasible solution. For finding the global minimizer of our continuous optimization problem, we employ the similar way as in [10,11]. In summary, the method is to add a quadratic penalty term associated with its equilibrium constraints and a logarithmic barrier term associated with box constraints $-1 \leqslant x_i \leqslant 1, i = 1,2,\ldots,n$, respectively, to the objective function, and then construct a global smoothing function. Since the generalized Fischer–Burmeister function ψ_p is quasi-linear, the quadratic penalty for equilibrium constraints will make the convexity of the global smoothing function more stronger. Particularly, we have shown that the global smoothing function is strictly convex in the whole domain for barrier parameter large enough or in a subset of its domain for penalty parameter large enough. According to the feature above, we use a global continuation algorithm defined in [11] via a sequence of unconstrained minimization for this function with varying penalty and barrier parameters. Although the idea is brought from [11], as will be seen, the theoretical background for such extension can not be easily carried over. Indeed, it needs some subtle analysis for extending the background materials. Without loss of generally, in this paper we consider the case that $S = \{-1,1\}^n$. By a transformation $z = (x+e)/2$ for the variable x and the unit vector e in $\mathbb{R}$, we can extend the conclusions to the case $S = \{0,1\}^n$.

2. Continuous formulation based on Φ_p function

In this section we will reformulate (1) as an equivalent continuous optimization based on the ϕ_p function. As will be seen, the following equivalence plays a key role which says that a binary constraint $t \in \{a,b\}$ with $a,b \in \mathbb{R}$ is equivalent to a complementarity condition (or equilibrium constraint), i.e.,

$$t \in \{a,b\} \quad \Longleftrightarrow \quad t-a \geqslant 0, \quad b-t \geqslant 0, \quad (t-a)(t-b) = 0.$$

With this, the unconstrained BQP problem in (1) can be recast as a mathematical programming problem with equilibrium constraints (MPEC)

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & (1+x_i, 1-x_i) = 0, \quad i = 1,2,\ldots,n, \\ & 1+x_i \geqslant 0, \quad 1-x_i \geqslant 0, \quad i = 1,2,\ldots,n. \end{array} \tag{6}$$

In fact, given any NCP-function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the property of NCP-functions (see [6]) yields that the equilibrium constraint in (6) is indeed equivalent to an equality constraint associated with ϕ:

$$(1+x_i, 1-x_i) = 0, \quad 1+x_i \geqslant 0, \quad 1-x_i \geqslant 0, \Longleftrightarrow \phi(1+x_i, 1-x_i) = 0. \tag{7}$$

Thus we reformulate the original BQP problem, which together with (6) and (7), as the following continuous optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \phi(1+x_i, 1-x_i) = 0, \quad i = 1,2,\ldots,n \\ & -1 \leqslant x_i \leqslant 1, \; i = 1,2,\ldots,n. \end{array} \tag{8}$$

Accordingly, the global minimizer of (8) is the solution of (1). Note that although the box constraints $-1 \leqslant x_i \leqslant 1, i = 1,2,\ldots,n$ in (8) are indeed redundant, we keep them on purpose. Actually, we shall see that such constraints play a crucial role in the construction of a global smoothing function for problem (8) as was shown in [9,10]. Generally speaking, most NCP-functions are non-differentiable, such as the popular Fischer–Burmeister function in (2), the generalized Fischer–Burmeister function in (3), as well as the minimum function

$$\phi_{\min}(a,b) = \min\{a,b\}.$$

However, it is very interesting to observe that, when specializing ϕ in (8) as the generalized Fischer–Burmeister function, we can reach smooth constraint functions

$$\phi_p(1+x_i, 1-x_i) = \sqrt[p]{|1+x_i|^p + |1-x_i|^p} - 2 = 0, \quad i = 1, 2, \ldots, n$$

and consequently some usual nonlinear programming solvers can be employed to design an effective algorithm for solving problem (8). In view of this, we in this paper pay attention to the following equivalent continuous formulations reformulated by the generalized Fischer–Burmeister function:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \phi_p(1+x_i, 1-x_i) = 0, \quad i = 1, 2, \ldots, n \\ & -1 \leqslant x_i \leqslant 1, \quad i = 1, 2, \ldots, n. \end{array} \tag{9}$$

We also note that using the equivalence that $x_i \in \{-1, 1\} \Longleftrightarrow x_i^2 = 1$ gives another another type of continuous optimization:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x_i^2 = 1, \quad i = 1, 2, \ldots, n \\ & -1 \leqslant x_i \leqslant 1, \quad i = 1, 2, \ldots, n. \end{array} \tag{10}$$

The formulation of (10) looks simple and friendly at first glance, nonetheless, the following remarkable advantages explain why we still stick to the smooth constrained optimization problem (9):

(i) The quasi-linearity of generalized Fischer–Burmeister function implies that it feasible set tends to be convex.
(ii) The equality constraint conditions $\phi_p(1+x_i, 1-x_i) = 0, i = 1, 2, \ldots, n$ have incorporated the equivalent formulation $x_i^2 = 1, i = 1, 2, \ldots, n$, of $x \in \{-1, 1\}$ with its relaxation formulation $-1 \leqslant x_i \leqslant 1, i = 1, 2, \ldots, n$, which indicates that, when solving (9) with a penalty function method, an implicit interior point constraint is additionally imposed on.
(iii) From Proposition 2.1 as below, we see that the quadratic penalty function of equality constraints is strictly convex in a very large region when the penalty parameter is large enough.

These advantages have great contributions to searching for an optimal solution or a favorable suboptimal solution of (1), which will be shown later. Before we prove the main proposition, we first introduce several technical lemmas which are important for building up the background materials of our extension.

Lemma 2.1. *Let f, g be real-valued functions from $\mathbb{R}$ to $\mathbb{R}_+$. Suppose f, g satisfy*

(i) $f'(x) > 0$ and $g'(x) < 0$ for all $x \in (a, b)$,
(ii) $f''(x) < 0$ and $g''(x) < 0$ for all $x \in (a, b)$,
(iii) $(fg)'(a) < 0$ and $f(a) \geqslant g(a)$.

Then $(fg)'(x) < 0$ for all $x \in (a, b)$.

Proof. To achieve our result, we need to verify two things: (i) $(fg)'(a) < 0$ and (ii) $(fg)'(x)$ is decreasing on $x \in (a, b)$. We proceed these verifications as below.

(i) From the assumptions and the chain rule, it is clear that

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a) < 0.$$

(ii) Since $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$, we see that in order to show $(fg)'(x)$ is decreasing on $x \in (a, b)$, it is enough to argue both $f'(x)g(x)$ and $f(x)g'(x)$ are decreasing on (a, b). We look into the first term first. Note that

$$(f'(x)g(x))' = f''(x)g(x) + f'(x)g'(x) \leqslant 0 \quad \forall x \in (a, b),$$

because $f''(x) < 0, g(x) \geqslant 0, f'(x) > 0$ and $g'(x) < 0$. This claims that $f'(x)g(x)$ is decreasing on $x \in (a, b)$. The decreasing of $f(x)g'(x)$ over (a, b) can be concluded similarly.

Thus, from all the above, the proof is complete. □

The conclusion of next lemma is simple and neat, however, its arguments are very tedious. Indeed the main idea behind is approximation.

Lemma 2.2. *Let ψ_p be defined as in (4). Then, $\psi_p''(1+t, 1-t)$ is positive at $t = \pm\sqrt{2^{\frac{1}{3}} - 1}$ for all $p \geqslant 2$.*

Proof. For symmetry, we only prove the case of $t = \sqrt{2^{\frac{1}{3}} - 1}$. First, from direct computations and simplifying the expression of ψ_p'', we have

$$\psi_p''(1+t,1-t) = \frac{((1+t)^p+(1-t)^p)^{\frac{1}{p}}}{(1+t)^2(1-t)^2((1+t)^p+(1-t)^p)^2} \times F(p,t), \tag{11}$$

where $F(p,t) = f_0(p,t)[f_1(p,t)+f_2(p,t)+f_3(p,t)]+f_4(p,t)$ with

$$f_0(p,t) = ((1+t)^p+(1-t)^p)^{\frac{1}{p}},$$
$$f_1(p,t) = (1-t)^2(t+1)^{2p},$$
$$f_2(p,t) = (t+1)^2(1-t)^{2p},$$
$$f_3(p,t) = (2t^2+4p-6)(t+1)^p(1-t)^p,$$
$$f_4(p,t) = (8-8p)(1-t^2)^p.$$

Since the first term on the right side of (11) is always positive for all $p \geqslant 2$, it suffices to show that $F\left(p,\sqrt{2^{\frac{1}{3}}-1}\right) > 0$ for all $p \geqslant 2$. However, it is very hard to claim this fact directly. Our strategy is to construct a function $A : \mathbb{R} \to \mathbb{R}$ such that

$$A(p) \leqslant F\left(p,\sqrt{2^{\frac{1}{3}}-1}\right) \quad \forall p \geqslant 2. \tag{12}$$

The special feature for $A(p)$ is that it is easier to verify $A(p) \geqslant 0$ for all $p \geqslant 2$ so that our goal could be reached. Now, we proceed the proof by carrying out the aforementioned two steps.

Step (1): Construct a function $A(\cdot)$ satisfying (12). Indeed, the function $F(\cdot,\cdot)$ is composed of f_0, f_1, f_2, f_3 and f_4, so for each f_i, we will construct a corresponding piecewise function a_i such that $a_i(p) \leqslant f_i\left(p,\sqrt{2^{\frac{1}{3}}-1}\right)$ for $i = 0,1,2,3,4$. Then, combining them together to build up the function $A(\cdot)$. For making the reader understand more easier, we will give some pictures during the process of proof.

(i) First, we explain how to set up $a_0(p)$. Notice that the second derivative of f_0 with respect to p is positive at $t = \sqrt{2^{\frac{1}{3}}-1}$ for all $p \geqslant 2, f_0$ is strictly convex at $t = \sqrt{2^{\frac{1}{3}}-1}$ for all $p \geqslant 2$ (the detailed arguments are provided in Appendix A). Hence, we consider a real piecewise function defined as

$$a_0(p) = \begin{cases} \frac{-1}{8}(p-2)+2^{\frac{2}{3}} & \text{if} \quad 2 \leqslant p \leqslant -8\sqrt{2^{\frac{1}{3}}-1}-6+8(2^{\frac{2}{3}}), \\ \sqrt{2^{\frac{1}{3}}-1}+1 & \text{if} \quad p \geqslant -8\sqrt{2^{\frac{1}{3}}-1}-6+8(2^{\frac{2}{3}}). \end{cases}$$

Fig. 1 depicts the relation between $a_0(p)$ and $f_0\left(p,\sqrt{2^{\frac{1}{3}}-1}\right)$. Besides, the following facts

$$a_0(2) = f_0\left(2,\sqrt{2^{\frac{1}{3}}-1}\right),$$
$$\lim_{p\to 2^+} a_0'(p) < \frac{d}{dp}f_0\left(2,\sqrt{2^{\frac{1}{3}}-1}\right),$$
$$a_0''(p) = 0 < \frac{d^2}{dp^2}f_0\left(p,\sqrt{2^{\frac{1}{3}}-1}\right)$$

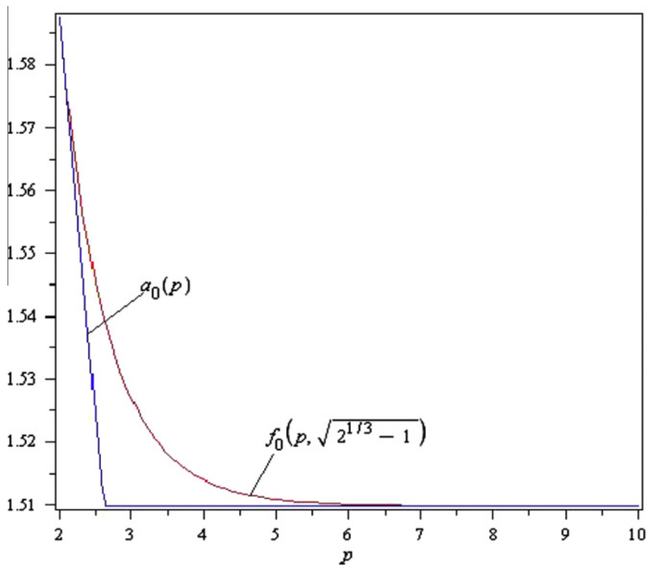

Fig. 1. The graphs of a_0 and f_0.

indicate the first part of function $a_0(p)$ is less than $f_0\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ for $2 < p \leqslant -8\sqrt{2^{\frac{1}{3}}-1} - 6 + 8(2^{\frac{2}{3}})$. On the other hand, another fact

$$\lim_{p\to\infty} f_0\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) = \sqrt{2^{\frac{1}{3}}-1} + 1$$

says that the second part of function $a_0(p)$ is less than or equal to $f_0\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ for $p \geqslant -8\sqrt{2^{\frac{1}{3}}-1} - 6 + 8(2^{\frac{2}{3}})$. Thus, we conclude that

$$a_0(p) \leqslant f_0\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) \quad \forall p \geqslant 2.$$

(ii) Secondly, we consider a quadratic function defined as

$$a_1(p) = \left(1 - \sqrt{2^{\frac{1}{3}}-1}\right)^2 \left(1 + \sqrt{2^{\frac{1}{3}}-1}\right)^4 \ln\left(1 + \sqrt{2^{\frac{1}{3}}-1}\right)(p-1)^2 + \left(1 - \sqrt{2^{\frac{1}{3}}-1}\right)^2 \left(1 + \sqrt{2^{\frac{1}{3}}-1}\right)^4 \left[1 - \ln\left(1 + \sqrt{2^{\frac{1}{3}}-1}\right)\right].$$

Fig. 2 depicts the relation between $a_1(p)$ and $f_1\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$. Again, using the following facts

$$a_1(2) = f_1\left(2, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$a_1'(2) = \frac{d}{dp} f_1\left(2, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$a_1''(p) \leqslant \frac{d^2}{dp^2} f_1\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) \quad \forall p \geqslant 2,$$

we immediately achieve

$$a_1(p) \leqslant f_1\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) \quad \forall p \geqslant 2.$$

(iii) Thirdly, we consider a function defined as

$$a_2(p) = \begin{cases} -\frac{1}{5}(p-2) + \left(1 - \sqrt{2^{\frac{1}{3}}-1}\right)^4 \left(1 + \sqrt{2^{\frac{1}{3}}-1}\right)^2 \\ \quad \text{if} \quad 2 \leqslant p \leqslant 12 + 20\left(2^{\frac{1}{3}} - 2^{\frac{2}{3}}\right) + \sqrt{2^{\frac{1}{3}}-1}\left(40(2^{\frac{1}{3}}) - 40 - 10(2^{\frac{2}{3}})\right), \\ 0 \\ \quad \text{if} \quad p \geqslant 12 + 20\left(2^{\frac{1}{3}} - 2^{\frac{2}{3}}\right) + \sqrt{2^{\frac{1}{3}}-1}\left(40(2^{\frac{1}{3}}) - 40 - 10(2^{\frac{2}{3}})\right). \end{cases}$$

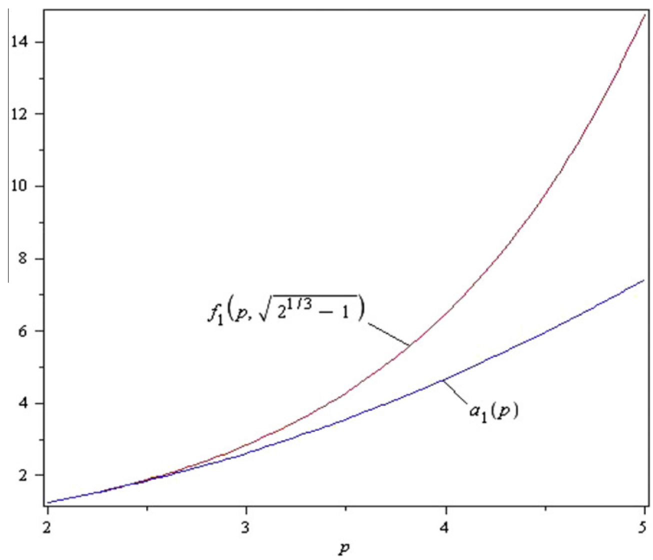

Fig. 2. The graphs of a_1 and f_1.

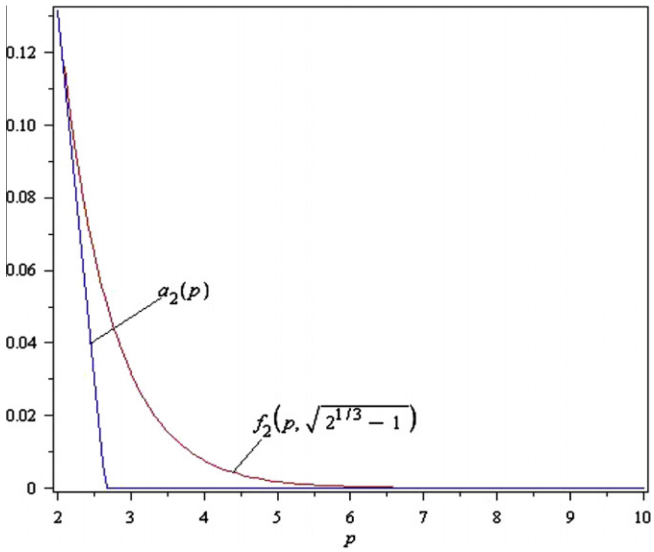

Fig. 3. The graphs of a_2 and f_2.

Fig. 3 depicts the relation between $a_2(p)$ and $f_2\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$. We observe that the function f_2 is positive and convex on $p \geqslant 2$, then the following facts

$$a_2(2) = f_2\left(2, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$\lim_{p\to 2^+} a_2'(p) < \frac{d}{dp} f_2\left(2, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$a_2''(p) = 0 < \frac{d^2}{dp^2} f_2\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) \quad \forall p > 2$$

yield $a_2(p) \leqslant f_2\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ for all $p \geqslant 2$.

(iv) Fourthly, we consider a real piecewise function defined as

$$a_3(p) = \begin{cases} \left[\sqrt{2-2^{\frac{1}{3}}}\left(24-12(2^{\frac{2}{3}})\right)+16(2^{\frac{2}{3}}-2^{\frac{1}{3}})-8\right]p & \\ \quad +\sqrt{2-2^{\frac{1}{3}}}(24(2^{\frac{2}{3}})-48)+40(2^{\frac{2}{3}}-2^{\frac{1}{3}})+20 & \text{if} \quad 2 \leqslant p \leqslant \frac{5}{2}, \\ -\left(\frac{4}{31}2^{\frac{1}{3}}+\frac{4}{31}\right)(2-2^{\frac{1}{3}})^{\frac{5}{2}}p+\left(\frac{72}{31}2^{\frac{1}{3}}+\frac{72}{31}\right)(2-2^{\frac{1}{3}})^{\frac{5}{2}} & \text{if} \quad \frac{5}{2} \leqslant p \leqslant 18, \\ 0 & \text{if} \quad p \geqslant 18. \end{cases}$$

Fig. 4 depicts the relation between $a_3(p)$ and $f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$. The relation is clear from the picture, however, we need to go through three subcases to verify it mathematically.

If $2 \leqslant p \leqslant \frac{5}{2}$, we compute $f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) = \left(2(2^{\frac{1}{3}})-8+4p\right)(2-2^{\frac{1}{3}})^p$. Moreover, we have

$$\frac{d}{dp} f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) = (2-2^{\frac{1}{3}})^p\left[4+\left(2(2^{\frac{1}{3}})-8+4p\right)\ln(2-2^{\frac{1}{3}})\right],$$
$$\frac{d^2}{dp^2} f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) = (2-2^{\frac{1}{3}})^p \ln(2-2^{\frac{1}{3}})\left[8+\left(2(2^{\frac{1}{3}})-8+4p\right)\ln(2-2^{\frac{1}{3}})\right].$$

Then, the following facts

$$a_3(2) = f_3\left(2, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$a_3\left(\frac{5}{2}\right) = f_3\left(\frac{5}{2}, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$\lim_{p\to 2^+} a_3'(p) \leqslant \frac{d}{dp} f_3\left(2, \sqrt{2^{\frac{1}{3}}-1}\right),$$

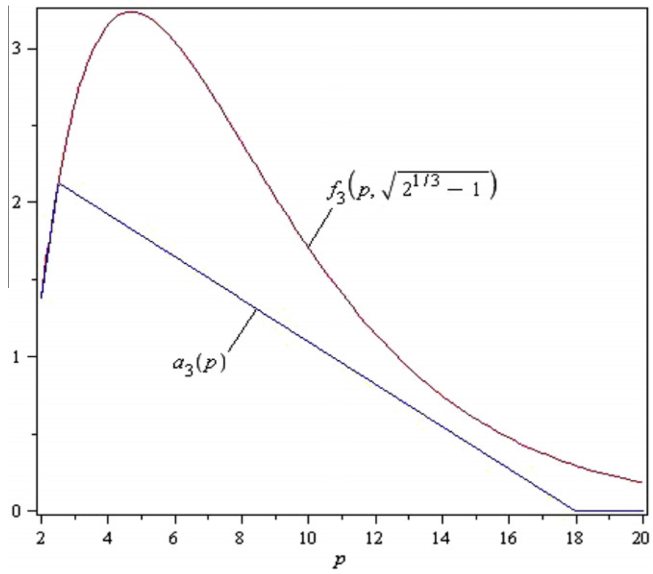

Fig. 4. The graphs of a_3 and f_3.

and $f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ being concave on $\left[2, \frac{5}{2}\right]$ imply $a_3(p) \leqslant f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ under this case.

If $\frac{5}{2} \leqslant p \leqslant 18$, using the facts that

$$a_3\left(\frac{5}{2}\right) = f_3\left(\frac{5}{2}, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$\lim_{p\to\frac{5}{2}^+} a_3'(p) \leqslant \frac{d}{dp} f_3\left(\frac{5}{2}, \sqrt{2^{\frac{1}{3}}-1}\right),$$

and $a_3(p) = f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ having only one solution at $p = \frac{5}{2}$, we obtain $a_3(p) \leqslant f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ under this case.

If $p \geqslant 18$, knowing $f_3(p) > 0$ for all p, then it is clear that $a_3(p) \leqslant f_3\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ under this case.

(v) Finally, notice that the second derivative of f_4 with respect to p is positive at $t = \sqrt{2^{\frac{1}{3}}-1}$ for all $p \geqslant \frac{-2+\ln(2-2^{\frac{1}{3}})}{\ln(2-2^{\frac{1}{3}})}$, and negative for $p \leqslant \frac{-2+\ln(2-2^{\frac{1}{3}})}{\ln(2-2^{\frac{1}{3}})}$, so f_4 is strictly convex at $t = \sqrt{2^{\frac{1}{3}}-1}$ for all $p \geqslant \frac{-2+\ln(2-2^{\frac{1}{3}})}{\ln(2-2^{\frac{1}{3}})}$ and strictly concave for all $p \leqslant \frac{-2+\ln(2-2^{\frac{1}{3}})}{\ln(2-2^{\frac{1}{3}})}$. Hence, we consider a real piecewise function defined as

$$a_4(p) = \begin{cases} -\frac{153}{50}p - 8(2-2^{\frac{1}{3}})^2 + \frac{153}{25} & \text{if} \quad 2 \leqslant p \leqslant \frac{5}{2}, \\ \left[-\frac{497}{50} + 16(2-2^{\frac{1}{3}})^2\right]p + \frac{583}{25} - 48(2-2^{\frac{1}{3}})^2 & \text{if} \quad \frac{5}{2} \leqslant p \leqslant 3, \\ -\frac{13}{10}p - \frac{13}{5} & \text{if} \quad 3 \leqslant p \leqslant \frac{49}{13}, \\ -\frac{15}{2} & \text{if} \quad p \geqslant \frac{49}{13}. \end{cases}$$

Fig. 5 depicts the relation between $a_4(p)$ and $f_4\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$. Again, we need to discuss several subcases to prove the relation mathematically.

For $2 \leqslant p \leqslant \frac{5}{2}$, the following facts

$$a_4(2) = f_4\left(2, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$\lim_{p\to 2^+} a_4'(p) < \frac{d}{dp} f_4\left(2, \sqrt{2^{\frac{1}{3}}-1}\right),$$
$$a_4''(p) = 0 < \frac{d^2}{dp^2} f_4\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$$

yield the first part of function $a_4(p)$ is less than $f_4\left(p, \sqrt{2^{\frac{1}{3}}-1}\right)$ under this case.

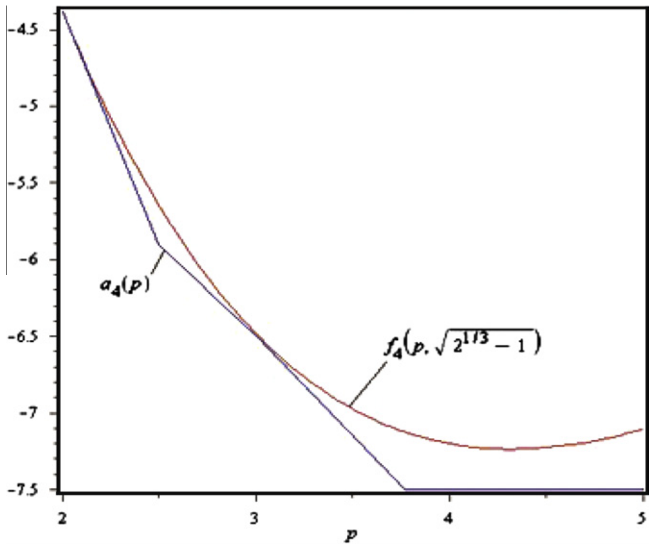

Fig. 5. The graphs of a_4 and f_4.

For $\frac{5}{2} \leqslant p \leqslant 3$, using the following facts

$$\begin{aligned} &a_4(3) < f_4\left(3, \sqrt{2^{\frac{1}{3}} - 1}\right), \\ &\lim_{p\to 3^-} a_4'(p) > \frac{d}{dp} f_4\left(3, \sqrt{2^{\frac{1}{3}} - 1}\right), \\ &a_4''(p) = 0 < \frac{d^2}{dp^2} f_4\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right). \end{aligned} \tag{13}$$

We have $a_4(p)$ is less than $f_4\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)$ under this case.

For $3 \leqslant p \leqslant \frac{49}{13}$, we know that

$$\lim_{p\to 3^+} a_4'(p) > \frac{d}{dp} f_4\left(3, \sqrt{2^{\frac{1}{3}} - 1}\right).$$

This together with (13) gives $a_4(p)$ is less than $f_4\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)$ under this case.

For $p \geqslant \frac{49}{13}$, from $f_4(p)$ being strictly convex for all $p \leqslant \frac{-2+\ln(2-2^{\frac{1}{3}})}{\ln(2-2^{\frac{1}{3}})}$ and being strictly concave for all $p \geqslant \frac{-2+\ln(2-2^{\frac{1}{3}})}{\ln(2-2^{\frac{1}{3}})}$, we know

$$\frac{d}{dp} f_4\left(\frac{-1+\ln(2-2^{\frac{1}{3}})}{\ln(2-2^{\frac{1}{3}})}\right) = 0 \quad \text{and} \quad \lim_{p\to\infty} f_4(p) = 0,$$

which lead to $f_4(p) > -\frac{15}{2}$ for all $p \leqslant 2$. Thus, $a_4(p) \leqslant f_4\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)$ under this case.

Now, we are ready to define a function $A : \mathbb{R} \to \mathbb{R}$ satisfying (12). As the mentioned idea, the function is defined by

$$A(p) = a_0(p)[a_1(p) + a_2(p) + a_3(p)] + a_4(p).$$

According to our constructions of $a_i(p)$, it is clear that $A(p) \leqslant F\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)$ for all $p \geqslant 2$. Fig. 6 shows the relation between $A(p)$ and $F\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)$.

Step (2): We will show that $A(p) \geqslant 0$ for all $p \geqslant 2$. Notice that $A(p)$ is piecewise smooth, hence $A'(p)$ is a piecewise function. Indeed, the expression of $A'(p)$ looks very ugly and tedious, we display it Appendix B. Furthermore, we also present an approximate expression for $A'(p)$ in Appendix C which helps us understand the structure of $A'(p)$. The key point is that from the expression of the $A'(p)$, we can verify the following facts:

$$\begin{aligned} &A(2) = 0, \\ &A\left(-8\sqrt{2^{\frac{1}{3}} - 1} - 6 + 8(2^{\frac{2}{3}})\right) > A\left(\frac{5}{2}\right) > 0, \end{aligned}$$

and

$$\begin{cases} A'(p) < 0 & \text{if} \quad p \in \left(\frac{5}{2}, -8\sqrt{2^{\frac{1}{3}} - 1} - 6 + 8(2^{\frac{2}{3}})\right), \\ A'(p) > 0 & \text{otherwise}, \end{cases}$$

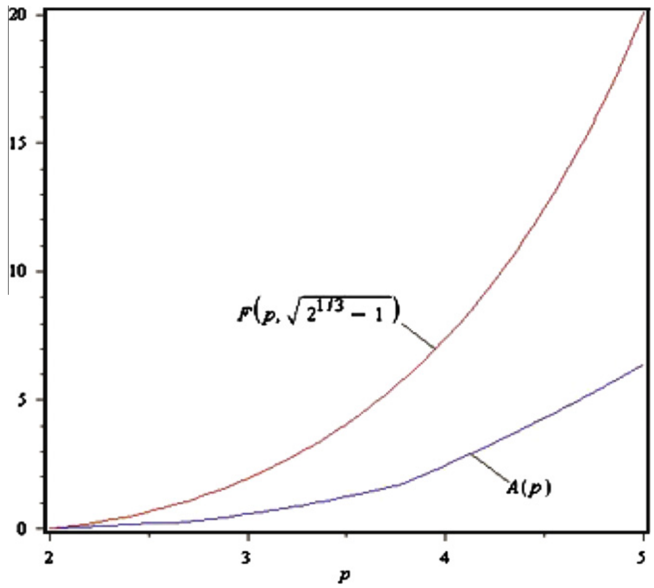

Fig. 6. The graphs of A and F.

with the exception of points of discontinuity. Thus, we conclude $A(p) \geqslant 0$ for all $p \geqslant 2$ and (12) is satisfied, which imply $F\left(p, \sqrt{2^{\frac{1}{3}}-1}\right) \geqslant 0$ for all $p \geqslant 2$. Then, the proof is complete. □

Lemma 2.3

(a) *Let f be a convex function defied on a convex set C in $\mathbb{R}^n$ and g be a nondecreasing convex function defined on an interval I in $\mathbb{R}$. Suppose $f(C) \subseteq I$. Then, the composite function $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$ is convex on C.*
(b) *Suppose $\phi_1 : \mathbf{U} \to \mathbb{R}$ is a twice continuously differentiable function with a compact set $\mathbf{U} \in \mathbb{R}^n$ and $\phi_2 : \mathbf{X} \to \mathbb{R}$ is a twice continuously differentiable function such that the minimum eigenvalue of its Hessian matrix $\nabla^2_{xx}\phi_2(x)$ is greater than $\varepsilon\,(>0)$ for all $x \in \mathbf{X}$, where $\mathbf{X} \subset \mathbf{U}$. Then there exists a constant $\hat{\beta} > 0$ such that $\phi_1 + \beta\phi_2$ is a strictly convex function on $\mathbf{X}$ for $\beta > \hat{\beta}$.*

Proof

(a) See [2, ChapIII, Lemma1.4].
(b) See [9, Theorem3.1]. □

Proposition 2.1. *Let ϕ_p and ψ_p be defined as in (3) and (4), respectively. Then, for any fixed $p \geqslant 2$, the following hold.*

(a) The function $\phi_p(1+t, 1-t)$ is strictly convex for all $t \in \mathbb{R}$.
(b) The function $\psi_p(1+t, 1-t)$ is strictly convex for all $t \notin \left[-\sqrt{2^{\frac{1}{3}}-1}, \sqrt{2^{\frac{1}{3}}-1}\right]$.

Proof

(a) It is known know that ϕ_p is a convex function [4–6]. Note that f is a composition of ϕ_p and an affine function. Thus, f is convex since it is a composition of a convex function and an affine function (the composition of two convex functions is not necessarily convex, however, our case does guarantee the convexity because one of them is affine).
(b) Due to the symmetry of $\psi_p(1+t, 1-t)$, it is enough to show that $\psi_p(1+t, 1-t)$ is strictly convex for $t \geqslant \sqrt{2^{\frac{1}{3}}-1}$. To proceed, we discuss two cases.
  (i) If $t \geqslant 1$, the function $\psi_p(1+t, 1-t)$ can be regard as a composite function of $\phi_p(1+t, 1-t)$ and $h(\cdot) = (\cdot)^2$. Because $h(\cdot)$ is nondecreasing convex function on $[0, \infty]$ and $\phi_p(1+t, 1-t)$ is positive strictly convex for $t \geqslant 1$, from Lemma 2.3, we obtain $\psi(1+t, 1-t)$ is strictly convex for $t \geqslant 2$.
  (ii) If $1 > t \geqslant \sqrt{2^{\frac{1}{3}}-1}$, we know that

$$-\psi_p'(1+t,1-t) = -\phi_p(1+t,1-t)\phi_p'(1+t,1-t),$$
$$-\psi_p''(1+t,1-t) = \left[-\phi_p'(1+t,1-t)\phi_p'(1+t,1-t) - \phi_p(1+t,1-t)\phi_p''(1+t,1-t)\right].$$

Then, it suffices to show that $-\psi_p''(1+t,1-t) < 0$ for $p \geqslant 2$. To this end, we compute the third derivative of $\phi_p(1+t,1-t)$ with respect to t and prove that it is negative. To see this,

$$\phi_p'''(1+t,1-t) = \frac{4\left[(1+t)^p + (1-t)^p\right]^{\frac{1}{p}}(1+t)^p(1-t)^p(p-1)}{(1+t)^3(t-1)^3\left[(1+t)^p + (1-t)^p\right]^3} \times T(p,t), \tag{14}$$

where T is a real valued function defined by

$$T(p,t) = (1+t)^p(2p-1-3t) - (1-t)^p(2p+3t-1).$$

It is not hard to verify the first term of the right side of (14) is always negative for all $p \geqslant 2$. Thus, we only need to show $T(p,t) > 0$ for all $p \geqslant 2$ which is equivalent to verifying $T(2,t) > 0$ and $T(p,t) > T(2,t)$ for all $p > 2$. These can be done as below.

(i) Because $T(2,t) = 6t - 6t^3$, it is clear $T(2,t) > 0$.
(ii) To show that $T(p,t) > T(2,t)$ for $p > 2$, we first argue that

$$(1+t)^p > (1-t)^{p-1}(2p+3t-1) \quad \forall p > 2, \tag{15}$$

It is equivalent to show that $\frac{(1+t)^p}{(1-t)^{p-1}(2p+3t-1)}$ is greater than 1 for all $p > 2$. Therefore, we consider the derivative of the following function with respect to p as follows:

$$\frac{d}{dp}\frac{(1+t)^p}{(1-t)^{p-1}(2p+3t-1)} = \frac{(1+t)^p}{(1-t)^{p-1}(2p+3t-1)^2} \times [(1-3t-2p)\ln(1-t) + (2p+3t-1)\ln(1+t) - 2]. \tag{16}$$

Observing both terms of the right side of (16) are positive for all $p > 2$ and using $\frac{(1+t)^p}{(-1+3t+2p)(1-t)^{p-1}} > 1$ when p = 2, we can achieve (15). Secondly, we know that

$$2p - 1 - 3t > 1 - t \quad \forall p > 2. \tag{17}$$

Combining (15) and (17), we have $T(p,t) \geqslant T(2,t)$. Hence,

$$\phi_p'''(1+t,1-t) < 0 \quad \forall p \geqslant 2.$$

Then, applying Lemma 2.1 gives the desired result for which we set $f(t) = -\phi_p(1+t,1-t)$ and $g(t) = \phi_p'(1+t,1-t)$. □

The result of Proposition 2.1(b) could be improved under some sense. More specifically, the interval where $\psi_p(1+t,1-t)$ is strictly convex varies as long as p changes. We originally wish to figure out the exact interval where $\psi_p(1+t,1-t)$ is strictly convex for each p. However, it is very hard to find a closed form depending p to reflect this feature (indeed, it

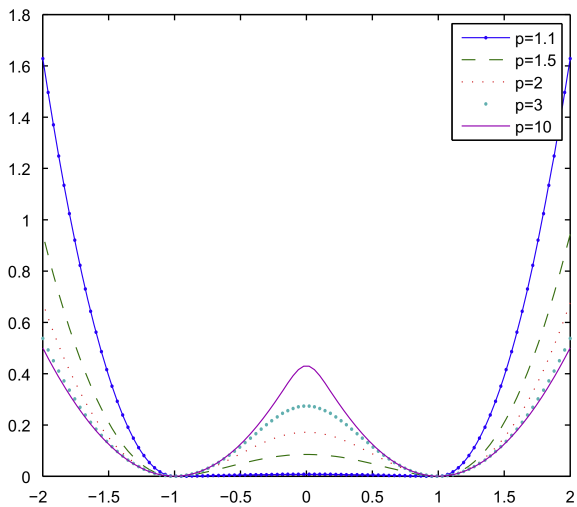

Fig. 7. The graphs of $\psi_p(1+t,1-t)$ for different p.

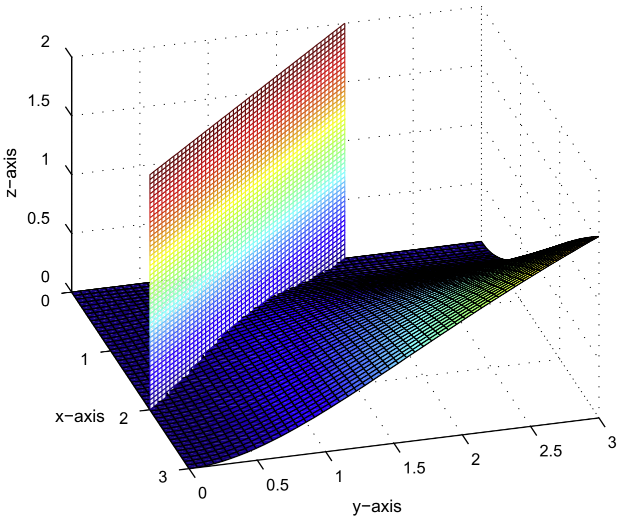

Fig. 8. The graph of $\psi_p(1+t, 1-t)$ with a fixed p.

may be not possible in our opinion). To compromise, we try to find such an appropriate common interval for all $p \geqslant 2$ as shown in Proposition 2.1(b). The following two figures (Figs. 7 and 8) depict the geometric view regarding what we just mentioned.

3. Global continuation algorithm for BQP

Due to the logarithmic barrier function being strictly convex and Proposition 2.1, we now introduce the quadratic penalty $\sum_{i=1}^n \psi_p(1+x_i, 1-x_i)$ for the equality constraints and the logarithmic barrier $-\sum_{i=1}^n [\ln(1+x_i) + \ln(1-x_i)]$ of the box constraints into the (9). Construct a global smoothing function

$$\phi(x, \alpha, \tau) = f(x) + \alpha \sum_{i=1}^n \psi_p(1+x_i, 1-x_i) - \tau \sum_{i=1}^n [\ln(1+x_i) + \ln(1-x_i)] \tag{18}$$

where $\tau > 0$ is a barrier parameter and $\alpha > 0$ is a penalty parameter. The next property indicates that the strictly convexity of function $\phi(x, \alpha, \tau)$ on $(-1, 1)^n$ when the barrier parameter is large enough, and the strictly convexity of function $\phi(x, \alpha, \tau)$ in a large subset of its domain for all $\tau > 0$.

Proposition 3.1. *Let $\phi(x, \alpha, \tau)$ be the function defined by (18). Then, the following hold.*

(a) There exists a constant $\hat{\tau} > 0$ such that if $\tau > \hat{\tau}$ and $\alpha > 0, \phi(x, \alpha, \tau)$ is strictly convex on $(-1, 1)^n$.
(b) There exists a constant $\hat{\alpha} > 0$ such that if $\alpha > \hat{\alpha}$ and $\tau > 0, \phi(x, \alpha, \tau)$ is strictly convex on the set $D := \left\{ x \in (-1, 1)^n \mid |x_i| > \sqrt{2^{\frac{1}{3}} - 1}, i = 1, 2, \ldots, n \right\}$.

Proof

(a) Let $X = (-1, 1)^n$ and denote

$$\phi_a(x) := f(x) + \alpha \sum_{i=1}^n \psi_p(1+x_i, 1-x_i),$$

$$\phi_b(x) := -\sum_{i=1}^n [\ln(1+x_i) + \ln(1-x_i)].$$

Then the expression of the Hessian matrix of $\phi_b(x)$ at any $x \in X$ is given by

$$\nabla^2_{xx} \phi_b(x) = \textbf{diag}\left(\frac{1}{(1-x_1)^2} + \frac{1}{(1+x_1)^2}, \ldots, \frac{1}{(1-x_n)^2} + \frac{1}{(1+x_n)^2} \right),$$

where $\mathrm{diag}(x)$ denotes a diagonal matrix with the components of x as the diagonal elements. Moreover, the function $\frac{1}{(1-x_i)^2}+\frac{1}{(1+x_i)^2}$ has minimum 2 at point $x_i=0$, and so every diagonal element of $\nabla^2_{xx}\phi_b(x)$ is at least 2. Thus, by letting $U=[-1,1]^n, \varepsilon=2$ and using Lemma 2.3(b) yield the desired result.

(b) Set $\phi_a=f(x)$ and

$$\phi_b=\sum_{i=1}^n\psi_p(1+x_i,1-x_i)-\frac{\tau}{\alpha}\sum_{i=1}^n[\ln(1+x_i)+\ln(1-x_i)].$$

From the proof of Lemma 2.2, it follows that

$$\nabla^2_{xx}\left(\sum_{i=1}^n\psi_p(1-x_i,1-x_i)\right)=\mathbf{diag}\Big(\psi''_p(1-x_1,1+x_1),\cdots,\psi''_p(1-x_n,1+x_n)\Big),$$

where $\psi''_p(1-x_i,1+x_i)$ can be found in (11). Now taking $f(t)=-\phi_p(1+t,1-t), g(t)=\phi'_p(1+t,1-t)$ and applying the proof (ii) of Lemma 2.1, we have

$$\psi''_p(1-t,1+t)>\psi''_2(1-t,1+t)\quad\forall p>2.$$

In addition, from [11] (Lemma 3.1), we also have

$$\psi''_b(1-t,1+t)=\frac{2\sqrt{(2t^2+2)^3}-8}{\sqrt{(2t^2+2)^3}}>0.0004\quad\forall|t|>0.51.$$

Therefore, the above two inequalities imply

$$\psi''_p(1-t,1+t)>0.0004\quad\forall\ |t|>0.51\quad\text{and}\quad\forall p\geqslant 2.$$

This indicates that every diagonal element of $\nabla^2_{xx}\psi_p$ is at least 0.0004. Using the fact that the Hessian matrix of $-\frac{\tau}{\alpha}\sum_{i=1}^n[\ln(1+x_i)+\ln(1-x_i)]$ is positive definite, we obtain that every diagonal element of $\nabla^2_{xx}\phi_b$ is at least 0.0004. Now taking

$$U=[-1,1]^n,\quad X=D\quad\text{and}\quad\varepsilon=0.0004$$

and applying Lemma 2.3 gives the desired conclusion. □

As remarked in [11], the result of Proposition 3.1 offers motivation to use the function $\phi(x,\alpha,\tau)$ to develop a global continuation algorithm for the constrained optimization problem (9). This method will generate a global optimal solution or at least a desirable local solution via a sequence of unconstrained minimization

$$\min_{x\in\mathbb{R}^n}\phi(x,\alpha_k,\tau_k)\tag{19}$$

with an increasing penalty parameter sequence $\{\alpha_k\}$ and a decreasing barrier parameter sequence $\{\tau_k\}$. Note that to ensure the strict convexity of $\phi(x,\alpha_k,\tau_k)$, we have to utilize a sufficiently large initial value τ_0 to start with the algorithm. As the iteration goes on, the convexity of logarithmic barrier $-\tau_k\sum_{i=1}^n[\ln(1+x_i)+\ln(1-x_i)]$ will become weak, but the strict convexity of $\phi(x,\alpha_k,\tau_k)$ can still be guaranteed due to the increasing of the penalty parameter α_k. This means that for each $k\in\mathrm{IN}$, the minimization problem (19) can be easily solved if we have skillful technique to adjust the parameter α and τ.

Algorithm 3.1

Step 0 Given parameters $\alpha_0,\tau_0,\sigma_1>1,\sigma_2\in(0,1)$ and $\epsilon>0$. Select a starting point $\hat{x}^0$ and set $k=0$.
Step 1 Solve the unconstrained minimization problem (19) with the starting point $\hat{x}^k$, and denote by x^k its optimal solution.
Step 2 If $\sqrt{\sum_{i=1}^n\psi_p(1-x_i^k,1-x_i^k)}\leqslant\epsilon$, terminate the algorithm, else go to Step 3.
Step 3 Update the parameters $\alpha_{k+1}=\sigma_1\alpha_k$ and $\tau_{k+1}=\sigma_2\tau_k$.
Step 4 Set $\hat{x}^{k+1}=\hat{x}^k, k=k+1$ and go to Step 1.

Is Algorithm 3.1 well-defined? To answer this, we give an existence theorem of solution for the unconstrained minimization problem (19). In fact, its proof can be found in [11] Lemma 3.2, we give a brief proof here for completeness.

Proposition 3.2. *Let $\phi(x,\alpha_k,\tau_k)$ be the function defined as in (18). Then, the following hold.*

(a) For each $k\in\mathrm{IN}$, the minimization problem (19) has a solution x^k.
(b) From (a), there exists an $\hat{\tau}$ such that the solution to problem (19) is unique when $\tau_k>\hat{\tau}$.

Proof

(a) We first show the existence of x^k for each $k \in \mathrm{IN}$. Let $X_1 = \left[-\frac{3}{4}, \frac{3}{4}\right]$. Since $\phi(x, \alpha_k, \tau_k)$ is continuous and X_1 is a compact set, there exist two real numbers L_1 and U_1 such that

$$L_1 \leqslant \phi(x, \alpha_k, \tau_k) \leqslant U_1 \quad \forall x \in X_1.$$

On the other hand, we note that $\phi(x, \alpha_k, \tau_k) \to +\infty$ when $x_{i_0} \to 1^-$ or $x_{i_0} \to 1^+$ for some $i_0 \in \{1, 2, \ldots, n\}$. Hence, the continuity of function $\phi(x, \alpha_k, \tau_k)$ implies that there exists an δ with $0 < \delta < 1/4$ such that

$$\phi(x, \alpha_k, \tau_k) \geqslant U_1 \quad \forall x \in ((-1, -1 + \delta] \cup [1 - \delta, 1))^n. \tag{20}$$

Let $X = [-1 + \delta, 1 - \delta]^n$. Again, $\phi(x, \alpha_k, \tau_k)$ being continuous on a compact set X implies that there exists an $\hat{x} \in X$ such that for each $k \in \mathrm{IN}$

$$\phi(\hat{x}, \alpha_k, \tau_k) \leqslant \phi(x, \alpha_k, \tau_k) \quad \forall x \in X.$$

Moreover, due to $X_1 \subseteq X$, we know

$$\phi(\hat{x}, \alpha_k, \tau_k) \leqslant U_1. \tag{21}$$

Combining (20) and (21) yields that

$$\phi(\hat{x}, \alpha_k, \tau_k) \leqslant \phi(x, \alpha_k, \tau_k) \quad \forall x \in (-1, 1)^n \setminus X.$$

Thus, together with (20), it shows that $\hat{x}$ is exactly the desired solution x^k.

(b) From conclusion of Proposition 3.1(a), $\phi(\hat{x}, \alpha_k, \tau_k)$ is strictly convex on $(-1, 1)^n$. Hence x^k is unique. □

4. Numerical experiments

In this section, we report numerical results of Algorithm 3.1 for solving the unconstrained binary quadratic programming problem. Our numerical experiments are carried out in Matlab (version 7.8) running on a PC Inter core 2 Q8200 of 2.33 GHz CPU and 2.00 GB Memory.

In our numerical experiments, we employ BFGS algorithm with strong Wolfe–Powell line search to solve the unconstrained minimization problem (19), and terminate the current iteration as long as x^k satisfies the following criterion:

$$\|\nabla_x \phi(x^k, \alpha_k, \tau_k)\| \leqslant 5.0e-3.$$

The values for the parameters involved in Algorithm 3.1 are chosen as follows:

$$\alpha_0 = 0, \quad \sigma_1 = 2, \quad \sigma_2 = 0.5, \quad \varepsilon = 1.0e-3,$$

and the initial barrier parameter τ_0 varies with the scale of problems (here we choose its value the same as that in [11]). The starting point $\hat{x}^0 = 0.9(1, 1, \ldots, 1)^T \in \mathbb{R}^n$ is used for all test problems. To obtain an integer solution x^* from the final iterate point $\hat{x}^*$ of Algorithm 3.1, we let

$$x_i^* = \begin{cases} -1 & \text{if } |\hat{x}_i^* + 1| \leqslant 1.0e-2 \\ 1 & \text{if } |\hat{x}_i^* - 1| \leqslant 1.0e-2 \end{cases} \quad \text{for } i = 1, 2, \ldots, n.$$

The test problems are all from the OR-Library and have the following formulation

$$\begin{array}{ll} \max & z^T Q z \\ \text{s.t.} & z_i \in \{0, 1\}, \quad i = 1, 2, \ldots, n. \end{array}$$

To solve these problems with Algorithm 3.1, we use the formula $z = (x + e)/z$ to transform them into the following formulation

$$\begin{array}{ll} -\min & -\frac{1}{4}x^T Q x - \frac{1}{2}x^T Q e - \frac{1}{4}e^T Q e \\ \quad \text{s.t.} & x_i \in \{-1, 1\}, \quad i = 1, 2, \ldots, n. \end{array}$$

The optimal values generated by Algorithm 3.1 with different p (p = 1.1,2,4,5,10,20,50,100) are listed in Tables 1–5 (see Appendix D), where '–' means that the algorithm fails to get an optimal solution when the maximum CPU time arrives. Moreover, to present the objective evaluation and comparison of the performance of Algorithm 3.1 with different p, we adopt the performance profile introduced in [7] as a means. In particular, we regard Algorithm 3.1 corresponding to a p as a solver and assume that there are n_s solvers and n_j test problems from the OR-Library collection $\mathcal{J}$. We are interested in using the optimal values calculated by Algorithm 3.1 as performance measure for different p. For each problem j and solver s, let

$$t_{j,s} := \text{ the optimal value of problem } j \text{ by solver } s, \quad \mu_{j,s} := \frac{1}{t_{j,s}}.$$

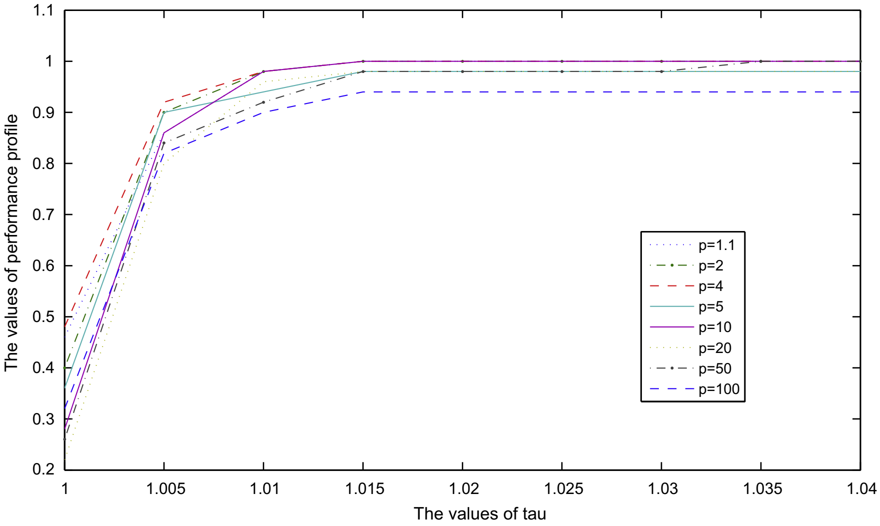

Fig. 9. Performance profile of the reciprocals of optimal values by Algorithm 3.1 with different p.

We compare the performance on problem j by solver s with the best performance by any one of the n_s solvers on this problem, i.e., we employ the performance ratio

$$r_{j,s} := \frac{\mu_{j,s}}{\min\{\mu_{j,s} : s \in \mathcal{S}\}} = \frac{\max\{t_{j,s} : s \in \mathcal{S}\}}{t_{j,s}},$$

where $\mathcal{S}$ is the set of eight solvers. An overall assessment of each solver is obtained from

$$\rho_s(\tau) := \frac{1}{n_j}\text{size}\{j \in \mathcal{J} : r_{j,s} \leqslant \tau\},$$

which is called the performance profile of the reciprocal of optimal solution obtained by Algorithm 3.1 for solver s.

Fig. 9 shows the performance profile of the reciprocals of optimal values obtained by Algorithm 3.1 in the range of $[1, 1.04]$ for eight solvers on 50 test problems. The eight solvers correspond to Algorithm 3.1 with $p = 1.1, p = 2, p = 4, p = 5, p = 10$ $, p = 20, p = 50$ and $p = 100$, respectively. From this figure, we see that Algorithm 3.1 are considerably efficient no matter which value of p is chosen. In fact, Algorithm 3.1 with the aforementioned p values can solve all the 50 test problems except for $p = 5, 20, 100$. Moreover, Algorithm 3.1 with $p = 4$ has the best numerical performance (has the highest probability of being the optimal solver) and the probability of its being the winner on a given BQP is around 0.48. Besides, $p = 1.1$ and $p = 2$ have a comparable performance with $p = 4$, please refer to Appendix D for more detailed numerical reports.

Appendix A

Here is the proof of the strictly convexity of $f_0\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)$ for all $p \geqslant 2$.

To see this, it is enough to verify that $\frac{d^2}{dp^2} f_0\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right) > 0$ for all $p > 2$. In fact,

$$\frac{d^2}{dp^2} f_0\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right) = f_0\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)\left[\frac{-\ln\left(f_0(p, \sqrt{2^{\frac{1}{3}} - 1})\right)^p}{p^2} + \frac{a^p \ln(a) + b^p \ln(b)}{p\left(f_0(p, \sqrt{2^{\frac{1}{3}} - 1})\right)^p}\right]^2 + \frac{1}{p^3\left(f_0\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)\right)^{2p-1}} \times M(p), \tag{22}$$

where $a = 1 + \sqrt{2^{\frac{1}{3}} - 1}, b = 1 - \sqrt{2^{\frac{1}{3}} - 1}$ and

$$\begin{aligned} M(p) = &\left[2\left(f_0\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)\right)^{2p} \ln\left(f_0\left(p, \sqrt{2^{\frac{1}{3}} - 1}\right)\right)^p - 2p(a)^{2p}(\ln a)\right] \\ &+ \left[p^2\left(2 - 2^{\frac{1}{3}}\right)^p (\ln b)^2 - 2p^2\left(2 - 2^{\frac{1}{3}}\right)^p (\ln a)(\ln b)\right] + \left[p^2\left(2 - 2^{\frac{1}{3}}\right)^p (\ln a)^2 - 2p\left(2 - 2^{\frac{1}{3}}\right)^p (\ln a)\right] \\ &+ \left[-2p\left(2 - 2^{\frac{1}{3}}\right)^p (\ln b) - 2p(b^{2p})(\ln b) - 2p(b^{2p})(\ln b)\right]. \end{aligned} \tag{23}$$

The value of $M(p)$ is always positive for all $p \geqslant 2$ since the four terms on the right-hand side of (23) are all positive. This together with the fact that all terms on the right-hand side of (22) are also positive yields the desired result. □

Appendix B

We present the expression of the function $A'(p)$ as below.

$$A'(p) = \begin{cases} I_1(p) & \text{if} \quad 2 \leqslant p < \frac{5}{2} \\ I_2(p) & \text{if} \quad \frac{5}{2} \leqslant p < -8\sqrt{2^{\frac{1}{3}}-1} - 6 + 8(2^{\frac{2}{3}}) \\ I_3(p) & \text{if} \quad -8\sqrt{2^{\frac{1}{3}}-1} - 6 + 8(2^{\frac{2}{3}}) \leqslant p \\ & \quad\; < 12 + 20(2^{\frac{1}{3}}) + \sqrt{2^{\frac{1}{3}}-1}\left[40(2^{\frac{1}{3}}) - 40 - 10(2^{\frac{2}{3}})\right] - 20(2^{\frac{2}{3}}) \\ I_4(p) & \text{if} \quad 12 + 20(2^{\frac{1}{3}}) + \sqrt{2^{\frac{1}{3}}-1}\left[40(2^{\frac{1}{3}}) - 40 - 10(2^{\frac{2}{3}})\right] - 20(2^{\frac{2}{3}}) \\ & \quad\; \leqslant p < 3 \\ I_5(p) & \text{if} \quad 3 \leqslant p < \frac{49}{13} \\ I_6(p) & \text{if} \quad \frac{49}{13} \leqslant p < 18 \\ I_7(p) & \text{if} \quad p \geqslant 18 \end{cases}$$

where

$$\begin{aligned} I_1(p) = \Big[& -\frac{1}{2}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right) - \ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}}) - 2\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)\sqrt{2^{\frac{1}{3}}-1} + \ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}}) \\ & + 2\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1} - \frac{1}{2}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1}\Big]p^2 \\ & + \Big[\frac{4}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1} - 6\sqrt{2^{\frac{1}{3}}-1} + \frac{41}{20} + \frac{35}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right) + 4(2^{\frac{1}{3}}) \\ & - \frac{26}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}}) - 4(2^{\frac{2}{3}}) + \frac{35}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + 3\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}}) \\ & - \frac{52}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)\sqrt{2^{\frac{1}{3}}-1} + \frac{2}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}})\Big]p - \frac{16}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right) \\ & + 18\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}}) - \frac{1004}{25} + 22(2^{\frac{1}{3}}) + \frac{9}{5}(2^{\frac{2}{3}}) - \frac{16}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} - 24(2^{\frac{1}{3}})\sqrt{2-2^{\frac{1}{3}}} \\ & + 12\sqrt{2-2^{\frac{1}{3}}} + \frac{32}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)\sqrt{2^{\frac{1}{3}}-1} - \frac{4}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}}) + \frac{16}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)\left(2^{\frac{1}{3}}\right) \\ & - \frac{8}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1} \end{aligned}$$

$$\begin{aligned} I_2(p) = \Big[& -\frac{1}{2}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right) - \ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}}) - 2\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)\sqrt{2^{\frac{1}{3}}-1} \\ & + \ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}} + 2\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1} - \frac{1}{2}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1}\Big]p^2 \\ & + \Big[\frac{4}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{6}{31}\sqrt{2^{\frac{1}{3}}-1} + \frac{1}{20} + \frac{35}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right) - \frac{26}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}}) \\ & + \frac{35}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} - \frac{6}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}}) + \frac{2}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}}) \\ & - \frac{52}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)\sqrt{2^{\frac{1}{3}}-1}\Big]p - \frac{16}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right) + \frac{6}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}}) + \frac{2679}{50} - 65(2^{\frac{1}{3}}) \\ & + \left(2^{\frac{2}{3}}\right)\frac{84}{5} - \frac{16}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} - \frac{60}{31}\sqrt{2-2^{\frac{1}{3}}} + \frac{24}{31}(2^{\frac{1}{3}})\sqrt{2-2^{\frac{1}{3}}} \\ & + \frac{32}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)\sqrt{2^{\frac{1}{3}}-1} - \frac{4}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{2}{3}}) \\ & + \frac{16}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}}) - \frac{8}{3}\ln\left(1+\sqrt{2^{\frac{1}{3}}-1}\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1} \end{aligned}$$

$$\begin{aligned}
I_3(p) = {} & \Big[-\frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) - \frac{56}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}}) + \frac{80}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) \\
& + \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} - \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} \\
& - \frac{16}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1}\Big]p - \frac{20}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} + 16(2^{\frac{2}{3}}) - 64(2^{\frac{1}{3}}) \\
& + \frac{28}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) - \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) + \frac{2693}{50} + \frac{12}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}}) \\
& - \frac{24}{31}\sqrt{2-2^{\frac{1}{3}}} - \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} + \frac{12}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) \\
& + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1}(2^{\frac{1}{3}}) - \frac{24}{31}\sqrt{2-2^{\frac{1}{3}}}\sqrt{2^{\frac{1}{3}}-1}
\end{aligned}$$

$$\begin{aligned}
I_4(p) = {} & \Big[-\frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) - \frac{56}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}}) + \frac{80}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) \\
& + \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} - \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} - \frac{16}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1}\Big]p \\
& - \frac{20}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} + 16(2^{\frac{2}{3}}) - 64(2^{\frac{1}{3}}) + \frac{28}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) - \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) \\
& + \frac{2693}{50} + \frac{12}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}}) - \frac{24}{31}\sqrt{2-2^{\frac{1}{3}}} - \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} + \frac{12}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) \\
& + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1}(2^{\frac{1}{3}}) - \frac{24}{31}\sqrt{2-2^{\frac{1}{3}}}\sqrt{2^{\frac{1}{3}}-1} + \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} + \frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
I_5(p) = {} & \Big[-\frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) - \frac{56}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}}) + \frac{80}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) \\
& + \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} - \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} - \frac{16}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1}\Big]p \\
& - \frac{20}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} + \frac{28}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) - \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) \\
& - \frac{7}{10} + \frac{12}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}}) - \frac{24}{31}\sqrt{2-2^{\frac{1}{3}}} - \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} + \frac{12}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) \\
& + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1}(2^{\frac{1}{3}}) - \frac{24}{31}\sqrt{2-2^{\frac{1}{3}}}\sqrt{2^{\frac{1}{3}}-1} + \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} + \frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
I_6(p) = {} & \Big[-\frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) - \frac{56}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}}) + \frac{80}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) \\
& + \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} - \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} - \frac{16}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1}\Big]p \\
& - \frac{20}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} + \frac{28}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) - \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) \\
& + \frac{6}{10} + \frac{12}{31}\sqrt{2-2^{\frac{1}{3}}}(2^{\frac{2}{3}}) - \frac{24}{31}\sqrt{2-2^{\frac{1}{3}}} - \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} + \frac{12}{31}\sqrt{2-2^{\frac{1}{3}}}2^{\frac{2}{3}}\sqrt{2^{\frac{1}{3}}-1} + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) \\
& + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1}(2^{\frac{1}{3}}) - \frac{24}{31}\sqrt{2-2^{\frac{1}{3}}}\sqrt{2^{\frac{1}{3}}-1} + \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} + \frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
I_7(p) = {} & \Big[-\frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) - \frac{56}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}}) + \frac{80}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) \\
& + \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} - \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} - \frac{16}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}})\sqrt{2^{\frac{1}{3}}-1}\Big]p \\
& - \frac{20}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1} + \frac{28}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) - \frac{40}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{1}{3}}) + \frac{6}{10} - \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} \\
& + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right) + \frac{4}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)(2^{\frac{2}{3}})\sqrt{2^{\frac{1}{3}}-1} + \frac{8}{3}\ln\left(\sqrt{2^{\frac{1}{3}}-1}+1\right)\sqrt{2^{\frac{1}{3}}-1}(2^{\frac{1}{3}}) + \frac{1}{5}\sqrt{2^{\frac{1}{3}}-1} + \frac{1}{5}
\end{aligned}$$

Appendix C

Here is an approximate expression for the function $A'(p)$.

$$A(p) \doteqdot \begin{cases} -0.1285995252p^2 + 1.021170584p - 1.237710478 & \text{if} \quad 2 \leqslant p < 2.5 \\ -0.1285995252p^2 + 1.430290495p - 2.872510153 & \text{if} \quad 2.5 \leqslant p < 2.62061219 \\ 1.035534493p - 2.203712390 & \text{if} \quad 2.620612190 \leqslant p < 2.658005104 \\ 1.035534493p - 1.901747484 & \text{if} \quad 2.658005104 \leqslant p < 3 \\ 1.035534493p - 2.025217119 & \text{if} \quad 3 \leqslant p < 3.769230769 \\ 1.035534493p - .7252171187 & \text{if} \quad 3.769230769 \leqslant p < 18 \\ 1.035534493p - .5177672467 & \text{if} \quad p \geqslant 18 \end{cases}$$

Appendix D

See Tables 1–5.

Table 1
Numerical results for BQPs with 50 variables ($\tau_0 = 20$).

Prob.	$p = 1.1$		$p = 2$		$p = 4$		$p = 5$		$p = 10$		$p = 20$		$p = 50$		$p = 100$	
	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time
1	2098	1.28	2098	1.58	2098	0.76	2098	1.10	2098	1.94	2098	1.22	2098	0.86	2098	1.69
2	3702	1.61	3702	1.03	3702	1.07	3702	1.62	3702	2.34	3702	0.93	3702	2.17	3702	1.26
3	4626	0.73	4626	0.62	4626	0.77	4626	0.81	4626	0.97	4626	0.80	4626	0.78	4626	0.98
4	3544	1.10	3544	1.71	3544	2.44	3544	0.96	3544	2.96	3544	2.83	3438	1.48	3544	0.96
5	4012	0.87	4012	0.64	4012	1.11	4012	1.14	4012	1.24	4012	1.20	4012	0.83	4012	0.84
6	3693	1.67	3693	1.97	3693	2.56	3693	1.01	3693	2.47	3693	2.96	3693	2.09	3693	1.05
7	4510	1.29	4510	1.28	4510	2.08	4510	1.06	4510	2.67	4510	1.24	4510	1.22	4510	1.10
8	4216	1.31	4216	1.07	4216	2.46	4212	2.51	4216	2.58	4212	2.70	4216	1.26	4216	2.51
9	3744	1.74	3780	1.94	3780	1.19	3748	1.46	3748	2.98	3744	1.45	3732	2.25	3732	2.45
10	3507	1.55	3505	1.78	3507	2.03	3461	1.86	3499	3.39	3507	1.89	3507	3.01	3505	1.67

Table 2
Numerical results for BQPs with 100 variables ($\tau_0 = 20$).

Prob.	$p = 1.1$		$p = 2$		$p = 4$		$p = 5$		$p = 10$		$p = 20$		$p = 50$		$p = 100$	
	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time
1	7840	4.55	7822	3.30	7840	5.08	7840	4.80	7812	4.19	7812	3.28	7838	6.22	–	–
2	11036	4.62	10994	4.57	10996	10.28	11018	2.38	11028	3.98	10994	3.01	11030	4.01	11018	3.01
3	12723	3.00	12723	3.68	12652	2.65	12723	3.83	12723	5.09	12723	2.97	12723	3.25	12723	4.95
4	10362	5.15	10368	3.15	10368	7.94	10362	3.01	10368	4.46	10368	3.18	10368	3.53	10368	3.37
5	9083	5.26	9083	7.03	9040	2.16	9040	2.87	9040	2.15	9040	4.61	9045	2.96	9045	4.11
6	10130	6.76	10101	3.84	10202	6.69	10168	9.10	10202	3.86	10101	4.93	10092	4.22	10184	5.13
7	10063	3.83	10098	3.23	10098	4.92	10094	6.10	10098	4.78	10094	3.00	10094	2.61	10094	5.02
8	11435	3.65	11419	5.27	11435	4.47	11435	4.67	11435	8.03	11415	5.20	11415	5.50	11435	4.23
9	11455	3.28	11455	2.76	11455	2.62	11455	4.18	11380	4.37	11357	5.97	11437	3.32	11380	4.93
10	12547	3.91	12565	2.16	12565	3.63	12565	5.53	12523	2.51	12503	2.84	12565	2.01	12547	1.72

Table 3
Numerical results for BQPs with 250 variables ($\tau_0 = 40$).

Prob.	$p = 1.1$		$p = 2$		$p = 4$		$p = 5$		$p = 10$		$p = 20$		$p = 50$		$p = 100$	
	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time
1	45607	32.30	45571	32.66	45499	31.32	45463	21.04	45434	22.26	45571	30.77	45374	26.65	45607	34.02
2	44329	19.60	44810	19.33	44355	28.74	44738	28.45	44505	23.19	44273	21.14	44337	24.72	44810	18.64
3	49037	32.43	49037	21.83	48978	20.27	48947	36.48	48964	30.14	-	-	48978	24.75	48951	22.41
4	41188	19.99	41219	19.35	41254	28.08	41215	30.41	41199	17.08	41202	35.10	41199	38.74	40993	22.62
5	47821	28.42	47758	22.36	47876	27.72	47877	28.58	47823	22.80	47877	18.82	47928	23.62	47937	22.05
6	40625	34.13	40996	20.91	40768	24.00	41006	35.94	41006	24.28	40771	21.94	40839	28.15	40679	28.56
7	46484	23.62	46757	15.56	46732	20.70	46713	19.38	46689	32.30	46753	19.60	46667	29.59	46753	15.38
8	35572	21.63	35294	21.40	35666	16.17	35666	28.37	35416	37.27	35473	27.77	35726	20.51	35282	28.27
9	48605	18.06	48605	20.50	48733	37.23	48562	22.18	48733	22.98	48683	31.82	48677	18.94	48788	29.38
10	40442	30.15	40252	32.09	40442	26.57	39992	35.33	40308	30.46	40252	26.37	40330	20.59	40288	43.26

Table 4
Numerical results for BQPs with 500 variables ($\tau_0 = 70$).

Prob.	$p = 1.1$		$p = 2$		$p = 4$		$p = 5$		$p = 10$		$p = 20$		$p = 50$		$p = 100$	
	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time
1	115822	158.2	116458	178.1	116563	130.4	116310	112.5	115163	181.1	115944	118.7	115966	171.3	115976	158.1
2	127997	147.9	127895	156.9	127796	150.2	128036	139.1	127994	121.7	128045	100.8	127795	135.6	127690	140.5
3	130812	128.6	130790	189.8	130523	222.6	130714	169.6	130295	172.7	130782	120.6	130782	140.3	130744	137.5
4	129802	199.1	129992	191.5	129805	180.4	129706	127.7	129711	142.7	129860	192.0	129521	217.9	129706	112.4
5	125173	160.7	125243	122.9	125245	138.1	125399	137.7	125251	125.8	125135	141.1	125151	145.9	125333	137.4
6	121611	164.5	121586	133.4	121535	169.0	121603	112.9	121225	134.5	121589	98.6	120773	152.7	121480	146.6
7	121769	165.6	121721	132.4	121475	167.5	122104	123.5	121744	129.4	121240	204.2	121941	170.8	121800	114.0
8	123526	157.0	122330	189.5	123360	159.1	123401	184.1	123001	188.8	123323	148.2	123301	163.4	123391	94.3
9	119968	160.4	119977	171.5	120748	168.7	119780	180.3	120100	209.6	120594	174.5	120263	147.6	120697	104.4
10	130109	156.8	130181	165.8	129977	189.4	130180	141.4	129977	164.2	129797	180.0	129635	137.9	130273	195.3

Table 5
Numerical results for BQPs with 1000 variables ($\tau_0 = 180$).

Prob.	$p = 1.1$		$p = 2$		$p = 4$		$p = 5$		$p = 10$		$p = 20$		$p = 50$		$p = 100$	
	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time	Obj.	Time
1	371013	948.3	370515	948.7	370481	1001.3	370033	946.5	370618	1197.6	370723	969.7	370269	780.3	–	–
2	354453	1347.9	352389	1066.4	353768	762.0	352769	1100.7	354031	1009.2	352245	863.3	352667	903.3	353886	1033.3
3	370097	864.1	369638	841.0	369400	1230.1	370984	909.6	368351	917.6	369895	1270.0	370537	908.2	369622	848.0
4	370058	987.3	370426	1162.3	370171	1047.4	369628	1421.5	370113	945.7	369860	976.4	369800	818.8	369906	889.1
5	352187	868.4	352299	1053.4	351881	763.4	352410	668.8	352064	1013.0	352296	706.2	351846	1271.9	351753	886.5
6	357631	1231.9	358436	809.5	358528	862.2	358580	766.5	358425	837.3	357771	1056.5	358808	935.6	–	–
7	370033	856.1	369437	962.4	370198	1133.1	370192	850.4	369967	794.6	370062	928.0	369615	802.5	369982	660.0
8	350389	869.7	350727	780.7	350781	931.3	351013	795.7	350346	986.9	350982	747.1	350931	810.2	350128	1012.0
9	349038	1130.4	348232	1136.8	349205	679.6	348911	710.8	347864	716.7	347510	1031.6	347469	1094.8	347541	1219.2
10	349729	704.5	350476	916.6	350593	970.3	–	–	350366	795.2	350096	742.3	349842	644.1	349583	974.3

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