NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
2021, VOL. 42, NO. 13, 1572–1585
https://doi.org/10.1080/01630563.2021.2006693

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Characterizations of Boundary Conditions on Some Non-Symmetric Cones

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ABSTRACT
In contrast to symmetric cone optimization, there has no unified framework for non-symmetric cone optimization. One main reason is that the structure of various non-symmetric cone differs case by case. Especially, their boundary conditions are usually mysterious. In this paper, we provide characterizations of boundary conditions on some non-symmetric cones, including p-order cone, ellipsoidal cone, power cone and general closed convex cone. These results will be key bricks for further investigations on non-symmetric cone optimization accordingly.

ARTICLE HISTORY
Received 25 June 2021
Revised 10 November 2021
Accepted 10 November 2021

KEYWORDS
Power cone; p-order cone; ellipsoidal cone; non-symmetric cone

1. Introduction

In contrast to symmetric cone optimization, there has no unified framework for non-symmetric cone optimization. One main reason is that the structure of various non-symmetric cone differs case by case. In the literature, the study regarding non-symmetric cones, focuses on homogeneous cones [1–3], matrix norm cones [4], p-order cones [5–8], hyperbolicity cones [9–11], circular cones [12, 13] and copositive cones [14], etc. In particular, there seems no systematic study due to the various features and very few algorithms are proposed to solve optimization problems with these non-symmetric cones constraints, except for some interior-point type methods [1, 8, 15, 16]. For instance, Xue and Ye [8] study an optimization problem of minimizing a sum of p-norms, in which two new barrier functions are introduced for p-order cones and a primal-dual potential reduction algorithm is presented. Chua [1] combines the T-algebra with the primal-dual interior-point algorithm to solve the homogeneous conic programming problems. In light of the concept of self-concordant barriers and the efficient computational experience of the long path-following steps, Nesterov [15] proposes a new predictor-corrector path-following method.

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Skajaa and Ye [16] investigate a homogeneous interior-point algorithm for non-symmetric convex conic optimization.

Besides the aforementioned interior-point type methods, are there any other possible algorithms that we can explore? To answer this question, we recall that one category of algorithms to deal with optimization problems rely on so-called complementarity functions, which play an important role in recasting the corresponding KKT conditions as a system of nonsmooth equations or an unconstrained minimization problem. Hence, looking for appropriate complementarity functions is an important issue from a computational viewpoint. In [17], the authors establish novel constructions of complementarity functions associated with symmetric cones, in which the decomposition associated with symmetric cones and the boundary conditions on symmetric cones are crucial in the analysis. Can these ideas be employed in non-symmetric cone setting? Indeed, for non-symmetric cones, their corresponding decompositions are figured out only in a few cases, see [13, 18]. The big hurdle is still the miscellaneous structures of non-symmetric cones. In view of this, we pay attention to the boundary conditions on some non-symmetric cones, which may help building up more useful links to explore possible algorithms for solving non-symmetric cone optimization.

It is well-known that the KKT conditions of an optimization problem is closely related to complementarity problem. More specifically, for a nonlinear programming, its KKT conditions can be rewritten as a nonlinear complementarity problem (NCP) as below: find a solution $x \in \mathbb{R}^n$ to the system

$$x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and F is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. There are essentially three popular ways to solve the NCP: (i) smoothing approach, (ii) merit functions approach, and (iii) projection-type approach. All of these approaches rely on so-called NCP-functions and their corresponding merit functions. A function $\phi : R^2 \to R$ is called a NCP-function if

$$\phi(a,b) = 0 \Longleftrightarrow a, b \geq 0 \text{ and } ab = 0.$$

From nonlinear programming problems to symmetric cone optimization, the above ideas can be employed if we extend the concepts of NCP-function and complementarity problem to the setting of symmetric cones. Accordingly, as a natural extension of the NCP, the symmetric cone complementarity problem (SCCP) is to find a point $v \in \mathbb{E}$ such that

$$v \in \mathcal{K}, \quad F(v) \in \mathcal{K} \text{ and } \langle v, F(v) \rangle = 0,$$

where F is an operator on R^n; $\mathbb{E}$ is a Euclidean Jordan algebra and $\mathcal{K}$ is the

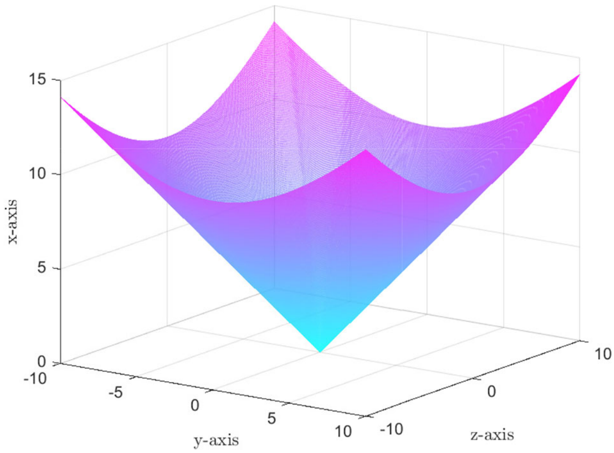

Figure 1. The graph of a 3-dimensional second-order cone.

corresponding symmetric cone in $\mathbb{E}$. In particular, for $\mathbb{E} = R^n$ and $\mathcal{K} = \mathcal{K}^n$, i.e., the second-order cone (SOC, also called Lorentz cone, see Figure 1), defined by

$$\mathcal{K}^n := \{x = (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n \geq ||\bar{x}||\},$$

then the SCCP reduces to the second-order cone complementarity problem (SOCCP), which is to find a point $z \in R^n$ satisfying

$$z \in \mathcal{K}^n, \quad F(z) \in \mathcal{K}^n \text{ and } \langle z, F(z) \rangle = 0.$$

There exists a special *Jordan product* associated with SOC of $x = (\bar{x}, x_n) \in R^{n-1} \times R$ and $y = (\bar{y}, y_n) \in R^{n-1} \times R,$ which is defined by

$$x \circ y := \left(y_n \bar{x} + x_n \bar{y}, \langle x, y \rangle \right).$$

Then, by [19, Proposition 2.1], there holds

$$x \in \mathcal{K}^n, y \in \mathcal{K}^n \text{ and } \langle x, y \rangle = 0 \iff x \in \mathcal{K}^n, y \in \mathcal{K}^n \text{ and } x \circ y = 0. \quad (1)$$

As a result, a so-called C-function (parallel to NCP-function) associated with SOC is a function $\varphi : R^n \times R^n \to R^n$ satisfying

$$\varphi(x, y) = 0 \iff x \in \mathcal{K}^n, y \in \mathcal{K}^n \text{ and } x \circ y = 0.$$

In the setting of non-symmetric cone, the corresponding complementarity problem becomes finding an element z such that

$$z \in \mathrm{K}, \quad F(z) \in \mathrm{K}^* \text{ and } \langle z, F(z) \rangle = 0,$$

where K^* means the dual cone of K. To tackle this non-symmetric cone complementarity problem, we observe that the following parts are key components to do the analysis.

i. $z = 0$ and $F(0) \in \mathrm{K}^*$.

ii. $z \in \mathrm{K}$ and $F(z) = 0$.
iii. $z \in \mathrm{K}\backslash\{0\}$, $F(z) \in \mathrm{K}^*\backslash\{0\}$.

For part (iii), it leads to investigate z being on the boundary of the cone $\partial\mathrm{K}$ and $F(z)$ being on the boundary of the dual cone $\partial\mathrm{K}^*$ (this will be elaborated more in Section 4). This is another reason why we look into the boundary conditions of non-symmetric cones. Apparently, our results on the boundary conditions of non-symmetric cones are connected to non-symmetric cone complementarity problem, and hence possible algorithms can be adopted in light of the complementarity problem. In other words, the results established in this paper will be key bricks for further investigations on non-symmetric cone optimization accordingly, which has important contribution to the development of non-symmetric cone optimization.

2. Boundary conditions on ellipsoidal cone

In this section, we provide characterizations of boundary conditions on ellipsoidal cone [20, 21]. Before showing out the characterizations, we present the boundary conditions on second-order cone in the below proposition, which are already studied in [17, Lemmas 2.3, 2.4 and 2.5]. However, we hereby offer an alternative proof without using the spectral decomposition of vectors. From which, the similar idea and technique will be applied to our subsequent analysis.

Proposition 2.1. *Let $x = (\bar{x}, x_n), y = (\bar{y}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ with $x_n \neq 0$ and $y_n \neq 0$. Then, $x \in \mathcal{K}^n$, $y \in \mathcal{K}^n$ and $x \circ y = 0$ if and only if x, y are both on the boundary of $\mathcal{K}^n$ (that is, $x_n = ||\bar{x}||, y_n = ||\bar{y}||$) and $y_n\bar{x} + x_n\bar{y} = 0$. Moreover, we have $\bar{x} \neq 0$, $\bar{y} \neq 0$ and $\bar{y} = -m\bar{x}$, where $m := \frac{||\bar{y}||}{||\bar{x}||}$.*

Proof. "$\Rightarrow$" Suppose that x and y are nonzero vectors in $\mathcal{K}^n$ with $x \circ y = 0$. Then, we have

$$y_n\bar{x} + x_n\bar{y} = 0 \,\text{and}\, 0 = \langle x, y\rangle = x_n y_n + \langle \bar{x}, \bar{y}\rangle,$$

which implies $x_n y_n = -\langle \bar{x}, \bar{y}\rangle$. Since $x, y \in \mathcal{K}^n$ and $x_n \neq 0$, $y_n \neq 0$, we have $x_n > 0$, $y_n > 0$, which yields

$$0 < x_n y_n = |\langle \bar{x}, \bar{y}\rangle| \leq ||\bar{x}|| \cdot ||\bar{y}|| \leq x_n y_n.$$

This means $x_n y_n = ||\bar{x}|| \cdot ||\bar{y}||$ and $x_n = ||\bar{x}||$, $y_n = ||\bar{y}||$; and by Cauchy-Schwartz inequality, one of $\bar{x}$ and $\bar{y}$ is a multiple of the other.

"$\Leftarrow$" It suffices to show that if $y_n\bar{x} + x_n\bar{y} = 0$ then $\langle x, y\rangle = 0$. Suppose $y_n\bar{x} + x_n\bar{y} = 0$, then $y_n\langle \bar{x}, \bar{y}\rangle + x_n||\bar{y}||^2 = 0$. Since $y_n = ||\bar{y}||$, $y_n\langle \bar{x}, \bar{y}\rangle + x_n y_n^2 = 0$ and hence $\langle x, y\rangle = x_n y_n + \langle \bar{x}, \bar{y}\rangle = 0$.

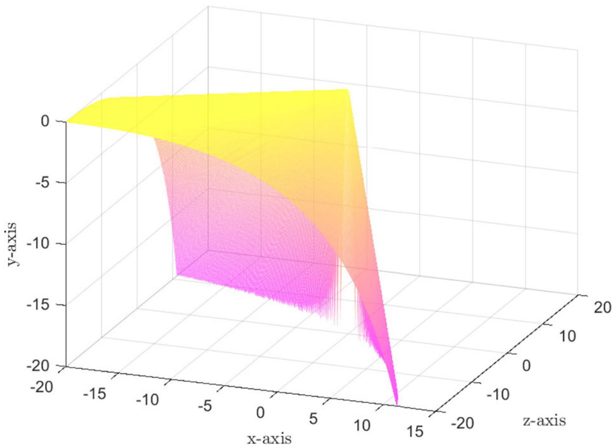

Figure 2. The graph of a 3-dimensional ellipsoidal cone.

Moreover, we have $\bar{x} \neq 0, \bar{y} \neq 0$, and

$$x_n \bar{y} = -y_n \bar{x} \quad \Rightarrow \quad \bar{y} = -\frac{y_n}{x_n}\bar{x} = -\frac{||\bar{y}||}{||\bar{x}||}\bar{x}.$$

Thus, the proof is complete. □

Remark 2.1. *Note that in Proposition 2.1 a vector $x = (\bar{x}, x_n)$ in the cone $\mathcal{K}^n$ with $x_n \neq 0$ is equivalent to say $x \in \mathcal{K}^n$ with $x \neq 0$. We use this just for convenience.*

Proposition 2.1 characterizes the boundary conditions on second-order cone, which is a symmetric cone, namely it is self-dual. Here, we offer another way to verify it without using the spectral decomposition of vectors. In fact, this kind of techniques will be employed to derive analogous conditions for nonsymmetric cones, including ellipsoidal cone, p-order cone, power cone, and also for general closed convex cones.

The ellipsoidal cone (see Figure 2) is the form of

$$\mathrm{K}_\varepsilon := \{x \in \mathbb{R}^n \mid x^T Q x \leq 0,\, u_n^T x \geq 0\},$$

where $Q \in \mathbb{R}^{n \times n}$ is a nonsingular symmetric matrix with a single negative eigenvalue λ_n corresponding to the unit eigenvector u_n. The dual cone K_ε^* is given as

$$\mathrm{K}_\varepsilon^* := \{y \in R^n \,|\, y^T Q^{-1} y \leq 0,\, u_n^T y \geq 0\}.$$

Both K_ε and K_ε^* are closed convex cone and it is obvious that K_ε is not a symmetric cone. The arising ellipsoidal cone complementarity problem (ECCP) is to find a point $x \in R^n$ such that

$$x \in \mathrm{K}_\varepsilon, \quad F(x) \in \mathrm{K}_\varepsilon^* \ \text{ and } \ \langle x, F(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable mapping.

As an important prototype, ellipsoidal cone is a natural generalization of second-order cone, circular cone and elliptic cone. More precisely, let

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2\theta \end{bmatrix} \text{ or } \begin{bmatrix} M^TM & 0 \\ 0 & -1 \end{bmatrix},$$

and $u_n = (0, \ldots, 0, 1)^T$, where I_{n-1} is the identity matrix of order $n-1$, M is any nonsingular matrix of order $n-1$, the ellipsoidal cone respectively reduces to the second-order cone:

$$\mathcal{K}^n := \{x = (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n \geq ||\bar{x}||\},$$

the circular cone:

$$\mathrm{L}_\theta := \{x = (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n \tan\theta \geq ||\bar{x}||\},$$

and the elliptic cone:

$$\mathrm{K}_M^n := \{x = (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n \geq ||M\bar{x}||\}.$$

Therefore, the ellipsoidal cone complementarity problem (ECCP) covers a range of nonsymmetric cone complementarity problems.

Since the ellipsoidal cone is described by a symmetric matrix Q, we can change the x-coordinate to the α-coordinate by an orthogonal matrix U^T, where columns of U are eigenvectors and the corresponding eigenvalues can be chose to satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots > 0 > \lambda_n.$$

Let $U := [u_1 \, u_2 \cdots u_n]$ and $\alpha := [\alpha_1, \alpha_2, \ldots, \alpha_n]^T = U^T x$, i.e., $\alpha_i = u_i^T x$ for $i = 1, 2, \ldots, n$, it follows that

$$x^T Q x = \sum_{i=1}^n \lambda_i \alpha_i^2 \ \text{ and } \ u_n^T x = \alpha_n,$$

and the ellipsoidal cone K_ε can be expressed as

$$\mathrm{K}_\varepsilon = \left\{ U\alpha \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \lambda_i \alpha_i^2 \leq 0 \ \text{ and } \ \alpha_n \geq 0 \right\}. \tag{2}$$

Similarly, the dual cone K_ε^* can be expressed as

$$\mathrm{K}_\varepsilon^* = \left\{ U\beta \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \lambda_i^{-1} \beta_i^2 \leq 0 \ \text{ and } \ \beta_n \geq 0 \right\}, \tag{3}$$

where $\beta := [\beta_1, \beta_2, \ldots, \beta_n]^T = U^T y$.

The product of x and y associated with the ellipsoidal cone K_ε is defined by

$$x \bullet y = \begin{bmatrix} w \\ \langle x, y \rangle \end{bmatrix} \text{ where } w := (w_1, \dots, w_{n-1})^T \text{ with } w_i = \beta_n \lambda_i^{\frac{1}{2}} \alpha_i - \lambda_n \alpha_n \lambda_i^{-\frac{1}{2}} \beta_i.$$

Let $D := \mathrm{diag}\left[\lambda_1^{-\frac{1}{2}}, \lambda_2^{-\frac{1}{2}}, \cdots, (-\lambda_n)^{-\frac{1}{2}}\right]$, the diagonal entries are singular values of Q, then D is nonsingular. To establish the boundary conditions on ellipsoidal cone, we need following lemma, which provide equivalent conditions involved in the Euclidean inner product $\langle \cdot, \cdot \rangle$, the Jordan product "$\circ$" and the product "$\bullet$". Indeed, this lemma is a direct consequence of [21, Theorem 2.3, 2.5]. Here, we give a new proof, which is neat and differs form the one in the literature.

Lemma 2.1. *For any $x, y \in \mathbb{R}^n$, the following are equivalent:*

a. $x \in \mathrm{K}_\varepsilon$, $y \in \mathrm{K}_\varepsilon^*$ *and* $\langle x, y \rangle = 0$.
b. $D^{-1}U^T x \in \mathcal{K}^n$, $DU^T y \in \mathcal{K}^n$ *and* $(D^{-1}U^T x) \circ (DU^T y) = 0$.
c. $x \in \mathrm{K}_\varepsilon$, $y \in \mathrm{K}_\varepsilon^*$ *and* $x \bullet y = 0$.

Proof. "(a) $\iff$ (b)" Note that

$$\begin{aligned} (D^{-1}U^T x) \circ (DU^T y) &= \begin{pmatrix} \lambda_1^{\frac{1}{2}} \alpha_1 \\ \vdots \\ \lambda_{n-1}^{\frac{1}{2}} \alpha_{n-1} \\ (-\lambda_n)^{\frac{1}{2}} \alpha_n \end{pmatrix} \circ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} \beta_1 \\ \vdots \\ (\lambda_{n-1})^{-\frac{1}{2}} \beta_{n-1} \\ (-\lambda_n)^{-\frac{1}{2}} \beta_n \end{pmatrix} \\ &= \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \langle \alpha, \beta \rangle \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \langle x, y \rangle \end{pmatrix} \end{aligned} \tag{4}$$

where the equalities are due to $x = U\alpha$, $y = U\beta$ and U being orthogonal. In addition, from (2) and (3), it follows that

$$\begin{aligned} x \in \mathrm{K}_\varepsilon &\iff \sum_{i=1}^n \lambda_i \alpha_i^2 \le 0 \text{ and } \alpha_n \ge 0 \\ &\iff \lambda_1 \alpha_1^2 + \cdots + \lambda_{n-1} \alpha_{n-1}^2 \le -\lambda_n \alpha_n^2 \text{ and } \alpha_n \ge 0 \\ &\iff D^{-1}U^T x \in \mathcal{K}^n \end{aligned}$$

Similarly, it can be verified that $y \in \mathrm{K}_\varepsilon^*$ if and only if $DU^T y \in \mathcal{K}^n$. Hence, conditions (a) and (b) are equivalent.

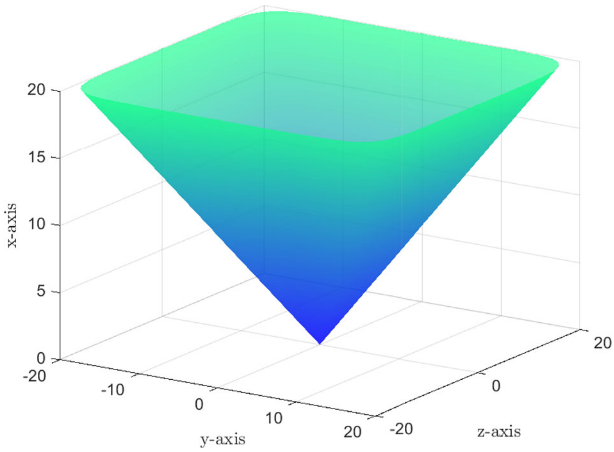

Figure 3. The graph of a 3-dimensional p-order cone with $p = 8$.

"(a) $\iff$ (c)" For $x \in \mathrm{K}_\varepsilon$, $y \in \mathrm{K}_\varepsilon^*$, it is sufficient to show that $\langle x, y\rangle = 0$ if and only if $x \bullet y = 0$. From the above equality (4), we see that

$$
\begin{aligned}
&\langle x, y\rangle = 0 \\
&\qquad \iff (D^{-1}U^T x) \circ (DU^T y) = 0, \\
&\qquad \iff \langle x, y\rangle = 0 \text{ and } (-\lambda_n)^{-\frac{1}{2}}\beta_n \begin{pmatrix} \lambda_1^{\frac{1}{2}}\alpha_1 \\ \vdots \\ \lambda_{n-1}^{\frac{1}{2}}\alpha_{n-1} \end{pmatrix} + (-\lambda_n)^{\frac{1}{2}}\alpha_n \begin{pmatrix} \lambda_1^{-\frac{1}{2}}\beta_1 \\ \vdots \\ (\lambda_{n-1})^{-\frac{1}{2}}\beta_{n-1} \end{pmatrix} = 0, \\
&\qquad \iff \langle x, y\rangle = 0 \text{ and } (-\lambda_n)^{-\frac{1}{2}}\beta_n\lambda_i^{\frac{1}{2}}\alpha_i + (-\lambda_n)^{\frac{1}{2}}\alpha_n\lambda_i^{-\frac{1}{2}}\beta_i = 0 \text{ for all } i = 1, \dots, n-1, \\
&\qquad \iff \langle x, y\rangle = 0 \text{ and } w_i = 0 \text{ for all } i = 1, \dots, n-1, \\
&\qquad \iff x \bullet y = 0.
\end{aligned}
$$

Then, the proof is complete. □

Now, in light of Lemma 2.1, we characterize the boundary conditions on ellipsoidal cone as below.

Proposition 2.2. *Let $x, y \in \mathbb{R}^n$. If x, y are nonzero vectors with $x \in \mathrm{K}_\varepsilon$, $y \in \mathrm{K}_\varepsilon^*$ and $\langle x, y\rangle = 0$, then x is on the boundary of K_ε and y is on the boundary of K_ε^*.*

Proof. Suppose that x, y are nonzero vectors with $x \in \mathrm{K}_\varepsilon y \in \mathrm{K}_\varepsilon^*$ and $\langle x, y\rangle = 0$. By Lemma 2.1(b), it gives $D^{-1}U^T x$, DU^{Ty} are nonzero vectors in $\mathcal{K}^n$ and $(D^{-1}U^T x) \circ (DU^T y) = 0$. Then, applying (1) and Proposition 2.1, both $D^{-1}U^T x$, DU^{Ty} are on the boundary of $\mathcal{K}^n$. Hence, we have

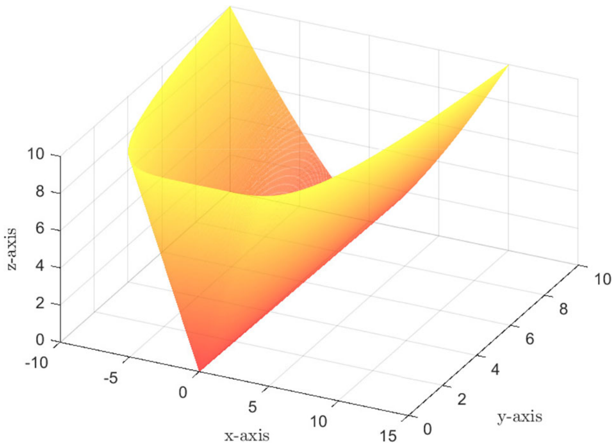

Figure 4. The graph of a 3-dimensional power cone with $\alpha = 0.2$.

$$\left((-\lambda_n)^{\frac{1}{2}}\alpha_n\right)^2 = \left(\lambda_1^{\frac{1}{2}}\alpha_1\right)^2 + \cdots + \left(\lambda_{n-1}^{\frac{1}{2}}\alpha_{n-1}\right)^2$$
$$\Rightarrow \lambda_1\alpha_1^2 + \cdots + \lambda_{n-1}\alpha_{n-1}^2 + \lambda_n\alpha_n^2 = 0,$$

which implies $U\alpha = x$ is on the boundary of K_ε. Similarly, it can be verified that $U\beta = y$ is on the boundary of K_ε^*. □

3. Boundary conditions on *p*-order cone and power cone

In this section, we establish the boundary conditions on p-order cone (see Figure 3) and power cone (see Figure 4), which are two popular cones in reality.

First, we quickly review the definition of p-order cone [22]. The p-order cone, denoted by K_p, is defined by

$$\mathrm{K}_p := \left\{x = (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n \geq ||\bar{x}||_p\right\},$$

and its dual cone is given by

$$\mathrm{K}_p^* := \left\{y = (\bar{y}, y_n) \in R^{n-1} \times R \mid y_n \geq ||\bar{y}||_q\right\} = \mathrm{K}_q,$$

where $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Both K_p and K_p^* are closed convex cone. Indeed, the p-order cone is also a generalization of the second-order cone, but there is no Jordan product for the setting of the p-order cone yet. There is a product for p-order cone defined in [22] as below

$$x \diamond y = \begin{bmatrix} w \\ \langle x, y \rangle \end{bmatrix}$$

where $w := (w_1, \ldots, w_{n-1})^T$ with $w_i = |x_n|^{\frac{p}{q}}|y_i| - |y_n||x_i|^{\frac{p}{q}}$ and $\bar{x} = (x_1, \ldots, x_{n-1})^T, \bar{y} = (y_1, \ldots, y_{n-1})^T$. When x, y satisfy POCCP (see [22]) and

$p = q = 2$, it is the Jordan product in the setting of second-order cone, but it is inappropriate to call it a "Jordan product", because it is not symmetric with respect to the inner product, that is, the condition $\langle x \diamond y, z\rangle = \langle y, x \diamond z\rangle$ is not satisfied. For example, for $p = 3$ and $q = \frac{3}{2}$, let $x = (1, 2), y = (1, 3)$ and $z = (1, 4)$ then $\langle x \diamond y, z\rangle = 29$ but $\langle y, x \diamond z\rangle = 27$. Moreover, the $\diamond$ product is even not commutative due to $(1, 2)\diamond(1, 3) \neq (1, 3)\diamond(1, 2)$. The following proposition describes the boundary conditions on p-order cone, and the proof is similar to that of Proposition 2.1.

Proposition 3.1. *Let $x = (\bar{x}, x_n) \in R^{n-1} \times R$, $y = (\bar{y}, y_n) \in R^{n-1} \times R$ with $x_n \neq 0$ and $y_n \neq 0$. If $x \in \mathrm{K}_p$, $y \in \mathrm{K}_p^*$ and $\langle x, y\rangle = 0$, then x is on the boundary of K_p and y is on the boundary of K_p^*, that is, $x_n = ||\bar{x}||_p$ and $y_n = ||\bar{y}||_q$.*

Proof. Suppose that x and y are nonzero vectors with $x \in \mathrm{K}_p$, $y \in \mathrm{K}_p^*$ and $\langle x, y\rangle = 0$, then $x_n y_n + \langle \bar{x}, \bar{y}\rangle = 0$. Applying Holder's inequality yields

$$0 < x_n y_n = |\langle \bar{x}, \bar{y}\rangle| \leq ||\bar{x}||_p ||\bar{y}||_q \leq x_n y_n,$$

which says $x_n y_n = ||\bar{x}||_p ||\bar{y}||_q$ and $x_n = ||\bar{x}||_p$, $y_n = ||\bar{y}||_q$. Then, the proof is complete. □

Another cone that has real applications is power cone, see [18] for more details. We describe its definition as below. Let $\alpha_1, \ldots, \alpha_m \in R$ be positive with $\alpha_1 + \cdots + \alpha_m = 1$. The power cone K_α is defined by

$$\mathrm{K}_\alpha := \left\{(x, y) \in R_+^m \times R^n \;\middle|\; x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} \geq ||y||\right\},$$

and its dual cone is given by

$$\mathrm{K}_\alpha^* := \left\{(u, v) \in R_+^m \times R^n \;\middle|\; \left(\frac{u_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{u_2}{\alpha_2}\right)^{\alpha_2} \cdots \left(\frac{u_m}{\alpha_m}\right)^{\alpha_m} \geq ||v||\right\}.$$

Following the similar techniques, the boundary conditions on power cone are established as below.

Proposition 3.2. *Let $(x, y) \in R_+^m \times R^n$ and $(u, v) \in R_+^m \times R^n$ be nonzero vectors with $\langle x, u\rangle \neq 0$. If $(x, y) \in \mathrm{K}_\alpha$, $(u, v) \in \mathrm{K}_\alpha^*$ and $\langle (x, y), (u, v)\rangle = 0$, then (x, y) is on the boundary of K_α and (u, v) is on the boundary of K_α^*.*

Proof. Suppose that $x_1 u_1 + \cdots + x_m u_m + y_1 v_1 + \cdots + y_n v_n = 0$, using Cauchy-Schwartz inequality, it leads to

$$0 < x_1 u_1 + \cdots + x_m u_m = -(y_1 v_1 + \cdots + y_n v_n) \leq ||y|| \cdot ||v||.$$

Then, we have

$$\begin{aligned}
||y|| \cdot ||v|| &\geq x_1u_1 + \cdots + x_mu_m \\
&= \alpha_1\left(x_1\frac{u_1}{\alpha_1}\right) + \cdots + \alpha_m\left(x_m\frac{u_m}{\alpha_m}\right) \\
&\geq (x_1\frac{u_1}{\alpha_1})^{\alpha_1}\cdots(x_m\frac{u_m}{\alpha_m})^{\alpha_m} \\
&= x_1^{\alpha_1}\cdots x_m^{\alpha_m}(\frac{u_1}{\alpha_1})^{\alpha_1}\cdots(\frac{u_m}{\alpha_m})^{\alpha_m} \\
&\geq ||y|| \cdot ||v|| \\
&> 0.
\end{aligned}$$

This indicates $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m} = ||y||$ and $(\frac{u_1}{\alpha_1})^{\alpha_1}(\frac{u_2}{\alpha_2})^{\alpha_2}\cdots(\frac{u_m}{\alpha_m})^{\alpha_m} = ||v||$, In other words, (x, y) is on the boundary of K_α and (u, v) is on the boundary of K_α^*. □

4. Boundary conditions on general closed convex cones

It is a natural question whether the aforementioned analysis can be extended to general closed convex cones. In general, suppose that $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ is an inner product space and $\mathbb{K}$ is a closed convex cone in $\mathbb{V}$. Let $\mathbb{K}^*$ denote the dual cone of $\mathbb{K}$, then the boundary conditions are established as follows.

Proposition 4.1. *Suppose that $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ is an inner product space, $\mathbb{K}$ is a closed convex cone in $\mathbb{V}$, and both $\mathbb{K}$ and its dual $\mathbb{K}^*$ are solid (i.e., their interiors are nonempty). Let $x, y \in \mathbb{V}$, and suppose $x \in \mathbb{K}\backslash\{0\}$, $y \in \mathbb{K}^*\backslash\{0\}$ and $\langle x, y\rangle = 0$, then x is on the boundary of $\mathbb{K}$ and y is on the boundary of $\mathbb{K}^*$.*

Proof. Suppose on the contrary that x is an interior point of $\mathbb{K}$, then there exists a radius $r>0$ such that $B_r(x) = x + B_r(0) \subseteq \mathbb{K}$. Consider the point $z = -\frac{r}{2}\frac{y}{||y||}$, then $||z||<r$ and hence $x + z \in B_r(x) \subseteq \mathbb{K}$. Thus, there holds

$$0 \leq \langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle = \langle z, y\rangle = -\frac{r}{2}||y||,$$

which is indeed a contradiction. Therefore, x is on the boundary of $\mathbb{K}$.

On the other hand, since $\mathbb{K}$ is a closed convex cone, we have $(\mathbb{K}^*)^* = \mathbb{K}$. Consider $y \in \mathbb{K}^*$ and $x \in \mathbb{K} = (\mathbb{K}^*)^*$ with $\langle x, y\rangle = 0$, as shown above, y must be on the boundary of $\mathbb{K}^*$. □

Remark 4.1. *The condition of $\mathbb{K}$ being solid is essential in the proof of Proposition 4.1. Here is a counterexample, which tells that Proposition 4.1 does not hold in the relative sense. The relative interior of $\mathbb{K}$ is given by*

$$\begin{aligned}\operatorname{relint}(\mathbb{K}):&\\ &= \{v\in\mathbb{K} \mid \textit{ for all } u\\ &\in \mathbb{K}, \textit{ there exists some } \lambda>1 \textit{ such that } \lambda v+(1-\lambda)u\in\mathbb{K}\}.\end{aligned}$$

Consider $\mathbb{K}=\{(0,0,z)\mid z\geq 0\}$ *the non-negative z-axis in* R^3*, then the dual cone is* $\mathbb{K}^*=\{(x,y,z)\mid x,y\in R, z\geq 0\}$*. Let* $u=(0,0,1)$ *and* $v=(1,1,0)$*, we have* $u\in\operatorname{relint}(\mathbb{K})$ *and* $v\in\partial\mathbb{K}^*$*, but* $\langle u,v\rangle=0$.

In fact, the condition of a closed convex cone in Proposition 4.1 can be relaxed to a closed set in an inner product space $(\mathbb{V},\langle\cdot,\cdot\rangle)$. A dual cone for any subset C in $\mathbb{V}$ can be defined as

$$C^*=\{y\in\mathbb{V}\mid\langle y,x\rangle\geq 0,\ \forall x\in C\}.$$

Notice that C^* is always a convex cone and the result of Proposition 4.1 still holds for the closed set C case.

Proposition 4.2. *Suppose that* $(\mathbb{V},\langle\cdot,\cdot\rangle)$ *is an inner product space,* C *is a closed set in* $\mathbb{V}$*. Let* $x,y\in\mathbb{V}$*, and suppose* $x\in C\backslash\{0\}$*,* $y\in C^*\backslash\{0\}$ *and* $\langle x,y\rangle=0$*, then* x *is on the boundary of* C *and* y *is on the boundary of* C^*.

Proof. Note that if C is a closed set with the empty interior, then $x\in C$ if and only if x belongs to boundary of C. Therefore, we only consider cases: (i) $\operatorname{int}(C)$ is nonempty and $x\in\operatorname{int}(C)$ or (ii) $\operatorname{int}(C^*)$ is nonempty and $y\in\operatorname{int}(C^*)$. In each case, with almost the same arguments as in the proof of Proposition 4.1, by replacing $\mathbb{K}$, $\mathbb{K}^*$ with C, C^*, respectively, we can verify that x is on the boundary of C, and y is on the boundary of C^*, without using the convex cone property of $\mathbb{K}$. □

Based on the observation of Proposition 4.1, we would like to propose an approach to solve the complementarity problem on general closed convex cones. Let $F:\mathbb{V}\to\mathbb{V}$ be a function defined on $\mathbb{V}$. Before to solve $z\in\mathbb{K}$ and $F(z)\in\mathbb{K}^*$, we divide it into the following three possibilities:

I. $z=0$ and $F(0)\in\mathbb{K}^*$.
II. $z\in\mathbb{K}$ and $F(z)=0$.
III. $z\in\mathbb{K}\backslash\{0\}, F(z)\in\mathbb{K}^*\backslash\{0\}$.

For solutions of types I and II, we do not need to check $\langle z,F(z)\rangle=0$ since it is automatically satisfied. In fact, for solution of type I, we only have to check $F(0)\in\mathbb{K}^*$, and this is simple. On the other hand, for solution of type II, this is an inclusion problem to find a $z\in F^{-1}(0)\cap\mathbb{K}$.

For solution of type III, combined with $\langle z,F(z)\rangle=0$, then we observe that z is on the boundary of the cone $\partial\mathbb{K}$ and $F(z)$ is on the boundary of the dual cone $\partial\mathbb{K}^*$ by Proposition 4.1. We can take second-order cone $\mathcal{K}^n$ as a concrete example. Thus, to find the solution is equivalent to solve the equations:

$$z_n = ||\bar{z}||, \quad F_n(z) = ||\bar{F}(z)|| \text{ and } \langle z, F(z)\rangle = 0.$$

A more fancy way to express the result is the following. Let $\mathcal{C} = \mathbb{K} \times \mathbb{K}^*$. It is easy to see that $\mathcal{C}$ is a closed convex cone in $\mathbb{V} \times \mathbb{V}$. Define

$$\mathcal{D} = \{(u, v) \in \partial\mathbb{K} \times \partial\mathbb{K}^* \mid u \neq 0, v \neq 0, \langle u, v\rangle = 0\}.$$

Now, we denote $\bar{\partial}\mathcal{C} := (\{0\} \times \mathbb{K}^*) \cup (\mathbb{K} \times \{0\}) \cup \mathcal{D}$. Suppose z is a solution to the below complementarity problem:

$$z \in \mathbb{K}, \quad F(z) \in \mathbb{K}^*, \quad \langle z, F(z)\rangle = 0.$$

Then, we have $(z, F(z)) \in \bar{\partial}\mathcal{C}$, and vice versa.

In summary, if we understand the boundary behavior of the cone and its dual more, then we understand the complementarity problem more. In most of the time, taking ellipsoidal cone, p-order cone and power cone for instances, the boundary conditions on these cones are always defined by algebraic equations. These are good and helpful for subsequent analysis and investigation.

Another research direction is studying on the property of the function F on the boundary behavior of the cone and its dual, which are also essential to the complementarity problem. Maybe knowledge from algebraic geometry would help. We leave it for future work.

Funding

The author's work is supported by Ministry of Science and Technology, Taiwan.

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