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NOVEL CONSTRUCTIONS OF COMPLEMENTARITY FUNCTIONS ASSOCIATED WITH SYMMETRIC CONES

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Abstract. We provide affirmative answers to two long-standing questions regarding symmetric cone complementarity problem: (i) Is there systematic way to construct complementarity functions associated with symmetric cone? (ii) Is it possible to utilize existing NCP-functions to construct complementarity functions for symmetric cone? More specifically, we present three different assumptions, under one of which, we can construct complementarity functions associated with symmetric cone. For the second question, we demonstrate how to write out complementarity functions associated with symmetric cone by using a given NCP-function. Especially, we construct simple complementarity functions in the settings of second-order cone and positive semidefinite cone, which are two special types of symmetric cones. This novel idea opens up a new approach in solving the complementarity problem based on NCP-functions.

1. Introduction

To tackle optimization problems, one category of algorithms rely on so-called complementarity functions, which plays an important role in recasting the corresponding KKT conditions as a system of nonsmooth equations or an unconstrained minimization problem. Therefore, looking for appropriate complementarity functions is an important issue from a computational viewpoint. This paper is aimed at finding ways to construct complementarity functions associated with symmetric cones so that they can be employed in solving general symmetric cone programs including nonlinear programming, second-order cone programming, and positive semidefinite programming.

It is well-known that complementarity problem arises from the KKT conditions of an optimization problem. For instance, for a nonlinear programming, its KKT conditions can be rewritten as a nonlinear complementarity problem (NCP), which is the problem of finding a point $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ F(x) \geq 0,\ \langle x, F(x) \rangle = 0, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \dots, F_n)^T$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. In order to solve nonlinear complementarity problem, the so-called NCP-function plays an important role. Formally, a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\phi(a,b) = 0 \quad \Longleftrightarrow \quad a, b \geq 0,\ ab = 0$$

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is called an NCP-function. During the past few decades, numerous NCP-functions have been proposed and extensively studied in the literature, see [1, 4–6, 8, 9, 13, 21, 25, 27] and references therein. Moreover, there are also some systematic ways to construct new NCP-functions, for instance in [1, 13, 25, 27]. There are a couple of features regarding NCP-functions which are worth pointing out. The first one is that an NCP-function cannot be differentiable and convex simultaneously as established in [28]. This fact is further extended to general complementarity function associated with any closed and convex cone [18]. The other feature is that, even though we have the differentiability of an NCP-function, the Newton method may not be applied directly because the Jacobian at a degenerate solution to the NCP may be singular, see [19, 20]. Nonetheless, the differentiability of an NCP-function is still useful since we can use some other methods relying on differentiability (like quasi Newton methods, neural network methods) and hence they can be used directly for solving the NCP.

The symmetric cone complementarity problem (SCCP), on the other hand, can be viewed as natural extension of the NCP. The SCCP is the problem of finding a point $z \in \mathbb{V}$ such that

$$z \in \mathcal{K},\ F(z) \in \mathcal{K},\ \langle z, F(z) \rangle = 0, \tag{1.2}$$

where $F : \mathbb{V} \to \mathbb{V}$ is a map, $\mathbb{V}$ is a Euclidean Jordan algebra and $\mathcal{K}$ is its corresponding symmetric cone defined in $\mathbb{V}$. We shall see more details in Section 2 regarding symmetric cone and Euclidean Jordan algebra. The SCCP (1.2) includes a few well-known complementarity problems as special cases. For example, when $\mathcal{K}$ is the nonnegative orthant $\mathbb{R}^n_+$, the problem (1.2) reduces to the NCP (1.1). When $\mathcal{K}$ is the second-order cone $\mathcal{K}^n$, the problem (1.2) is known as the second-order cone complementarity problem (SOCCP), see [10, 11, 17, 26, 30, 31]. When $\mathcal{K}$ is the positive semidefinite cone $\mathcal{S}^n_+$, the problem (1.2) is the positive semidefinite complementarity problem (SDCP), see [32, 35]. Likewise, there is a need for a corresponding complementarity function for the SCCP (C-function for short) when tackling the SCCP (1.2). Under the symmetric cone setting, we call a function $\varphi : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ a complementarity function associated with symmetric cone (again, C-function for short) if it satisfies

$$\varphi(x, y) = 0 \quad \Longleftrightarrow \quad x \in \mathcal{K},\ y \in \mathcal{K},\ x \circ y = 0,$$

where $x \circ y$ means the Jordan product of x and y (which will be recalled in Section 2).

In view of the importance of complementarity functions for the SCCP (1.2), many researchers have paid significant attention to extending some existing NCP-functions to serve as C-functions in the general symmetric cone setting. In particular, Gowda at el. [15] established that the following two popular functions are C-functions for the SCCP:

$$\begin{aligned} \varphi_{\rm FB}(x, y) &= \left(x^2 + y^2\right)^{1/2} - (x + y) \\ \varphi_{\rm NR}(x, y) &= x - (x - y)_+ \end{aligned}$$

which are called Fischer-Burmeister and natural residual functions, respectively, where $(\cdot)_+$ means the metric or orthogonal projection onto $\mathcal{K}$. Kong at el. [22]

studied the vector-valued type of implicit Lagrangian function, and proved that it is a C-function for the SCCP. Liu et al. [24] successfully extended those families of NCP-functions proposed by Luo-Tseng in [25] to the symmetric cone setting. In addition, Pan and Chen [29], Kum and Lim [23] generalized some penalized complementarity functions to the symmetric cone setting. Following these research directions, there are two natural and long-standing questions to ask regarding the construction of complementarity functions for the symmetric cone complementarity problem: (i) Is there a systematic way to construct complementarity functions associated with symmetric cone? (ii) Is it possible to employ existing NCP-functions to generate complementarity functions for symmetric cone? These two problems are indeed long-standing questions in the literature complementarity functions. The main purpose and contribution of this paper lie on providing affirmative answers for the aforementioned questions.

More specifically, we present two methods for the constructions of C-functions in the symmetric cone setting. The first method is inspired by a class of NCP-functions investigated by Mangasarian in [27], which is stated below.

Proposition 1.1. *Assume that $\theta : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, that is, $a > b \Longleftrightarrow \theta(a) > \theta(b)$, and let $\theta(0) = 0$. Then, the function*

$$\phi(a,b) := \theta(|a-b|) - \theta(a) - \theta(b)$$

is an NCP-function.

In [27], Mangasarian provided two examples of θ, namely $\theta(z) = z|z|$ and $\theta(z) = z$. Accordingly, they induce the following NCP-functions:

$$\begin{aligned} \phi_{\text{Man1}}(a,b) &= (a-b)^2 - b|b| - a|a|, \\ \phi_{\text{Man2}}(a,b) &= |a-b| - b - a. \end{aligned}$$

Motivated by Proposition 1.1, as will be seen Section 3, we define a class of vector-valued functions to induce C-function associated with symmetric cone. Moreover, we develop some various kinds of composition forms of C-functions.

The second method is built upon existing NCP-functions. As mentioned earlier, extension of NCP-functions to serve as C-functions for the SCCP are studied by many researchers. Our novel idea is to employ existing NCP-functions (real-valued functions) to construct vector-valued C-functions in the symmetric cone setting. It is known that there exists around fifty NCP-functions in the literature. In turn, our idea opens up an innovative way to obtain plenty of C-functions. We believe that this result is a good contribution to the literature, which paves bricks for subsequent analysis regarding the SCCP through NCP-functions. In particular, we shall demonstrate general forms of C-functions using NCP-functions for the SCCP. Especially, we construct C-functions in two special symmetric cones including second-order cone and positive semidefinite cone based on explicit formulas of the inner product (Jordan product). This novel idea is outspread a new direction in tackling complementarity problems via minimization problems related to NCP-functions.

2. Preliminaries

In this section, we review some background materials and properties about symmetric cones which are needed for subsequent analysis. Most of these contents can be found in [7, 15, 16, 33, 34].

Let $(\mathbb{V}, \langle .,. \rangle)$ be a finite dimensional inner product space over $\mathbb{R}$ and $\mathbb{V} \times \mathbb{V} \to \mathbb{V}$, $(x, y) \mapsto x \circ y$ satisfying the following three conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$ where $x^2 := x \circ x$;
(iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in \mathbb{V}$.

A triple $(\mathbb{V}, \circ, \langle .,. \rangle)$ satisfying the above three conditions is called a *Euclidean Jordan algebra*. An element $e \in \mathbb{V}$ is a *unit* element if $x \circ e = x$ for all $x \in \mathbb{V}$. In $\mathbb{V}$, the set of squares $\mathcal{K} := \{x^2 \mid x \in \mathbb{V}\}$ is said to be a *symmetric cone*. It is known that $\mathcal{K}$ is a self-dual closed convex cone.

For any $x, y \in \mathbb{V}$, we write $x \succeq_{\mathcal{K}} y$ if $x - y \in \mathcal{K}$ and write $x \succ_{\mathcal{K}} y$ if $x - y \in \mathrm{int}(\mathcal{K})$. In other words, we have $x \succeq_{\mathcal{K}} 0$ if and only if $x \in \mathcal{K}$ and $x \succ_{\mathcal{K}} 0$ if and only if $x \in \mathrm{int}(\mathcal{K})$. For $x \in \mathbb{V}$, we define $m(x) := \min\{k > 0 \,|\, \{e, x, \dots, x^k\}$ is linearly independent$\}$ and call the number $r := \max\{m(x) \,|\, x \in \mathbb{V}\}$ the *rank* of $\mathbb{V}$. An element $c \in \mathbb{V}$ is an *idempotent* if $c^2 = c$, while it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. One says that a finite set $\{e_1, e_2, \dots, e_m\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ if } i \neq j \text{ and } \sum_{i=1}^{m} e_i = e.$$

Note that $\langle e_i, e_j \rangle = \langle e_i \circ e_j, e \rangle = 0$ whenever $i \neq j$. We have the following spectral decomposition theorem.

Theorem 2.1 ([16, Theorem III.1.2]). *Let $\mathbb{V}$ be a Euclidean Jordan algebra with rank r. Then, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e_1, e_2, \dots, e_r\}$ and real numbers $\lambda_1(x), \lambda_2(x), \dots, \lambda_r(x)$ such that*

$$x = \lambda_1(x) e_1 + \cdots + \lambda_r(x) e_r. \tag{2.1}$$

Here, $\lambda_i(x)$ are called the eigenvalues of x.

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. A vector-valued function $f^{\mathrm{sc}} : \mathbb{V} \to \mathbb{V}$ associated with the Euclidean Jordan algebra [3, 33] (*SC*-function for short) is defined by

$$f^{\mathrm{sc}}(x) := \sum_{i=1}^{r} f(\lambda_i(x)) e_i = f(\lambda_1(x)) e_1 + \cdots + f(\lambda_r(x)) e_r,$$

where x is defined in (2.1). This is also called a Löwner function. For example, if we take $f(t) = (t)_+ := \max(0, t)$ for $t \in \mathbb{R}$, then $f^{\mathrm{sc}}(x)$ becomes the projection operator onto $\mathcal{K}$:

$$(x)_+ = (\lambda_1(x))_+ e_1 + \cdots + (\lambda_r(x))_+ e_r.$$

Similarly, when $f(t) = (t)_- := \min(0, t)$ for $t \in \mathbb{R}$, $f^{\rm sc}(x)$ means the projection operator onto $-\mathcal{K}$:

$$(x)_- = (\lambda_1(x))_- e_1 + \cdots + (\lambda_r(x))_- e_r.$$

We also have the following facts

$$x = (x)_+ + (x)_-, \ |x| = (x)_+ - (x)_-.$$

Besides, it is known that $x \in \mathcal{K}$ if and only if $\lambda_i(x) \geq 0$ for all $i = 1, \ldots, r$. Thoughout the remaining paper, for $x \in \mathbb{V}$ we denote $x \circ |x|$ by $x|x|$; for any $x, y \in \mathbb{V}$, $\lambda_i(x)$ and $\lambda_i(y)$, $i = 1, \ldots, r$ are arranged in the increasing order $\lambda_1(x) \leq \cdots \leq \lambda_r(x)$ and $\lambda_1(y) \leq \cdots \leq \lambda_r(y)$, respectively.

It is worth writing out the spectral decomposition given in (2.1) for two special symmetric cones. To see this, we now look into two special examples of Euclidean Jordan algebra (see in [3, 14, 15]).

Example 2.2. The algebra $\mathscr{S}^n$ of $n \times n$ real symmetric matrices. Let $\mathbb{S}^{n\times n}$ be the set of all $n \times n$ real symmetric matrices with the inner product and Jordan product given by

$$\langle X, Y \rangle := \mathrm{trace}(XY) \text{ and } X \circ Y := \frac{1}{2}(XY + YX) \ \forall X, Y \in \mathbb{S}^{n\times n}.$$

Then, $(\mathbb{S}^{n\times n}, \circ, \langle ., . \rangle)$ is a Euclidean Jordan algebra and we write it as $\mathscr{S}^n$. The cone of squares $\mathbb{S}^{n\times n}_+$ in $\mathscr{S}^n$ is the set of all positive semidefinite matrices in $\mathbb{S}^{n\times n}$.

Note that the rank of $\mathbb{S}^{n\times n}$ is n and the identity matrix is the unit element. Given any $X \in \mathbb{S}^{n\times n}$, there exists an orthogonal matrix U with columns $u_1, u_2, \ldots, u_n$ and a real diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $X = UDU^T$. Clearly, there holds

$$X = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T$$

which is the spectral decomposition of X. In particular, $\{u_1 u_1^T, \ldots, u_n u_n^T\}$ is a Jordan frame.

Example 2.3. The Jordan spin algebra $\mathscr{L}^n$. Consider $\mathbb{R}^n$ $(n > 1)$ with inner product $\langle \cdot, \cdot \rangle$ and Jordan product defined by

$$x \circ y := (\langle x, y \rangle, y_1 \bar{x}_2 + x_1 \bar{y}_2).$$

for any $x = (x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, \bar{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Then, $(\mathbb{R}^n, \circ, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra, which is denoted by $\mathscr{L}^n$. The cone of squares, denoted by $\mathscr{L}^n_+$, is called *Lorentz cone* (or second-order cone or ice-cream cone) which is given by $\mathscr{L}^n_+ := \{(x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, x_1 \geq \|\bar{x}_2\|\}$.

It is clear that the unit element in $\mathscr{L}^n$ is $e = (1, 0, \ldots, 0)$. For each $x = (x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, a spectral decomposition of x associated with $\mathscr{L}^n_+$ is given by

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)},$$

where $\lambda_1(x)$, $\lambda_2(x)$ are the spectral values and $e_1 \equiv u_x^{(1)}$, $e_2 \equiv u_x^{(2)}$ are their corresponding spectral vectors of x. There are explicit expressions for $\lambda_i(x)$ and $u_x^{(i)}$ below:

$$\lambda_i(x) = x_1 + (-1)^i ||\bar{x}_2||, \tag{2.2}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{\bar{x}_2}{||\bar{x}_2||}\right) & \text{if } \bar{x}_2 \neq 0, \\ \frac{1}{2}\left(1, (-1)^i w_2\right) & \text{if } \bar{x}_2 = 0, \end{cases} \tag{2.3}$$

for $i = 1, 2$, with w_2 being any vector in $\mathbb{R}^{n-1}$ satisfying $||w_2|| = 1$. If $\bar{x}_2 \neq 0$, the decomposition is unique.

Now, we present a few technical lemmas which are crucial to our subsequent analysis. The first one is a useful monotone property, which is proved in [15, Proposition 8].

Lemma 2.4. *For any $x, y \in \mathcal{K}$, if $x \succeq_{\mathcal{K}} 0$, $y \succeq_{\mathcal{K}} 0$ and $x \succeq_{\mathcal{K}} y$, then $x^{1/2} \succeq_{\mathcal{K}} y^{1/2}$.*

The second lemma includes a few properties regarding positive semidefinite matrices which can be found in [14, 32]. For any $X \in \mathbb{S}^{n\times n}$, we denote $X \succeq_{\mathbb{S}^{n\times n}} 0$ by $X \succeq 0$.

Lemma 2.5. *Let X, Y be $n \times n$ matrices in $\mathbb{S}^{n\times n}$. Then, the following hold:*

- (a) *$X \succeq 0 \Rightarrow UXU^T \succeq 0$ for any orthogonal matrix U.*
- (b) *$X \succeq 0$, $Y \succeq 0 \Rightarrow \langle X, Y\rangle \geq 0$.*
- (c) *$X \succeq 0$, $Y \succeq 0$, $\langle X, Y\rangle = 0 \Rightarrow XY = YX = 0$.*
- (d) *If $X \succeq 0$, $Y \succeq 0$, then $\langle X, Y\rangle = 0 \Longleftrightarrow XY = 0$.*
- (e) *Given X and Y in $\mathbb{S}^{n\times n}$ with $XY = YX$, there exists an orthogonal matrix U, diagonal matrices D and E such that $X = UDU^T$ and $Y = UEU^T$.*

The next three lemmas describe the boundary behavior of Lorentz cone.

Lemma 2.6. *Let $x = (x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, \bar{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Then,*

$$x \succeq_{\mathscr{L}^n_+} 0, \ y \succeq_{\mathscr{L}^n_+} 0 \ \text{ and } \ x \circ y = 0$$

if and only if the following hold

- (i) *If $\bar{x}_2 \neq 0$ and $\bar{y}_2 \neq 0$, then x, y are both on the boundary of $\mathscr{L}^n_+$, share the same spectral vectors, and can be expressed as*

$$\begin{aligned} x &= \lambda_2(x) \cdot u_x^{(2)} = 2x_1 \cdot \frac{1}{2}\left(1, \frac{\bar{x}_2}{||\bar{x}_2||}\right), \\ y &= \lambda_2(y) \cdot u_y^{(2)} = 2y_1 \cdot \frac{1}{2}\left(1, -\frac{\bar{x}_2}{||\bar{x}_2||}\right), \end{aligned}$$

  with $\langle u_x^{(2)}, u_y^{(2)}\rangle = 0$ or $u_x^{(2)} \circ u_y^{(2)} = 0$.
- (ii)] *If $\bar{x}_2 = 0$ or $\bar{y}_2 = 0$, then it goes to the trivial cases that $x = 0$ and $y \in \mathscr{L}^n_+$ or $x \in \mathscr{L}^n_+$ and $y = 0$.*

Proof. The idea for the proof is very similar to [17, Proposition 2.1]. For the sake of completeness, we provide the details.

"$\Leftarrow$" The proof of this direction is trivial.

"$\Rightarrow$" From $x \succeq_{\mathscr{L}^n_+} 0$, $y \succeq_{\mathscr{L}^n_+} 0$ and $x \circ y = (\langle x, y\rangle, x_1\bar{y}_2 + y_1\bar{x}_2) = 0$, we have

$$\langle x, y\rangle = x_1 y_1 + \bar{x}_2^T \bar{y}_2 = 0, \ x_1 \geq ||\bar{x}_2||, \ y_1 \geq ||\bar{y}_2||. \tag{2.4}$$

To proceed, we discuss two cases.

(i) If $\bar{x}_2 \neq 0$ and $\bar{y}_2 \neq 0$, then equation (2.4) implies $-\bar{x}_2^T \bar{y}_2 = x_1 y_1 \geq ||\bar{x}_2|| ||\bar{y}_2||$. Since $-\bar{x}_2^T \bar{y}_2 \leq ||\bar{x}_2|| ||\bar{y}_2||$, it leads to $x_1 y_1 = -\bar{x}_2^T \bar{y}_2 = ||\bar{x}_2|| ||\bar{y}_2||$. Hence $x_1 = ||\bar{x}_2||, y_1 = ||\bar{y}_2||$; otherwise, if $x \in \mathrm{int}(\mathscr{L}_+^n)$ or $y \in \mathrm{int}(\mathscr{L}_+^n)$ then $x_1 y_1 > ||\bar{x}_2|| ||\bar{y}_2||$, which is impossible. This means x and y are both on the boundary of $\mathscr{L}_+^n$. Using the facts that the second component of $x \circ y$ is zero, i.e $x_1 \bar{y}_2 + y_1 \bar{x}_2 = 0$, and the fact that $x_1 = ||\bar{x}_2||$, $y_1 = ||\bar{y}_2||$, these yield that

$$x = \lambda_2(x) \cdot u_x^{(2)} = (x_1 + ||\bar{x}_2||) \cdot \frac{1}{2}\Big(1, \frac{\bar{x}_2}{||\bar{x}_2||}\Big) = 2x_1 \cdot \frac{1}{2}\Big(1, \frac{\bar{x}_2}{||\bar{x}_2||}\Big)$$

and

$$y = \lambda_2(y) \cdot u_y^{(2)} = (y_1 + ||\bar{y}_2||) \cdot \frac{1}{2}\Big(1, \frac{\bar{y}_2}{||\bar{y}_2||}\Big) = 2y_1 \cdot \frac{1}{2}\Big(1, -\frac{\bar{x}_2}{||\bar{x}_2||}\Big),$$

where x and y can be viewed as sharing the same spectral vectors $\{u_x^{(2)}, u_y^{(2)}\}$ with $u_x^{(2)} = \frac{1}{2}(1, \frac{\bar{x}_2}{||\bar{x}_2||})$, $u_y^{(2)} = \frac{1}{2}(1, -\frac{\bar{x}_2}{||\bar{x}_2||}) = u_x^{(1)}$ and $\langle u_x^{(2)}, u_y^{(2)} \rangle = u_x^{(2)} \circ u_y^{(2)} = 0$.

(ii) If $\bar{x}_2 = 0$, from equation (2.4), we obtain $x_1 y_1 = 0$. It leads to $x_1 = 0$ or $y_1 = 0$. For $x_1 = 0$, then we have $x = 0$ and y can be any element in $\mathscr{L}_+^n$. For $y_1 = 0$, then $\bar{y}_2$ must be 0 from the third inequality of (2.4), which means $y = 0$ and x can be any element in $\mathscr{L}_+^n$ in this case. Similar to the case $\bar{y}_2 = 0$. $\square$

Lemma 2.7. *Let $x = (x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, \bar{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with $\bar{x}_2 \neq 0$, $\bar{y}_2 \neq 0$. Then,*

$$x \succeq_{\mathscr{L}_+^n} 0, \ y \succeq_{\mathscr{L}_+^n} 0 \ \text{ and } \ x \circ y = 0$$

if and only if $x_1 = \|\bar{x}_2\|$, $y_1 = \|\bar{y}_2\|$, and $x_1 \bar{y}_2 + y_1 \bar{x}_2 = 0$.

Proof. This is an immediate consequence of Lemma 2.6. $\square$

Lemma 2.8. *Let $x = (x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, \bar{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with $\bar{x}_2 \neq 0$, $\bar{y}_2 \neq 0$. If $x \succeq_{\mathscr{L}_+^n} 0$, $y \succeq_{\mathscr{L}_+^n} 0$ and $x \circ y = 0$, then $\bar{y}_2 = -m\bar{x}_2$, where $m := \frac{\|\bar{y}_2\|}{\|\bar{x}_2\|}$. Moreover,*

$$\begin{aligned} \bar{y}_2 = -m\bar{x}_2 \iff & \text{ there exists } k \in \{2, \ldots, n\} \text{ such that } y_k = -m x_k \neq 0 \\ & \text{ and } y_l x_k = x_l y_k \text{ for all } l \in \{2, \ldots, n\}. \end{aligned} \tag{2.5}$$

Proof. From case (i) in the proof of Lemma 2.6, we see that $\bar{x}_2^T \bar{y}_2 = -||\bar{x}_2|| ||\bar{y}_2||$, which further implies

$$\frac{\bar{x}_2^T \bar{y}_2}{||\bar{y}_2||} = -||\bar{x}_2|| \iff \frac{\bar{x}_2^T \bar{y}_2}{||\bar{y}_2||} = -\frac{\bar{x}_2^T \bar{x}_2}{||\bar{x}_2||} \iff \frac{\bar{y}_2}{||\bar{y}_2||} = -\frac{\bar{x}_2}{||\bar{x}_2||} \iff \bar{y}_2 = -\frac{||\bar{y}_2||}{||\bar{x}_2||}\bar{x}_2.$$

Letting $m := \frac{\|\bar{y}_2\|}{\|\bar{x}_2\|}$, it implies $\bar{y}_2 = -m\bar{x}_2$.

Next, we prove the relation (2.5).

"$\Rightarrow$" Since $\bar{y}_2 = -m\bar{x}_2$, and $\bar{x}_2 \neq 0$, $\bar{y}_2 \neq 0$, there exists $k \in \{2, \ldots, n\}$ such that $x_k \neq 0$, $y_k \neq 0$ and $y_k = -m x_k$. In addition, $y_l = -m x_l$ for all $l \in \{2, \ldots, n\}$. Multiplying by $-m x_k$ both sides of this equation, we have

$$y_l(-m x_k) = -m x_l(-m x_k) = -m x_l y_k.$$

Thus, we prove that $y_l x_k = x_l y_k$.

"$\Leftarrow$" Since $y_l x_k = x_l y_k$ and $y_k = -m x_k \neq 0$, it yields $y_l x_k = x_l(-m x_k)$. This implies that $y_l = -m x_l$ for all $l \in \{2, \ldots, n\}$. Hence, $\bar{y}_2 = -m\bar{x}_2$. □

3. First construction method of C-functions

This section is devoted to establishing assumptions under which we can construct a C-function in the setting of symmetric cone. We shall provide three different assumptions, each of which leads to a possible construction way of C-function. Moreover, the C-function can be extended to general Euclidean Jordan algebras.

3.1. **A general form of C-functions.** There exist some systematic ways [1,13] to construct NCP-functions, which usually exploits the fact that $a \geq 0, b \geq 0, ab = 0$ implies either $a = 0$ or $b = 0$. Unfortunately, this phenomenon does not occur in the symmetric cone setting. We note the fact that from [15, Proposition 6], we have

$$x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ x \circ y = 0 \Longleftrightarrow x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ \langle x, y \rangle = 0.$$

The main hurdle for a symmetric cone $\mathcal{K}$ is that

$$x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ x \circ y = 0 \ \text{ does not imply that } \ x = 0 \text{ or } y = 0.$$

Nonetheless, through the following assumption, it may remedy the above deficiency.

Assumption 3.1. A function $\theta : \mathbb{R}^n \to \mathbb{R}^n$ is said to satisfy Assumption 3.1 if

(i): $x \succeq_{\mathcal{K}} 0$ if and only if $\theta(x) \succeq_{\mathcal{K}} 0$.

(ii): for any $x, y \succeq_{\mathcal{K}} 0$, $x \circ y = 0$ if and only if $\theta(x) \circ \theta(y) = 0$.

Assumption 3.1(i) is a slightly weaker than the strictly increasing property mentioned in Proposition 1.1, whereas Assumption 3.1(ii) is used to adjust the expression in a general symmetric cone setting.

Theorem 3.1. *Suppose that $\theta : \mathbb{R}^n \to \mathbb{R}^n$ satisfies Assumption 3.1. Then, the function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$\varphi(x, y) := |\theta(x) - \theta(y)| - \theta(x) - \theta(y)$$

is a C-function in the symmetric cone setting.

Proof. It suffices to verify that $\varphi(x, y) = 0$ if and only if $x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ x \circ y = 0$.

"$\Rightarrow$" Assume that $\varphi(x, y) = 0$, we observe

$$\begin{aligned}
& \varphi(x, y) = |\theta(x) - \theta(y)| - \theta(x) - \theta(y) = 0 \\
\Longrightarrow\ & |\theta(x) - \theta(y)| = \theta(x) + \theta(y) \\
\Longrightarrow\ & |\theta(x) - \theta(y)|^2 = (\theta(x) + \theta(y))^2 \\
\Longrightarrow\ & \theta(x)^2 - 2\theta(x) \circ \theta(y) + \theta(y)^2 = \theta(x)^2 + 2\theta(x) \circ \theta(y) + \theta(y)^2 \\
\Longrightarrow\ & \theta(x) \circ \theta(y) = 0.
\end{aligned} \tag{3.1}$$

Letting $\omega = |\theta(x) - \theta(y)|$ gives $\omega^2 = \theta(x)^2 - 2\theta(x) \circ \theta(y) + \theta(y)^2 = \theta(x)^2 + \theta(y)^2$. Thus, we have $\omega^2 \succeq_{\mathcal{K}} \theta(x)^2$ and $\omega^2 \succeq_{\mathcal{K}} \theta(y)^2$. This leads to $\omega \succeq_{\mathcal{K}} \theta(x)$ and $\omega \succeq_{\mathcal{K}} \theta(y)$ by applying Lemma 2.4. Since $\varphi(x, y) = 0$, $\omega = \theta(x) + \theta(y)$, it follows that $\theta(x) = \omega - \theta(y) \succeq_{\mathcal{K}} 0$ and $\theta(y) = \omega - \theta(x) \succeq_{\mathcal{K}} 0$. Using Assumption 3.1(i) of θ, we obtain $x, y \succeq_{\mathcal{K}} 0$. Then, we further have $x \circ y = 0$ from Assumption 3.1(ii).

"$\Leftarrow$" Suppose that $x \succeq_{\mathcal{K}} 0, y \succeq_{\mathcal{K}} 0, x \circ y = 0$ and θ satisfies Assumption 3.1. Then it is clear that $\theta(x) \succeq_{\mathcal{K}} 0$, $\theta(y) \succeq_{\mathcal{K}} 0$ and $\theta(x) \circ \theta(y) = 0$. On the other hand, we have

$$\begin{aligned}\varphi(x,y) &= [(\theta(x)-\theta(y))^2]^{1/2} - (\theta(x)+\theta(y))\\ &= [\theta(x)^2 - 2\theta(x)\circ\theta(y) + \theta(y)^2]^{1/2} - (\theta(x)+\theta(y))\\ &= [\theta(x)^2 + 2\theta(x)\circ\theta(y) + \theta(y)^2]^{1/2} - (\theta(x)+\theta(y))\\ &= |\theta(x)+\theta(y)| - (\theta(x)+\theta(y)) = 0,\end{aligned}$$

where $\theta(x)+\theta(y) \in \mathcal{K}$ due to [15, Proposition 6]. □

What are some examples of $\theta(\cdot)$ function that satisfy Assumption 3.1? Indeed, in light of Theorem 2.1 and note that $x \in \mathcal{K}$ if and only if $\lambda_i(x) \geq 0$ for all $i = 1, \ldots, r$, we can confirm that the following functions satisfy Assumption 3.1 in their domain:

$$\begin{aligned}\theta_1(z) &= z,\\ \theta_2(z) &= z^p, \text{ where } p \text{ is positive odd integer},\\ \theta_3(z) &= z|z|,\\ \theta_4(z) &= z^{1/2}, \text{ where } \theta_4 : \mathcal{K} \to \mathcal{K}.\end{aligned}$$

Hence, by Theorem 3.1, these functions corresponds to C-functions φ_1, φ_2, φ_3, and φ_4 which are listed below.

$$\begin{aligned}\varphi_1(x,y) &= |x-y| - (x+y) = -\frac{1}{2}\varphi_{\rm NR}(x,y);\\ \varphi_2(x,y) &= |x^p - y^p| - x^p - y^p, \text{ where } p \text{ is positive odd integer};\\ \varphi_3(x,y) &= \big|x|x| - y|y|\big| - x|x| - y|y|;\\ \varphi_4(x,y) &= |x^{1/2} - y^{1/2}| - x^{1/2} - y^{1/2}, \text{ where } \varphi_4 : \mathcal{K}\times\mathcal{K} \to \mathcal{K}.\end{aligned}$$

3.2. **Composition form of C-functions.** In this subsection, we explore composition forms of C-functions. More specifically, given a $\theta(\cdot)$ function satisfying Assumption 3.1 and any C-function φ, the composition function $\varphi(\theta(x), \theta(y))$ is a C-function as well.

Theorem 3.2. *Suppose that $\theta : \mathbb{R}^n \to \mathbb{R}^n$ satisfies Assumption 3.1. Then, for any C-function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the composition function $\varphi(\theta(x), \theta(y))$ is also a C-function.*

Proof. "$\Leftarrow$" If $x \succeq_{\mathcal{K}} 0$, $y \succeq_{\mathcal{K}} 0, x \circ y = 0$ and θ satisfies Assumption 3.1, we have $\theta(x) \succeq_{\mathcal{K}} 0$ and $\theta(y) \succeq_{\mathcal{K}} 0$ by Assumption 3.1(i) and $\theta(x) \circ \theta(y) = 0$ by Assumption 3.1(ii). Then, it follows that $\varphi(\theta(x), \theta(y)) = 0$ since φ is a C-function.

"$\Rightarrow$" If $\varphi(\theta(x), \theta(y)) = 0$, we have $\theta(x), \theta(y) \succeq_{\mathcal{K}} 0$ and $\theta(x) \circ \theta(y) = 0$ since φ is a C-function. Again, applying Assumption 3.1 yields $x, y \succeq_{\mathcal{K}} 0$ and $x \circ y = 0$. □

Since those functions θ_1, θ_2, θ_3, θ_4 shown in Section 3.1 satisfy Assumption 3.1, we can use them and apply Theorem 3.2 to obtain more C-functions. For example, if we take the Fischer-Burmeister function

$$\varphi_{\rm FB}(x,y) = (x^2+y^2)^{1/2} - (x+y),$$

then we achieve the following C-functions accordingly:

$$\begin{aligned}
\tilde{\varphi}_1(x,y) &= \varphi_{\rm FB}(x,y);\\
\tilde{\varphi}_2(x,y) &= (x^{2p}+y^{2p})^{1/2}-(x^p+y^p),\ \text{where } p \text{ is positive odd integer;}\\
\tilde{\varphi}_3(x,y) &= ((x|x|)^2+(y|y|)^2)^{1/2}-(x|x|+y|y|);\\
\tilde{\varphi}_4(x,y) &= (x+y)^{1/2}-(x^{1/2}+y^{1/2}),\ \text{where } \tilde{\varphi}_4:\mathcal{K}\times\mathcal{K}\to\mathcal{K}.
\end{aligned}$$

In fact, item (i) and (ii) in Assumption 3.1 can be combined together as a complementarity property, which is slightly weaker than Assumption 3.1.

Assumption 3.2. A function $\theta:\mathbb{R}^n\to\mathbb{R}^n$ is said to satisfy Assumption 3.2 if

$$x\succeq_{\mathcal{K}}0,\ y\succeq_{\mathcal{K}}0,\ x\circ y=0 \iff \theta(x)\succeq_{\mathcal{K}}0,\ \theta(y)\succeq_{\mathcal{K}}0,\ \theta(x)\circ\theta(y)=0.$$

It is noted that Assumption 3.2 is sufficient for Theorem 3.2. The following is a weaker version of the composition form.

Theorem 3.3. *Suppose that $\theta:\mathbb{R}^n\to\mathbb{R}^n$ satisfies Assumption 3.2. Then, for any C-function $\varphi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, the composition function $\varphi(\theta(x),\theta(y))$ is also a C-function.*

Proof. The proof is straightforward. Since φ is a C-function and θ satisfies Assumption 3.2, we have

$$\begin{aligned}
&\varphi(\theta(x),\theta(y))=0\\
\iff\ &\theta(x)\succeq_{\mathcal{K}}0,\ \theta(y)\succeq_{\mathcal{K}}0,\ \theta(x)\circ\theta(y)=0\\
\iff\ &x\succeq_{\mathcal{K}}0,\ y\succeq_{\mathcal{K}}0,\ x\circ y=0.
\end{aligned}$$

Hence, $\varphi(\theta(x),\theta(y))$ is also a C-function. $\square$

If we choose $\theta(z)=z$, then the composition function $\varphi(\theta(x),\theta(y))$ in Theorem 3.3 goes back to the original C-function $\varphi(x,y)$. If we choose $\varphi_1(x,y)=x-(x-y)_+=\varphi_{\rm NR}(x,y)$, $\varphi_2(x,y)=(x^2+y^2)^{1/2}-(x+y)=\varphi_{\rm FB}(x,y)$, composing them with different $\theta(\cdot)$ leads to various C-functions.

(1) Let $\theta(z)=z^p$ where p is positive odd integer. Then, applying Theorem 3.3 implies that

$$\begin{aligned}
\varphi_1(\theta(x),\theta(y)) &= x^p-(x^p-y^p)_+,\\
\varphi_2(\theta(x),\theta(y)) &= \left(x^{2p}+y^{2p}\right)^{1/2}-(x^p+y^p),
\end{aligned}$$

are also C-functions.

(2) Let $\theta(z)=z|z|$. Then, applying Theorem 3.3 implies that

$$\begin{aligned}
\varphi_1(\theta(x),\theta(y)) &= x|x|-\big(x|x|-y|y|\big)_+,\\
\varphi_2(\theta(x),\theta(y)) &= \left((x|x|)^2+(y|y|)^2\right)^{1/2}-(x|x|+y|y|),
\end{aligned}$$

are also C-functions.

We next introduce a special class of functions, which also satisfy Assumption 3.2. Therefore, we can generate many $\theta(\cdot)$ functions from it and use them with Theorem 3.3.

Proposition 3.4. *For any real-valued function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:*

(i) $t \geq 0$ *if and only if* $f(t) \geq 0$*;*
(ii) $t = 0$ *if and only if* $f(t) = 0$*,*

the vector-valued function $f^{\rm sc} : \mathbb{R}^n \to \mathbb{R}^n$ associated with $\mathcal{K}$, defined by

$$f^{\rm sc}(x) = f(\lambda_1(x))e_1 + \cdots + f(\lambda_r(x))e_r \quad \forall x \in \mathbb{V},$$

satisfies Assumption 3.2. Here, $\lambda_i(x)$ and $\{e_i\}$ for $i = 1, 2, \ldots, r$ are the spectral values and the spectral vectors of x, respectively.

Proof. Let $x, y \in \mathbb{V}$, the spectral decompositions of x and y are given by

$$x = \sum_{i=1}^{r} \lambda_i(x)e_i \text{ and } y = \sum_{j=1}^{r} \lambda_i(y)f_j.$$

Then, we have

$$f^{\rm sc}(x) = \sum_{i=1}^{r} f(\lambda_i(x))e_i \text{ and } f^{\rm sc}(y) = \sum_{j=1}^{r} f(\lambda_j(y))f_j.$$

From the above properties (i)-(ii) of f, we obtain

$$\begin{aligned}
& x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ x \circ y = 0 \\
\Longleftrightarrow\ & x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ \langle x, y\rangle = 0 \\
\Longleftrightarrow\ & \lambda_i(x) \geq 0,\ \lambda_i(y) \geq 0,\ \sum_{i,j=1}^{r} \lambda_i(x)\lambda_j(y) \langle e_i, f_j\rangle = 0 \\
\Longleftrightarrow\ & \lambda_i(x) \geq 0,\ \lambda_i(y) \geq 0,\ \lambda_i(x)\lambda_j(y) = 0 \text{ or } \langle e_i, f_j\rangle = 0, i, j = 1, \ldots, r \\
\Longleftrightarrow\ & f(\lambda_i(x)) \geq 0,\ f(\lambda_i(y)) \geq 0,\ f(\lambda_i(x))f(\lambda_j(y)) = 0 \text{ or } \langle e_i, f_j\rangle = 0, i, j = 1, \ldots, r \\
\Longleftrightarrow\ & f^{\rm sc}(x) \succeq_{\mathcal{K}} 0,\ f^{\rm sc}(y) \succeq_{\mathcal{K}} 0,\ \sum_{i,j=1}^{r} f(\lambda_i(x))f(\lambda_j(y)) \langle e_i, f_j\rangle = 0 \\
\Longleftrightarrow\ & f^{\rm sc}(x) \succeq_{\mathcal{K}} 0,\ f^{\rm sc}(y) \succeq_{\mathcal{K}} 0,\ \langle f^{\rm sc}(x), f^{\rm sc}(y)\rangle = 0 \\
\Longleftrightarrow\ & f^{\rm sc}(x) \succeq_{\mathcal{K}} 0,\ f^{\rm sc}(y) \succeq_{\mathcal{K}} 0,\ f^{\rm sc}(x) \circ f^{\rm sc}(y) = 0.
\end{aligned}$$

Note that we have $\langle e_i, f_j\rangle \geq 0$ since e_i, f_j belong to the symmetric cone $\mathcal{K}$ which is self-dual. Thus, it is clear that Assumption 3.2 is satisfied and the proof is complete. □

We list a couple of examples of f mentioned in Proposition 3.4. The first one is $f(t) = t^p$ with positive odd number p. It is clear that the properties (i) and (ii) are held. Hence, its corresponding SC-function reduces to the regular function $f^{\rm sc}(x) = x^p$. The second one is $f(t) = \dfrac{t}{t^2+1}$, which also possesses (i) $t \geq 0$ if and only if $f(t) \geq 0$; and (ii) $t = 0$ if and only if $f(t) = 0$. Then, in light of Proposition 3.4, its SC-function satisfies Assumption 3.2. This means we can employ this $f^{\rm sc}$

function as a choice of $\theta(\cdot)$ function in Theorem 3.3 to generate C-functions below:

$$\theta(x) = f^{sc}(x) = \frac{\lambda_1(x)}{\lambda_1(x)^2+1}e_1 + \cdots + \frac{\lambda_r(x)}{\lambda_r(x)^2+1}e_r.$$

where $x \in \mathbb{V}$, $\lambda_i(x)$ for $i = 1, 2, \ldots, r$ are spectral values of x, and $\{e_i\}_{i=1}^r$ is a Jordan frame.

In fact, Assumption 3.2 can be extended to the two functions version below:

Assumption 3.3. The functions $\theta_1, \theta_2 : \mathrm{IR}^n \to \mathrm{IR}^n$ is said to satisfy Assumption 3.3 if

$$x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ x \circ y = 0 \ \Leftrightarrow\ \theta_1(x) \succeq_{\mathcal{K}} 0,\ \theta_2(y) \succeq_{\mathcal{K}} 0,\ \theta_1(x) \circ \theta_2(y) = 0.$$

Using Assumption 3.3, Theorem 3.3 can be naturally extended to a more general case as follows.

Theorem 3.5. *Suppose that $\theta_1, \theta_2 : \mathrm{IR}^n \to \mathrm{IR}^n$ satisfy Assumption 3.3. Then, for any C-function $\varphi : \mathrm{IR}^n \times \mathrm{IR}^n \to \mathrm{IR}^n$, the composition function $\varphi(\theta_1(x), \theta_2(y))$ is also a C-function.*

Proof. The proof is straightforward. Since φ is a C-function and θ_1, θ_2 satisfy Assumption 3.3, it is easy to verify that

$$\begin{aligned} & \varphi(\theta_1(x), \theta_2(y)) = 0 \\ \Longleftrightarrow\ & \theta_1(x) \succeq_{\mathcal{K}} 0,\ \theta_2(y) \succeq_{\mathcal{K}} 0,\ \theta_1(x) \circ \theta_2(y) = 0 \\ \Longleftrightarrow\ & x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ x \circ y = 0. \end{aligned}$$

Hence, we show that $\varphi(\theta_1(x), \theta_2(y))$ is also a C-function. □

Here are examples of $\theta_1(\cdot)$ and $\theta_2(\cdot)$ in Theorem 3.5:

$$\theta_1(x) = x^3 + x \quad \text{and} \quad \theta_2(y) = y|y|.$$

Composing these two functions with the natural residual function $\varphi_{\mathrm{NR}}(x, y) = x - (x-y)_+$ yields

$$\varphi_{\mathrm{NR}}(\theta_1(x), \theta_2(y)) = x^3 + x - (x^3 + x - y|y|)_+$$

which is a C-function due to Theorem 3.5. Note that the conclusion of Theorem 3.5 is still true if we exchange the position of θ_1 and θ_2 in the composition.

There is another surprising result that if we switch the roles of φ and θ in Theorem 3.5, the goal is still achieved.

Theorem 3.6. *Suppose that $\theta : \mathrm{IR}^n \to \mathrm{IR}^n$ satisfies $z = 0$ if and only if $\theta(z) = 0$. Then, for any C-function $\varphi : \mathrm{IR}^n \times \mathrm{IR}^n \to \mathrm{IR}^n$, the composition function $\theta\big(\varphi(\cdot, \cdot)\big)$ is also a C-function.*

Proof. Since φ is a C-function and θ satisfies $z = 0$ if and only if $\theta(z) = 0$, we have

$$\theta(\varphi(x, y)) = 0 \quad \Longleftrightarrow \quad \varphi(x, y) = 0 \quad \Leftrightarrow \quad x \succeq_{\mathcal{K}} 0,\ y \succeq_{\mathcal{K}} 0,\ x \circ y = 0.$$

This proves that $\theta(\varphi(x, y))$ is also a C-function. □

The following are some examples of $\theta(\cdot)$ mentioned in Theorem 3.6:

(1) $\theta(z) = z^p$, where p is a positive integer;

(2) $\theta(z) = |z|$;
(3) $\theta(z) = f^{\mathrm{sc}}(z)$ where $f^{\mathrm{sc}}(z)$ is the SC-function induced from a real-valued function f with $t = 0$ if and only if $f(t) = 0$.

4. Second construction method of C-functions

The main idea of the second construction method of C-functions is employing the existing NCP-functions (real-valued functions) to produce C-functions (vector-valued functions). This is a novel idea which indicates that the existing NCP-functions (about 50 of them) can be used to engender a bunch of C-functions.

4.1. **Using NCP-functions to construct C-functions in symmetric cone setting.** As mentioned earlier, the C-function is a vector-valued function whereas the NCP-function is only a real-valued function. How to extend an NCP-function into a C-function has been an open question in the past few decades. In this work, we shall demonstrate this in detail in symmetric cones.

Theorem 4.1. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be an NCP-function. For any $x \in \mathbb{V}$ and $y \in \mathbb{V}$, the following $\Phi : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ defined by*

$$\Phi(x, y) := \sum_{i,j=1}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j$$

is a C-function, where $\{e_i\}_{i=1}^r$, $\{f_j\}_{j=1}^r$ are Jordan frames of x and y, respectively.

Proof. We note the fact that $\Phi(x, y)$ is well-defined by using the uniquely defined spectral decomposition theorem in [2] which is proved in Appendix.

Let $x \in \mathbb{V}$ and $y \in \mathbb{V}$, the spectral decomposition of x and y are given by

$$x = \sum_{i=1}^{r} \lambda_i(x) e_i \text{ and } y = \sum_{j=1}^{r} \lambda_j(y) f_j.$$

By definition of C-function, it suffices to show that $\Phi(x, y) = 0 \Longleftrightarrow x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x, y\rangle = 0$. Indeed,

"$\Rightarrow$" Since $\langle e_i, f_j\rangle \geq 0$, we have that

$$\begin{aligned}
\Phi(x, y) = 0 &\Longleftrightarrow \sum_{i,j=1}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j = 0 \\
&\Longrightarrow \left\langle \sum_{i,j=1}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j, e \right\rangle = 0 \\
&\Longleftrightarrow \sum_{i,j=1}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) \langle e_i, f_j\rangle = 0 \\
&\Longleftrightarrow \phi^2(\lambda_i(x), \lambda_j(y)) = 0 \text{ or } \langle e_i, f_j\rangle = 0,\ i, j = 1, \ldots, r \\
&\Longleftrightarrow \phi(\lambda_i(x), \lambda_j(y)) = 0 \text{ or } \langle e_i, f_j\rangle = 0,\ i, j = 1, \ldots, r \\
&\Longleftrightarrow \lambda_i(x) \geq 0,\ \lambda_j(y) \geq 0, \lambda_i(x)\lambda_j(y) = 0 \text{ or } \langle e_i, f_j\rangle = 0,\ i, j = 1, \ldots, r
\end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow x \in \mathcal{K},\ y \in \mathcal{K},\ \sum_{i,j=1}^{r} \lambda_i(x)\lambda_j(y)\langle e_i, f_j\rangle = 0\\
&\Longleftrightarrow x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x, y\rangle = 0.
\end{aligned}$$

"$\Leftarrow$" By the above equivalences, we obtain

$$\begin{aligned}
x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x, y\rangle = 0 &\Longleftrightarrow \phi^2(\lambda_i(x), \lambda_j(y)) = 0 \text{ or } \langle e_i, f_j\rangle = 0,\ i,j = 1,\dots,r\\
&\Longleftrightarrow \phi^2(\lambda_i(x), \lambda_j(y)) = 0 \text{ or } e_i \circ f_j = 0,\ i,j = 1,\dots,r\\
&\Longrightarrow \sum_{i,j=1}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j = 0\\
&\Longleftrightarrow \Phi(x,y) = 0.
\end{aligned}$$

Thus, we achieve the desired result. □

In fact, if $\mathbb{V} \equiv \mathbb{R}$, then a C-function $\Phi(x,y)$ reduces to an NCP-function $\phi^2(x,y)$. It is clear that we can write out components of $\Phi(x,y)$ in Theorem 4.1 in the second-order cone case. Let $x = (x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, \bar{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

$$\Phi(x,y) = \begin{pmatrix} a + b\bar{u}_2^T\bar{v}_2 \\ c\bar{u}_2 + d\bar{v}_2 \end{pmatrix}, \tag{4.1}$$

where

$$\bar{u}_2 = \begin{cases} \frac{\bar{x}_2}{\|\bar{x}_2\|} & \text{if } \bar{x}_2 \neq 0 \\ \omega & \text{otherwise,} \end{cases} \qquad \bar{v}_2 = \begin{cases} \frac{\bar{y}_2}{\|\bar{y}_2\|} & \text{if } \bar{y}_2 \neq 0 \\ \vartheta & \text{otherwise,} \end{cases}$$

with any vector $\omega, \vartheta \in \mathbb{R}^{n-1}$ such that $\|\omega\| = 1$, $\|\vartheta\| = 1$, and

$$\begin{aligned}
a &= \frac{\phi^2(\lambda_1(x),\lambda_1(y)) + \phi^2(\lambda_1(x),\lambda_2(y)) + \phi^2(\lambda_2(x),\lambda_1(y)) + \phi^2(\lambda_2(x),\lambda_2(y))}{4},\\
b &= \frac{\phi^2(\lambda_1(x),\lambda_1(y)) - \phi^2(\lambda_1(x),\lambda_2(y)) - \phi^2(\lambda_2(x),\lambda_1(y)) + \phi^2(\lambda_2(x),\lambda_2(y))}{4},\\
c &= \frac{-\phi^2(\lambda_1(x),\lambda_1(y)) - \phi^2(\lambda_1(x),\lambda_2(y)) + \phi^2(\lambda_2(x),\lambda_1(y)) + \phi^2(\lambda_2(x),\lambda_2(y))}{4},\\
d &= \frac{-\phi^2(\lambda_1(x),\lambda_1(y)) + \phi^2(\lambda_1(x),\lambda_2(y)) - \phi^2(\lambda_2(x),\lambda_1(y)) + \phi^2(\lambda_2(x),\lambda_2(y))}{4}.
\end{aligned}$$

Example 4.2. We consider the FB function $\phi_{\rm FB}(a,b) = \sqrt{a^2+b^2} - (a+b)$ for all $(a,b) \in \mathbb{R} \times \mathbb{R}$. Then the corresponding C-function is

$$\Phi_{\rm FB}(x,y) = \sum_{i,j=1}^{r} \phi_{\rm FB}^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j.$$

It is easy to see that

$$\Phi_{\rm FB}(x,y) = 0 \quad \Longleftrightarrow \quad \varphi_{\rm FB}(x,y) = (x^2+y^2)^{1/2} - (x+y) = 0.$$

According to [31, Section 3], a formula of components of $\varphi_{\rm FB}(x,y)$ is very complicated which says that its subgradient formula is also complicated. However, using an explicit formula of $\Phi(x,y)$ given by (4.1), we see that computing the subgradient of $\Phi_{\rm FB}(x,y)$ becomes easier. Thus, $\Phi_{\rm FB}(x,y)$ might be easier in implementing numerical experiment comparing to $\varphi_{\rm FB}(x,y)$ for solving the SOCCP.

Next, we denote the set $I = \{(i,j) \in \{1,\ldots,r\} \mid \langle e_i, f_j\rangle = 0\}$, then C-functions can be constructed by the following theorem.

Theorem 4.3. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be an NCP-function. For any $x \in \mathbb{V}$ and $y \in \mathbb{V}$, the following $\Phi^1, \Phi^2 : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ defined by*

$$\Phi^1(x,y) := \sum_{(i,j)\notin I}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j$$

$$\Phi^2(x,y) := \sum_{(i,j)\notin I}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i$$

are C-functions, where $I = \{(i,j) \in \{1,\ldots,r\} \mid \langle e_i, f_j\rangle = 0\}$, $\{e_i\}_{i=1}^r$, $\{f_j\}_{j=1}^r$ are Jordan frames of x and y, respectively.

Proof. Using the proof of Theorem 4.1, it is clear that $\Phi^1(x,y), \Phi^2(x,y)$ are well-defined and $\Phi^1(x,y)$ is a C-function. We will now prove $\Phi^2(x,y)$ is a C-function. We note the fact that $\langle e_i, f_j\rangle > 0$ for $(i,j) \notin I$, $e_i \in \mathcal{K}$ and $\langle e_i, e_i\rangle > 0$ for all $i = 1,\ldots,r$. We have

$$\begin{aligned}
\Phi^2(x,y) = 0 \iff\ & \sum_{(i,j)\notin I}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i = 0 \\
\iff\ & \left\langle \sum_{(i,j)\notin I}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i, e \right\rangle = 0 \\
\iff\ & \left\langle \sum_{(i,j)\notin I}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i, \sum_{i=1}^{r} e_i \right\rangle = 0 \\
\iff\ & \sum_{(i,j)\notin I}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) \langle e_i, e_i\rangle = 0 \\
\iff\ & \phi^2(\lambda_i(x), \lambda_j(y)) = 0,\ (i,j) \notin I \\
\iff\ & \phi(\lambda_i(x), \lambda_j(y)) = 0,\ (i,j) \notin I \\
\iff\ & \lambda_i(x) \geq 0,\ \lambda_j(y) \geq 0,\ \lambda_i(x)\lambda_j(y) = 0,\ (i,j) \notin I \\
\iff\ & x \in \mathcal{K},\ y \in \mathcal{K},\ \sum_{(i,j)\notin I}^{r} \lambda_i(x)\lambda_j(y) \langle e_i, f_j\rangle = 0 \\
\iff\ & x \in \mathcal{K},\ y \in \mathcal{K},\ \sum_{(i,j)\notin I}^{r} \lambda_i(x)\lambda_j(y) \langle e_i, f_j\rangle \\
& + \sum_{(i,j)\in I}^{r} \lambda_i(x)\lambda_j(y) \langle e_i, f_j\rangle = 0 \\
\iff\ & x \in \mathcal{K},\ y \in \mathcal{K},\ \sum_{i,j=1}^{r} \lambda_i(x)\lambda_j(y) \langle e_i, f_j\rangle = 0
\end{aligned}$$

$$\iff \quad x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x, y\rangle = 0.$$

Thus, we obtain the desired result. □

Note that from $\Phi^2(x,y)$ in Theorem 4.3, we obtain that

$$\Phi_1^2(x,y) := \sum_{(i,j)\notin I}^{r} \phi^2(\lambda_i(x),\lambda_j(y))f_j \text{ and } \Phi_2^2(x,y) := \sum_{(i,j)\notin I}^{r} \phi^2(\lambda_i(x),\lambda_j(y))(e_i+f_j).$$

are also C-functions.

We will now establish C-functions for the special case of two commutative elements x and y. Suppose x and y share the same Jordan frame, that is,

$$x = \lambda_1(x)e_1(x) + \cdots + \lambda_r(x)e_r(x) \text{ and } y = \lambda_{\sigma(1)}(y)e_1(x) + \cdots + \lambda_{\sigma(r)}(y)e_r(x),$$

where $\sigma : \{1,\dots,r\} \to \{1,\dots,r\}$ is a permutation which re-permutes the eigenvalues of y corresponding the Jordan frame $\{e_1(x),\dots,e_r(x)\}$. Let $\Omega(x) = \{(x,y) \in \mathbb{V}\times\mathbb{V} \mid x \text{ and } y \text{ operator commute}\}$. Define $\Omega = \cup_{x\in\mathbb{V}}\Omega(x)$.

Theorem 4.4. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be an NCP-function. The function $\widetilde{\Phi} : \Omega \subset \mathbb{V}\times\mathbb{V} \to \mathbb{V}$ defined by*

$$\widetilde{\Phi}(x,y) := \sum_{i=1}^{r} \phi(\lambda_i(x),\lambda_{\sigma(i)}(y))e_i(x)$$

is a C-function, where $\{e_1,e_2,\dots,e_r\}$ is a Jordan frame of x and y.

Proof. Clearly, $\widetilde{\Phi}(x,y)$ is well-defined. It suffices to show that $\widetilde{\Phi}(x,y) = 0 \iff x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x,y\rangle = 0$. Indeed, we have

$$\begin{aligned}
\widetilde{\Phi}(x,y) = 0 &\iff \sum_{i=1}^{r} \phi(\lambda_i(x),\lambda_{\sigma(i)}(y))e_i = 0 \\
&\iff \left\langle \sum_{i=1}^{r} \phi(\lambda_i(x),\lambda_{\sigma(i)}(y))e_i, \sum_{i=1}^{r} \phi(\lambda_i(x),\lambda_{\sigma(i)}(y))e_i \right\rangle = 0 \\
&\iff \sum_{i=1}^{r} \phi^2(\lambda_i(x),\lambda_{\sigma(i)}(y)) \langle e_i, e_i\rangle = 0 \\
&\iff \phi^2(\lambda_i(x),\lambda_{\sigma(i)}(y)) = 0,\ i = 1,\dots,r \\
&\iff \phi(\lambda_i(x),\lambda_{\sigma(i)}(y)) = 0,\ i = 1,\dots,r \\
&\iff \lambda_i(x) \ge 0,\ \lambda_i(y) \ge 0, \lambda_i(x)\lambda_{\sigma(i)}(y) = 0,\ i = 1,\dots,r \\
&\iff x \in \mathcal{K},\ y \in \mathcal{K},\ \sum_{i=1}^{r} \lambda_i(x)\lambda_{\sigma(i)}(y) \langle e_i, e_i\rangle = 0 \\
&\iff x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x,y\rangle = 0,
\end{aligned}$$

where $\langle e_i, e_i\rangle > 0$ and $\langle e_i, e_j\rangle = 0$ whenever $i \ne j$. The proof is complete. □

Based on Theorem 4.4, we will show that $\widetilde{\Phi}(x,y)$ retrieves the existing C-functions in the special case of two commutative elements x and y. In particular, we focus on two popular NCP-functions, the FB and NR functions:

$$\begin{aligned}\phi_{\rm FB}(a,b) &= \left(a^2+b^2\right)^{1/2}-(a+b),\\ \phi_{\rm NR}(a,b) &= a-(a-b)_+\end{aligned}$$

for any $(a,b)\in\mathbb{R}\times\mathbb{R}$. For any $(x,y)\in\Omega$, the corresponding C-functions are

$$\begin{aligned}\widetilde{\Phi}_{\rm FB}(x,y) &= \sum_{i=1}^r \phi_{\rm FB}(\lambda_i(x),\lambda_{\sigma(i)}(y))e_i,\\ \widetilde{\Phi}_{\rm NR}(x,y) &= \sum_{i=1}^r \phi_{\rm NR}(\lambda_i(x),\lambda_{\sigma(i)}(y))e_i.\end{aligned}$$

Then

$$\begin{aligned}\widetilde{\Phi}_{\rm FB}(x,y) &\equiv \varphi_{\rm FB}(x,y),\\ \widetilde{\Phi}_{\rm NR}(x,y) &\equiv \varphi_{\rm NR}(x,y).\end{aligned}$$

Indeed, since x and y operator commute, we have

$$x^2+y^2=\sum_{i=1}^r\lambda_i^2(x)e_i+\sum_{i=1}^r\lambda_{\sigma(i)}^2(y)e_i \text{ and } \left(x^2+y^2\right)^{1/2}=\sum_{i=1}^r\left(\lambda_i^2(x)+\lambda_{\sigma(i)}^2(y)\right)^{1/2}e_i,$$

and

$$(x-y)_+=\sum_{i=1}^r\left(\lambda_i(x)-\lambda_{\sigma(i)}(y)\right)_+e_i.$$

Hence,

$$\begin{aligned}\widetilde{\Phi}_{\rm FB}(x,y) &= \sum_{i=1}^r\left(\left(\lambda_i^2(x)+\lambda_{\sigma(i)}^2(y)\right)^{1/2}-(\lambda_i(x)+\lambda_{\sigma(i)}(y))\right)e_i\\ &= \sum_{i=1}^r\left(\lambda_i^2(x)+\lambda_{\sigma(i)}^2(y)\right)^{1/2}e_i-(\sum_{i=1}^r\lambda_i(x)e_i+\sum_{i=1}^r\lambda_{\sigma(i)}(y)e_i)\\ &= \left(x^2+y^2\right)^{1/2}-(x+y)=\varphi_{\rm FB}(x,y),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Phi}_{\rm NR}(x,y) &= \sum_{i=1}^r\left(\lambda_i(x)-\left(\lambda_i(x)-\lambda_{\sigma(i)}(y)\right)_+\right)e_i\\ &= \sum_{i=1}^r\lambda_i(x)e_i-\sum_{i=1}^r\left(\lambda_i(x)-\lambda_{\sigma(i)}(y)\right)_+e_i\\ &= x-(x-y)_+=\varphi_{\rm NR}(x,y).\end{aligned}$$

Similar arguments apply for other C-functions in the literature.

Remark 4.5. (i) Note that if $e_i(x) = f_i(y)$ for $i = 1, \dots, r$, that is, $\sigma(i) = i$, then

$$y = \lambda_1(y) f_1 + \cdots + \lambda_r(y) f_r = \lambda_1(y) e_1 + \cdots + \lambda_r(y) e_r.$$

Moreover, we can conclude that

$$x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x, y \rangle = 0 \Longleftrightarrow \lambda_i(x) \geq 0\ \lambda_i(y) \geq 0,\ \lambda_i(x)\lambda_i(y) = 0.$$

Suppose that $x \neq 0$, we have $\lambda_r(x) > 0$. Then the above relation implies that $\lambda_r(y) = 0$, i.e. $y = 0$.

(ii) From Theorem 4.4 and [15, Proposition 6], we have that for any $x \in \mathbb{V}$, $y \in \mathbb{V}$, $x \neq 0$, $y \neq 0$, there holds

$$x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x, y \rangle = 0 \Longleftrightarrow \lambda_1(x) = 0,\ \lambda_1(y) = 0,\ \langle x, y \rangle = 0. \tag{4.2}$$

Indeed, it is enough to prove that $x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x, y \rangle = 0 \Longrightarrow \lambda_1(x) = 0,\ \lambda_1(y) = 0,\ \langle x, y \rangle = 0$. According to [15, Proposition 6], we have that x and y operator commute which together with the proof of Theorem 4.4 imply

$$\begin{aligned} \lambda_i(x) \geq 0,\ \lambda_i(y) \geq 0,\ \lambda_i(x)\lambda_{\sigma(i)}(y) = 0,\ i = 1, \dots, r& \\ \Longrightarrow \lambda_1(x)\lambda_{\sigma(1)}(y) + \cdots + \lambda_r(x)\lambda_{\sigma(r)}(y) = 0.& \end{aligned}$$

Using the rearrangement inequality, we obtain

$$0 = \lambda_1(x)\lambda_{\sigma(1)}(y) + \cdots + \lambda_r(x)\lambda_{\sigma(r)}(y) \geq \lambda_1(x)\lambda_r(y) + \cdots + \lambda_r(x)\lambda_1(y) \geq 0$$

which yields

$$\lambda_1(x)\lambda_r(y) = 0,\ \lambda_r(x)\lambda_1(y) = 0 \Longrightarrow \lambda_1(x) = 0,\ \lambda_1(y) = 0,$$

where $\lambda_r(x) > 0$ and $\lambda_r(y) > 0$ due to $x \neq 0$, $y \neq 0$.

(iii) Using the relation (4.2), the following holds

$$\begin{aligned} & x \in \mathcal{K},\ y \in \mathcal{K},\ \langle x, y \rangle = 0 \\ \Longleftrightarrow\ & \phi(\lambda_1(x), \lambda_1(y)) = 0,\ \langle x, y \rangle = 0 \ \text{ or} \\ & \phi(\lambda_1(x),\ \lambda_r(y)) = 0,\ \phi(\lambda_r(x), \lambda_1(y)) = 0,\ \langle x, y \rangle = 0, \end{aligned} \tag{4.3}$$

where ϕ is an NCP-function.

(iv) We observe that the construction of a general form of C-functions based on NCP-functions open a new approach in tackling the SCCP through the spectral eigenvalues and spectral vectors(Jordan frame). In our work, we have found out a new direction for solving the SOCCP and SDCP based on NCP-functions via the relation (4.3) that they can be formulated as the minimization problem. Furthermore, there are explicit expressions of the inner product (Jordan product) in two special symmetric cones, that is, second-order cone and positive semidefinite cone. Then, a simple form of C-functions for these two special cases can be generated by using the relation (4.3). We will demonstrate these in the next Section.

4.2. **Using NCP-functions to construct C-functions in second-order cone setting.** As discussed in Remark 4.5(iii), a simple form of C-functions can be constructed in the settings of second-order cone and positive semidefinite cone based on the relation (4.3). Using Lemma 2.6, Lemma 2.7, and Lemma 2.8, we first show how to do it in the SOC setting.

Theorem 4.6. *Assume that $x, y \neq 0$ and let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be an NCP-function. For any $x = (x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, \bar{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the following vector-valued function $\Phi^1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$\Phi^1(x, y) := \begin{pmatrix} \phi(\lambda_1(x), \lambda_1(y)) \\ x_1\bar{y}_2 + y_1\bar{x}_2 \end{pmatrix}$$

is a C-function in the second-order cone setting.

In particular, for solutions (x, y) to the SOCCP, there holds $\|\bar{x}_2\| y_k = -\|\bar{y}_2\| x_k \neq 0$ for some $k \in \{2, \ldots, n\}$ when $\bar{x}_2 \neq 0$ and $\bar{y}_2 \neq 0$. Then the following vector-valued function $\Phi^2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi^2(x, y) := \begin{pmatrix} \phi(\lambda_1(x), \lambda_2(y)) \\ \phi(\lambda_2(x), \lambda_1(y)) \\ \bar{y}_3 x_k - \bar{x}_3 y_k \end{pmatrix}$$

is a C-function in the second-order cone setting.
Here,

$$\bar{x}_3 := (x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)^T \in \mathbb{R}^{n-2},$$
$$\bar{y}_3 := (y_2, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)^T \in \mathbb{R}^{n-2};$$

and $\lambda_i(x)$, $\lambda_i(y)$ for $i = 1, 2$ are the spectral values of x and y associated with second-order cone, respectively.

Proof. For $\bar{x}_2 = 0$ or $\bar{y}_2 = 0$, from Lemma 2.6, we know that $x = 0$ or $y = 0$. Then, it is easy to verify

$$x \succeq_{\mathscr{L}^n_+} 0,\ y \succeq_{\mathscr{L}^n_+} 0,\ x \circ y = 0 \quad \Longleftrightarrow \quad \Phi^1(x, y) = 0 \text{ and } \Phi^2(x, y) = 0.$$

Therefore, we only may focus on the case of $\bar{x}_2 \neq 0$ and $\bar{y}_2 \neq 0$.

(i) We first prove that $\Phi^1(x, y)$ is a C-function. To proceed, we note a fact that for any $x \in \mathscr{L}^n_+, y \in \mathscr{L}^n_+$ we have

$$\lambda_1(x) = 0,\ x_1\bar{y}_2 + y_1\bar{x}_2 = 0 \quad \Longleftrightarrow \quad \lambda_1(y) = 0,\ x_1\bar{y}_2 + y_1\bar{x}_2 = 0.$$

This fact together with Lemma 2.7 yields

$$\begin{aligned} \Phi^1(x, y) = 0 \quad &\Longleftrightarrow \quad \left\{ \begin{array}{l} \phi(\lambda_1(x), \lambda_1(y)) = 0 \\ x_1\bar{y}_2 + y_1\bar{x}_2 = 0 \end{array} \right. \\ &\Longleftrightarrow \quad \left\{ \begin{array}{l} \lambda_1(x)\lambda_1(y) = 0,\ \lambda_1(x) \geq 0,\ \lambda_1(y) \geq 0 \\ x_1\bar{y}_2 + y_1\bar{x}_2 = 0 \end{array} \right. \\ &\Longleftrightarrow \quad \left\{ \begin{array}{l} \lambda_1(x) = 0,\ \lambda_1(y) = 0 \\ x_1\bar{y}_2 + y_1\bar{x}_2 = 0 \end{array} \right. \\ &\Longleftrightarrow \quad x \succeq_{\mathscr{L}^n_+} 0,\ y \succeq_{\mathscr{L}^n_+} 0,\ x \circ y = 0. \end{aligned}$$

Thus, $\Phi^1(x,y)$ is a C-function.

(ii) We now show that $\Phi^2(x,y)$ is a C-function. Applying Lemma 2.7 and Lemma 2.8, it follows that

$$\begin{aligned}
x \succeq_{\mathscr{L}^n_+} 0,\ y \succeq_{\mathscr{L}^n_+} 0,\ x \circ y = 0 \quad &\Rightarrow \quad \begin{cases} \lambda_1(x) = 0 \\ \lambda_1(y) = 0 \\ \bar{y}_2 = -\frac{\|\bar{y}_2\|}{\|\bar{x}_2\|}\bar{x}_2 \end{cases} \\
&\Rightarrow \quad \begin{cases} \phi(\lambda_1(x), \lambda_2(y)) = 0 \\ \phi(\lambda_2(x), \lambda_1(y)) = 0 \\ \|\bar{x}_2\| y_k = -\|\bar{y}_2\| x_k \neq 0 \text{ for some } k \in \{2,\ldots,n\} \\ y_l x_k = x_l y_k \ \text{ for all } \ l \in \{2,\ldots,n\} \end{cases} \\
&\Rightarrow \quad \begin{cases} \phi(\lambda_1(x), \lambda_2(y)) = 0 \\ \phi(\lambda_2(x), \lambda_1(y)) = 0 \\ \|\bar{x}_2\| y_k = -\|\bar{y}_2\| x_k \neq 0 \text{ for some } k \in \{2,\ldots,n\} \\ \bar{y}_3 x_k - \bar{x}_3 y_k = 0 \end{cases} \\
&\Rightarrow \quad \Phi^2(x,y) = 0.
\end{aligned}$$

Conversely, suppose that $\Phi^2(x,y) = 0$. Due to ϕ being an NCP-function, we obtain

$$\begin{aligned}
\begin{cases} \phi(\lambda_1(x), \lambda_2(y)) = 0 \\ \phi(\lambda_2(x), \lambda_1(y)) = 0 \\ \bar{y}_3 x_k - \bar{x}_3 y_k = 0 \end{cases} \quad &\Rightarrow \quad \begin{cases} \lambda_1(x) \geq 0,\ \lambda_2(x) \geq 0,\ \lambda_1(y) \geq 0,\ \lambda_2(y) \geq 0 \\ \lambda_1(x)\lambda_2(y) = 0 \\ \lambda_2(x)\lambda_1(y) = 0 \end{cases} \\
&\Rightarrow \quad \begin{cases} x \in \mathscr{L}^n_+,\ y \in \mathscr{L}^n_+ \\ \lambda_1(x)\lambda_2(y) + \lambda_2(x)\lambda_1(y) = 0. \end{cases} \qquad (4.4)
\end{aligned}$$

Note that $\lambda_i(x) = x_1 + (-1)^i \|\bar{x}_2\|$ and $\lambda_i(y) = y_1 + (-1)^i \|\bar{y}_2\|$ for $i = 1, 2$. Hence, we have

$$\begin{aligned}
\lambda_1(x)\lambda_2(y) &= x_1 y_1 - \|\bar{x}_2\|\|\bar{y}_2\| + x_1\|\bar{y}_2\| - y_1\|\bar{x}_2\|, \\
\lambda_2(x)\lambda_1(y) &= x_1 y_1 - \|\bar{x}_2\|\|\bar{y}_2\| - x_1\|\bar{y}_2\| + y_1\|\bar{x}_2\|.
\end{aligned}$$

This fact together with (4.4) and Lemma 2.8 lead to

$$\lambda_1(x)\lambda_2(y) + \lambda_2(x)\lambda_1(y) = 2(x_1 y_1 - \|\bar{x}_2\|\|\bar{y}_2\|) = 2(x_1 y_1 + \bar{x}_2^T \bar{y}_2) = 0.$$

It follows that $\langle x, y \rangle = 0$. Thus, $\Phi^2(x,y)$ is a C-function. $\square$

Although in practice we can not know which k in advance when applying $\Phi^2(x,y)$, but we can take those k satisfying $x_k \neq 0$, $y_k \neq 0$ in turn when implementing $\Phi^2(x,y)$. Note also that in Theorem 4.6, the component $x_1\bar{y}_2 + y_1\bar{x}_2$ of $\Phi^1(x,y)$ is a vector in $\mathbb{R}^{n-1}$ while the component $\bar{y}_3 x_k - \bar{x}_3 y_k$ of $\Phi^2(x,y)$ is a vector in $\mathbb{R}^{n-2}$. Therefore, both ranges of $\Phi^1(x,y)$ and $\Phi^2(x,y)$ are $\mathbb{R}^n$. It is well-known that there have plenty of NCP-functions in the literature. According to Theorem 4.6, we can convert them into C-functions associated with second-order cone. We illustrate this using two NCP-functions in the following example.

Example 4.7. We consider two popular NCP-functions as follows:

$$\phi_{\text{FB}}(a,b) = \sqrt{a^2+b^2} - (a+b) \text{ and } \phi_{\text{NR}}(a,b) = a - (a-b)_+,\ \forall (a,b) \in \mathbb{R} \times \mathbb{R}.$$

In light of Theorem 4.6, it is not hard to see that

$$\Phi^1_{\rm FB}(x,y)=\begin{pmatrix}\phi_{\rm FB}(\lambda_1(x),\lambda_1(y))\\ x_1\bar{y}_2+y_1\bar{x}_2\end{pmatrix},\quad \Phi^1_{\rm NR}(x,y)=\begin{pmatrix}\phi_{\rm NR}(\lambda_1(x),\lambda_1(y))\\ x_1\bar{y}_2+y_1\bar{x}_2\end{pmatrix}$$

and

$$\Phi^2_{\rm FB}(x,y)=\begin{pmatrix}\phi_{\rm FB}(\lambda_1(x),\lambda_2(y))\\ \phi_{\rm FB}(\lambda_2(x),\lambda_1(y))\\ y_lx_k-x_ly_k\end{pmatrix},\quad \Phi^2_{\rm NR}(x,y)=\begin{pmatrix}\phi_{\rm NR}(\lambda_1(x),\lambda_2(y))\\ \phi_{\rm NR}(\lambda_2(x),\lambda_1(y))\\ \bar{y}_3x_k-\bar{x}_3y_k\end{pmatrix}$$

are C-functions, where $\|\bar{x}_2\|y_k=-\|\bar{y}_2\|x_k\neq 0$ for some $k\in\{2,\dots,n\}$ when $\bar{x}_2\neq 0$ and $\bar{y}_2\neq 0$,

$\bar{x}_3=(x_2,\dots,x_{k-1},x_{k+1},\dots,x_n)^T\in{\rm I\!R}^{n-2}$, $\bar{y}_3=(y_2,\dots,y_{k-1},y_{k+1},\dots,y_n)^T\in{\rm I\!R}^{n-2}$,

and $\lambda_i(x)$, $\lambda_i(y)$ for $i=1,2$ are spectral values of x and y, respectively.

Indeed, we can further conclude that

$$\Phi^i_{\rm FB}(x,y)=0,\ i=1,2 \quad\Longleftrightarrow\quad \varphi_{\rm FB}(x,y)=(x^2+y^2)^{1/2}-(x+y)=0$$

and

$$\Phi^i_{\rm NR}(x,y)=0,\ i=1,2 \quad\Longleftrightarrow\quad \varphi_{\rm NR}(x,y)=x-(x-y)_+=0.$$

To see this, by definition of C-function and Lemma 2.7, for $\bar{x}_2\neq 0$ and $\bar{y}_2\neq 0$, we have

$$\begin{aligned}\varphi_{\rm FB}(x,y)=0 &\Longleftrightarrow x\in\mathscr{L}^n_+,\ y\in\mathscr{L}^n_+,\ x\circ y=0\\ &\Longleftrightarrow \lambda_1(x)=0,\ \lambda_1(y)=0,\ x_1\bar{y}_2+y_1\bar{x}_2=0\\ &\Longleftrightarrow \Phi^1_{\rm FB}(x,y)=0.\end{aligned}$$

For $\bar{x}_2=0$ or $\bar{y}_2=0$, it is easy to check $\varphi_{\rm FB}(x,y)=0\Longleftrightarrow\Phi^1_{\rm FB}(x,y)=0$ by definition of C-function and Lemma 2.6. Similar arguments apply for other cases. The above discussions indicate that $\Phi^i_{\rm FB}(x,y)$ are C-functions and equivalent to the traditional complementarity function $\varphi_{\rm FB}(x,y)$; $\Phi^i_{\rm NR}(x,y)$ are C-functions and equivalent to the traditional complementarity function $\varphi_{\rm NR}(x,y)$.

Remark 4.8. (i) In Theorem 4.6, if ϕ is a continuously differentiable NCP-function, then $\Phi^1(x,y)$ and $\Phi^2(x,y)$ are continuously differentiable C-functions when $\bar{x}_2\neq 0$ and $\bar{y}_2\neq 0$. Let $y=F(x)$, where $F:{\rm I\!R}^n\to{\rm I\!R}^n$ is continuously differentiable. Then the first row of the Jacobian $J\Phi^1(x,F(x))$ and the first and second row of the Jacobian $J\Phi^2(x,F(x))$ are described by

$$\begin{aligned}\left(J\Phi^1(x,F(x))\right)_1 &= \frac{\partial\phi}{\partial\lambda_1(x)}\nabla\lambda_1(x)^T+\frac{\partial\phi}{\partial\lambda_1(F(x))}\left(DF(x)\nabla\lambda_1(F(x))\right)^T,\\ \left(J\Phi^2(x,F(x))\right)_1 &= \frac{\partial\phi}{\partial\lambda_1(x)}\nabla\lambda_1(x)^T+\frac{\partial\phi}{\partial\lambda_2(F(x))}\left(DF(x)\nabla\lambda_2(F(x))\right)^T,\\ \left(J\Phi^2(x,F(x))\right)_2 &= \frac{\partial\phi}{\partial\lambda_2(x)}\nabla\lambda_2(x)^T+\frac{\partial\phi}{\partial\lambda_1(F(x))}\left(DF(x)\nabla\lambda_1(F(x))\right)^T,\end{aligned}$$

when $x \in \mathrm{bd}(\mathscr{L}_+^n)\backslash\{0\}$ and $F(x) \in \mathrm{bd}(\mathscr{L}_+^n)\backslash\{0\}$. Since ϕ is continuously differentiable, it can be seen that

$$\frac{\partial\phi}{\partial\lambda_1(x)}(0,0)=0,\ \frac{\partial\phi}{\partial\lambda_1(F(x))}(0,0)=0,\ \frac{\partial\phi}{\partial\lambda_2(x)}(\lambda_2(x),0)\neq 0$$

$$\text{and } \frac{\partial\phi}{\partial\lambda_2(F(x))}(0,\lambda_2(F(x)))\neq 0$$

when $x \in \mathrm{bd}(\mathscr{L}_+^n)\backslash\{0\}$ and $F(x) \in \mathrm{bd}(\mathscr{L}_+^n)\backslash\{0\}$. Thus, for $x \in \mathrm{bd}(\mathscr{L}_+^n)\backslash\{0\}$ and $F(x) \in \mathrm{bd}(\mathscr{L}_+^n)\backslash\{0\}$, the first row of the Jacobian $J\Phi^1(x, F(x))$ is zero and the first and second row of the Jacobian $J\Phi^2(x, F(x))$ are nonzero. In summary, when we apply Newton method to solve the SOCCP, $\Phi^2(x, F(x))$ is a better choice than $\Phi^1(x, F(x))$.

(ii) Note that it is not easy to write out an explicit formula of components of the existing C-functions in the literature. However, we see that there are explicit expressions of components of $\Phi^1(x, y)$ and $\Phi^2(x, y)$ in Theorem 4.6 which leads to easier in computing their subgradient compared to the existing C-functions such as the complicated formula of B-subgradient of the FB C-function [30, Proposition 3.1]. Thus, using $\Phi^1(x, y)$ and $\Phi^2(x, y)$ to deal with the SOCCP may be easier in implementing numerical simulation.

(iii) Regarding Remark 4.5(ii)-(iii), we propose a new direction to tackle the SOCCP which can be solved by the following unconstrained minimization problem

$$\min_{x\in\mathbb{R}^n} \phi^2(\lambda_1(x), \lambda_1(F(x))) + \langle x, F(x)\rangle^2$$

or

$$\min_{x\in\mathbb{R}^n} \phi^2(\lambda_1(x), \lambda_2(F(x))) + \phi^2(\lambda_2(x), \lambda_1(F(x))) + \langle x, F(x)\rangle^2,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a map.

Next, we have a compact equivalence of SOCCP when $F(x)$ is a SOC-function.

Theorem 4.9. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be an NCP-function. Suppose that $F(x)$ is a SOC-function induced from function $f : \mathbb{R} \to \mathbb{R}$, which means $F(x)$ can be written:*

$$F(x) = f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)}.$$

Then, there holds

$$\begin{aligned} & x \in \mathscr{L}_+^n,\ F(x) \in \mathscr{L}_+^n,\ \langle x, F(x)\rangle = 0 \\ \Longleftrightarrow\quad & \Phi^3(x) := \begin{pmatrix} \phi(\lambda_1(x), f(\lambda_1(x))) \\ \phi(\lambda_2(x), f(\lambda_2(x))) \\ 0 \end{pmatrix} = 0 \end{aligned}$$

where $x = (x_1, \bar{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\lambda_i(x)$, $u_x^{(i)}$ for $i = 1, 2$ are the spectral values and the spectral vectors of x, respectively.

Proof. We will prove for the case $\Phi^3(x, F(x))$. Assume that $F(x)$ can be written:

$$F(x) = f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)}.$$

Hence, we have

$$
\begin{aligned}
& x \in \mathscr{L}^n_+,\ F(x) \in \mathscr{L}^n_+,\ \langle x, F(x)\rangle = 0 \\
\iff\ & \lambda_i(x) \geq 0,\ f(\lambda_i(x)) \geq 0,\ \lambda_1(x) f(\lambda_1(x)) + \lambda_2(x) f(\lambda_2(x)) = 0 \\
\iff\ & \lambda_i(x) \geq 0,\ f(\lambda_i(x)) \geq 0,\ \lambda_1(x) f(\lambda_1(x)) = 0,\ \lambda_2(x) f(\lambda_2(x)) = 0 \\
\iff\ & \phi(\lambda_1(x), f(\lambda_1(x))) = 0,\ \phi(\lambda_2(x), f(\lambda_2(x))) = 0 \\
\iff\ & \Phi^3(x) = 0.
\end{aligned}
$$

□

4.3. **Using NCP-functions to construct C-functions in positive semidefinite cone setting.** In this section, by using Lemma 2.5 and noting that $\mathbb{S}^{n\times n} \cong \mathrm{I\!R}^{\frac{n(n+1)}{2}}$, we show how to construct C-functions based on given NCP-functions in the setting of positive semidefinite cone. We introduce the following notations for convenience. For any $X, Y \in \mathbb{S}^{n\times n}$, we denote

$$
X := \big[\, \mathbf{x}_1 \mid \ \ldots \ \mid \mathbf{x}_n \,\big], \quad Y := \big[\, \mathbf{y}_1 \mid \ \ldots \ \mid \mathbf{y}_n \,\big],
$$

where $\mathbf{x}_i$ and $\mathbf{y}_i$ for $i = 1, \ldots, n$ are column vectors of matrices X and Y, respectively.

Theorem 4.10. *Let $\phi : \mathrm{I\!R}^2 \to \mathrm{I\!R}$ be an NCP-function. For any $X, Y \in \mathbb{S}^{n\times n}$, the following two functions $\Phi^i : \mathbb{S}^{n\times n} \times \mathbb{S}^{n\times n} \to \mathrm{I\!R}^{\frac{n(n+1)}{2}}$, $i = 1, 2$, given by*

$$
\Phi^1(X, Y) := \begin{pmatrix} \phi(\lambda_1(X), \lambda_1(Y)) \\ \mathbf{x}_1^T \mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n^T \mathbf{y}_n \\ \mathbf{0} \end{pmatrix}
$$

$$
\Phi^2(X, Y) := \begin{pmatrix} \phi(\lambda_1(X), \lambda_n(Y)) \\ \phi(\lambda_n(X), \lambda_1(Y)) \\ \mathbf{x}_1^T \mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n^T \mathbf{y}_n \\ \mathbf{0} \end{pmatrix}
$$

are C-functions. Here, the zero vector in $\Phi^1(X, Y)$ belongs to $\mathrm{I\!R}^{\frac{(n+1)(n-2)}{2}}$ whereas the zero vector in $\Phi^2(X, Y)$ belongs to $\mathrm{I\!R}^{\frac{n^2-n-4}{2}}$. In addition, $\lambda_i(X)$, $\lambda_i(Y)$ for $i = 1, \ldots, n$ are eigenvalues of matrices X, Y, which are arranged in the increasing order $\lambda_1(X) \leq \cdots \leq \lambda_n(X)$ and $\lambda_1(Y) \leq \cdots \leq \lambda_n(Y)$, respectively.

Proof. First, according to Lemma 2.5 and $\lambda_1(X) \leq \cdots \leq \lambda_n(X)$, $\lambda_1(Y) \leq \cdots \leq \lambda_n(Y)$, we have

$$
\begin{aligned}
& X \succeq 0, Y \succeq 0, \langle X, Y\rangle = 0 \\
\iff\ & X \succeq 0, Y \succeq 0, XY = 0 \\
\iff\ & \lambda_1(X) \geq 0, \lambda_1(Y) \geq 0, \text{ and } XY = 0.
\end{aligned} \tag{4.5}
$$

Suppose that $X = 0$ or $Y = 0$, it is easy to see that

$$X \succeq 0, Y \succeq 0, \langle X, Y\rangle = 0 \quad \Longleftrightarrow \quad \Phi^1(X,Y) = 0 \text{ and } \Phi^2(X,Y) = 0.$$

Therefore, it suffices to consider the case of $X \neq 0$ and $Y \neq 0$. Suppose that $\Phi^1(X,Y) = 0$. Noting that $\langle X, Y\rangle = \text{trace}(XY) = \sum_{i=1}^n \mathbf{x}_i^T \mathbf{y}_i$, we have

$$\begin{aligned}
\Phi^1(X,Y) = 0 \quad &\Rightarrow \quad \begin{cases} \phi(\lambda_1(X), \lambda_1(Y)) = 0 \\ \mathbf{x}_i^T \mathbf{y}_i = 0, \ i = 1, \dots, n \end{cases} \\
&\Rightarrow \quad \begin{cases} \lambda_1(X) \geq 0, \lambda_1(Y) \geq 0 \\ \sum_{i=1}^n \mathbf{x}_i^T \mathbf{y}_i = 0 \end{cases} \\
&\Rightarrow \quad X \succeq 0, Y \succeq 0, \langle X, Y\rangle = 0.
\end{aligned}$$

Conversely, from (4.5), we know that

$$X \succeq 0, Y \succeq 0, \langle X, Y\rangle = 0 \quad \Rightarrow \quad \lambda_1(X) \geq 0, \ \lambda_1(Y) \geq 0, \text{ and } XY = 0. \tag{4.6}$$

We now claim that

$$(4.6) \quad \Rightarrow \quad \lambda_1(X) = 0, \lambda_1(Y) = 0, \text{ and } XY = 0. \tag{4.7}$$

By contradiction, suppose that $\lambda_1(X) > 0$. Hence, $\lambda_i(X) > 0$ for all $i = 1, \dots, n$. It follows that $\det(X) = \lambda_1(X) \dots \lambda_n(X) > 0$, that is, X is nonsingular matrix. Multiplying both sides of $XY = 0$ by X^{-1} leads to $Y = 0$, which contradicts the fact that $Y \neq 0$. Thus, $\lambda_1(X) = 0$. Similarly, we can argue that $\lambda_1(Y) = 0$. This says that $\lambda_1(X)\lambda_1(Y) = 0$, and then $\phi(\lambda_1(X), \lambda_1(Y)) = 0$. On the other hand, since $XY = 0$, $\mathbf{x}_i^T \mathbf{y}_i = 0$ for all $i = 1, \dots, n$. All the above concludes $\Phi^1(X,Y) = 0$.

For the case of $\Phi^2(X,Y)$, likewise, we also have

$$\begin{aligned}
\Phi^2(X,Y) = 0 \quad &\Rightarrow \quad \begin{cases} \phi(\lambda_1(X), \lambda_n(Y)) = 0 \\ \phi(\lambda_n(X), \lambda_1(Y)) = 0 \\ \mathbf{x}_i^T \mathbf{y}_i = 0, \ i = 1, \dots, n \end{cases} \\
&\Rightarrow \quad \begin{cases} \lambda_1(X) \geq 0, \lambda_1(Y) \geq 0 \\ \sum_{i=1}^n \mathbf{x}_i^T \mathbf{y}_i = 0 \end{cases} \\
&\Rightarrow \quad X \succeq 0, Y \succeq 0, \langle X, Y\rangle = 0.
\end{aligned}$$

Conversely, suppose that $X \succeq 0, Y \succeq 0, \langle X, Y\rangle = 0$. Hence $\lambda_n(X) > 0$ and $\lambda_n(Y) > 0$. From (4.6) and (4.7), we have

$$\lambda_1(X) = 0, \ \lambda_1(Y) = 0, \text{ and } XY = 0.$$

This yields $\lambda_1(X)\lambda_n(Y) = 0$ and $\lambda_n(X)\lambda_1(Y) = 0$, which further imply that $\phi(\lambda_1(X), \lambda_n(Y)) = 0$ and $\phi(\lambda_n(X), \lambda_1(Y)) = 0$. Moreover, since $XY = 0$, $\mathbf{x}_i^T \mathbf{y}_i = 0$ for all $i = 1, \dots, n$. Thus, we conclude that $\Phi^2(X,Y) = 0$. $\square$

Note that both $\Phi^1(X,Y)$ and $\Phi^2(X,Y)$ yield vectors in $\mathbb{R}^{\frac{n(n+1)}{2}}$. Therefore, they could be viewed as matrix-valued function. In fact, there exist a lot of matrix expressions for $\Phi^1(X,Y)$ and $\Phi^2(X,Y)$. For instance,

$$\Phi^1(X,Y) \equiv \begin{pmatrix} \mathbf{x}_1^T\mathbf{y}_1 & \phi(\lambda_1(X),\lambda_1(Y)) & 0 & \dots & 0 \\ \phi(\lambda_1(X),\lambda_1(Y)) & \mathbf{x}_2^T\mathbf{y}_2 & 0 & \dots & 0 \\ 0 & 0 & \mathbf{x}_3^T\mathbf{y}_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{x}_n^T\mathbf{y}_n \end{pmatrix}$$

$$\Phi^2(X,Y) \equiv \begin{pmatrix} \mathbf{x}_1^T\mathbf{y}_1 & \phi(\lambda_1(X),\lambda_n(Y)) & \phi(\lambda_n(X),\lambda_1(Y)) & \dots & 0 \\ \phi(\lambda_1(X),\lambda_n(Y)) & \mathbf{x}_2^T\mathbf{y}_2 & 0 & \dots & 0 \\ \phi(\lambda_n(X),\lambda_1(Y)) & 0 & \mathbf{x}_3^T\mathbf{y}_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{x}_n^T\mathbf{y}_n \end{pmatrix}.$$

Example 4.11. We consider the FB function $\phi_{\rm FB}(a,b) = \sqrt{a^2+b^2} - (a+b)$ for all $(a,b) \in \mathbb{R} \times \mathbb{R}$. Their corresponding C-functions are

$$\Phi^1_{\rm FB}(X,Y) = \begin{pmatrix} \phi_{\rm FB}(\lambda_1(X),\lambda_1(Y)) \\ \mathbf{x}_1^T\mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n^T\mathbf{y}_n \\ \mathbf{0} \end{pmatrix}$$

$$\Phi^2_{\rm FB}(X,Y) = \begin{pmatrix} \phi_{\rm FB}(\lambda_1(X),\lambda_n(Y)) \\ \phi_{\rm FB}(\lambda_n(X),\lambda_1(Y)) \\ \mathbf{x}_1^T\mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n^T\mathbf{y}_n \\ \mathbf{0} \end{pmatrix},$$

where $\lambda_i(X)$, $\lambda_i(Y)$ for $i = 1,\dots,n$ are eigenvalues of matrices X, Y, which are arranged in the increasing order $\lambda_1(X) \le \cdots \le \lambda_n(X)$ and $\lambda_1(Y) \le \cdots \le \lambda_n(Y)$, respectively.

Likewise, in the setting of positive semidefinite cone, it is easy to see that

$$\Phi^i_{\rm FB}(X,Y) = 0,\ i = 1,2 \iff \varphi_{\rm FB}(X,Y) = (X^2+Y^2)^{1/2} - (X+Y) = 0.$$

This feature indicates that $\Phi^i_{\rm FB}(X,Y)$ are C-functions and equivalent to the traditional complementarity functions $\varphi_{\rm FB}(X,Y)$.

Remark 4.12. There are some other possible forms equivalent to $\Phi^1(X,Y)$ and $\Phi^2(X,Y)$ in Theorem 4.10 without having a lot of zeros. For instance, we could define

$$\widetilde{\Phi}^1(X,Y) := \begin{pmatrix} \phi(\lambda_1(X),\lambda_1(Y)) \\ \mathbf{x}_1^T\mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n^T\mathbf{y}_n \\ \mathbf{v}^1_{XY} \end{pmatrix}$$

$$\widetilde{\Phi}^2(X,Y) := \begin{pmatrix} \phi(\lambda_1(X),\lambda_n(Y)) \\ \phi(\lambda_n(X),\lambda_1(Y)) \\ \mathbf{x}_1^T\mathbf{y}_1 \\ \vdots \\ \mathbf{x}_n^T\mathbf{y}_n \\ \mathbf{v}_{XY}^2 \end{pmatrix},$$

where $\lambda_i(X)$, $\lambda_i(Y)$ for $i = 1,\dots,n$ are eigenvalues of matrices X, Y, which are arranged in the increasing order. Here, $\mathbf{v}_{XY}^1 \in \mathbb{R}^{\frac{(n+1)(n-2)}{2}}$ and $\mathbf{v}_{XY}^2 \in \mathbb{R}^{\frac{n^2-n-4}{2}}$ may have many alternative forms, one pair of them is

$$\mathbf{v}_{XY}^1 := \left(\mathbf{x}_1^T\mathbf{y}_2,\dots,\mathbf{x}_1^T\mathbf{y}_n,\mathbf{x}_2^T\mathbf{y}_3,\dots,\mathbf{x}_{n-2}^T\mathbf{y}_n\right)^T,$$

$$\mathbf{v}_{XY}^2 := \left(\mathbf{x}_1^T\mathbf{y}_2,\dots,\mathbf{x}_1^T\mathbf{y}_n,\mathbf{x}_2^T\mathbf{y}_3,\dots,\mathbf{x}_{n-2}^T\mathbf{y}_{n-1}\right)^T.$$

Again, there are many matrix forms for $\widetilde{\Phi}^1(X,Y)$ and $\widetilde{\Phi}^2(X,Y)$. We hereby provide two matrix forms as follows:

$$\widetilde{\Phi}^1(X,Y) \equiv \begin{pmatrix} \mathbf{x}_1^T\mathbf{y}_1 & \phi(\lambda_1(X),\lambda_1(Y)) & \mathbf{x}_1^T\mathbf{y}_2 & \cdots & \mathbf{x}_1^T\mathbf{y}_{n-1} \\ \phi(\lambda_1(X),\lambda_1(Y)) & \mathbf{x}_2^T\mathbf{y}_2 & \mathbf{x}_1^T\mathbf{y}_n & \cdots & \mathbf{x}_2^T\mathbf{y}_{n-1} \\ \mathbf{x}_1^T\mathbf{y}_2 & \mathbf{x}_1^T\mathbf{y}_n & \mathbf{x}_3^T\mathbf{y}_3 & \cdots & \mathbf{x}_3^T\mathbf{y}_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_1^T\mathbf{y}_{n-1} & \mathbf{x}_2^T\mathbf{y}_{n-1} & \mathbf{x}_3^T\mathbf{y}_{n-1} & \cdots & \mathbf{x}_n^T\mathbf{y}_n \end{pmatrix}$$

$$\widetilde{\Phi}^2(X,Y) \equiv \begin{pmatrix} \mathbf{x}_1^T\mathbf{y}_1 & \phi(\lambda_1(X),\lambda_n(Y)) & \phi(\lambda_n(X),\lambda_1(Y)) & \cdots & \mathbf{x}_1^T\mathbf{y}_{n-2} \\ \phi(\lambda_1(X),\lambda_n(Y)) & \mathbf{x}_2^T\mathbf{y}_2 & \mathbf{x}_1^T\mathbf{y}_{n-1} & \cdots & \mathbf{x}_2^T\mathbf{y}_{n-2} \\ \phi(\lambda_n(X),\lambda_1(Y)) & \mathbf{x}_1^T\mathbf{y}_{n-1} & \mathbf{x}_3^T\mathbf{y}_3 & \cdots & \mathbf{x}_3^T\mathbf{y}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_1^T\mathbf{y}_{n-2} & \mathbf{x}_2^T\mathbf{y}_{n-2} & \mathbf{x}_3^T\mathbf{y}_{n-2} & \cdots & \mathbf{x}_n^T\mathbf{y}_n \end{pmatrix}.$$

Note that it might be difficult in using $\Phi^1(X,Y)$ and $\Phi^2(X,Y)$ to define a merit function $\frac{1}{2}\|\Phi(X,Y)\|^2$ for solving the SDCP due to the implicitness of eigenvalues of a real symmetric matrix. Thus, we propose a new direction to deal with the SDCP through NCP-functions. More precisely, we will present a form of optimization problem for the SDCP. Let $F : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ be a map. The SDCP is to find a matrix $X \in \mathbb{S}^{n\times n}$ such that

$$X \in \mathbb{S}^{n\times n}_+,\ F(X) \in \mathbb{S}^{n\times n}_+,\ \langle X, F(X)\rangle = 0. \tag{4.8}$$

According to the relation (4.2), the SDCP (4.8) is equivalent to find a matrix $X \in \mathbb{S}^{n\times n}$ such that

$$\lambda_1(X) = 0,\ \lambda_1(F(X)) = 0,\ \langle X, F(X)\rangle = 0$$

when $X \neq 0$ and $F(X) \neq 0$. One knows that

$$\lambda_1(X) = \min_{\|u\|=1} u^T X u \quad \text{and} \quad \lambda_1(F(X)) = \min_{\|v\|=1} v^T F(X) v.$$

Then, for the case $X \in \mathrm{bd}(\mathbb{S}^{n\times n}_+)$ and $F(X) \in \mathrm{bd}(\mathbb{S}^{n\times n}_+)$, the SDCP (4.8) becomes the following bilevel optimization problem:

$$\begin{array}{ll} \min & f(X, \lambda_1(X), \lambda_1(F(X))) := (\lambda_1(X))^2 + (\lambda_1(F(X)))^2 + \langle X, F(X)\rangle^2 \\ \text{s.t.} & \lambda_1(X) = \min\limits_{\|u\|=1} u^T X u \quad \text{and} \quad \lambda_1(F(X)) = \min\limits_{\|v\|=1} v^T F(X) v,\ X \in \mathbb{S}^{n\times n}. \end{array}$$

If the minimal value is zero, then there exists a matrix $X \in \mathbb{S}^{n\times n}$ satisfying $\lambda_1(X) = 0,\ \lambda_1(F(X)) = 0,\ \langle X, F(X)\rangle = 0$ which is a solution of the SDCP. We see that the above problem does not provide the solution for the cases $X = 0$ and $F(X) \in \mathrm{int}(\mathbb{S}^{n\times n}_+)$ or $X \in \mathrm{int}(\mathbb{S}^{n\times n}_+)$ and $F(X) = 0$. However, this will not happen if we use the same technique for $\Phi^1(X, F(X))$ and $\Phi^2(X, F(X))$. Note that

$$\Phi^1(X, F(X)) = 0 \iff \begin{array}{c} \langle X, F(X)\rangle = 0, \text{ and} \\ \phi(\lambda_1(X), \lambda_1(F(X))) = 0, \end{array}$$

or

$$\Phi^2(X, F(X)) = 0 \iff \begin{array}{c} \langle X, F(X)\rangle = 0, \\ \phi(\lambda_1(X), \lambda_n(F(X))) = 0 \text{ and } \phi(\lambda_n(X), \lambda_1(F(X))) = 0, \end{array}$$

where ϕ is a given NCP-function. Then, we have the corresponding bilevel optimization problems:

$$\begin{array}{ll} \min & f(X, \lambda_1(X), \lambda_1(F(X))) := (\phi(\lambda_1(X), \lambda_1(F(X))))^2 + \langle X, F(X)\rangle^2 \\ \text{s.t.} & \lambda_1(X) = \min\limits_{\|u\|=1} u^T X u \quad \text{and} \quad \lambda_1(F(X)) = \min\limits_{\|v\|=1} v^T F(X) v,\ X \in \mathbb{S}^{n\times n}. \end{array}$$

or

$$\begin{array}{ll} \min & f(X, \lambda_1(X), \lambda_1(F(X))) := (\phi(\lambda_1(X), \lambda_n(F(X))))^2 \\ & \qquad\qquad + (\phi(\lambda_n(X), \lambda_1(F(X))))^2 + \langle X, F(X)\rangle^2 \\ \text{s.t.} & \lambda_1(X) = \min\limits_{\|u\|=1} u^T X u,\ \lambda_n(X) = \max\limits_{\|u\|=1} u^T X u,\ \lambda_1(F(X)) = \min\limits_{\|v\|=1} v^T F(X) v, \\ & \text{and } \lambda_n(F(X)) = \max\limits_{\|u\|=1} u^T F(X) u,\ X \in \mathbb{S}^{n\times n}. \end{array}$$

Therefore, if the minimal value is zero, then there exists a matrix $X \in \mathbb{S}^{n\times n}$ satisfying $\lambda_1(X) \geq 0,\ \lambda_1(F(X)) \geq 0,\ \langle X, F(X)\rangle = 0$.

Next, we introduce an equivalence based on a special type of matrix-valued functions. To this end, we recall that for a real-valued function $f : \mathbb{R} \to \mathbb{R}$, there is a corresponding matrix-valued function defined by

$$f^{\text{matr}}(X) = f(\lambda_1(X))U_1 + \cdots + f(\lambda_n(X))U_n,$$

where X has the spectral decomposition $X = \lambda_1(X)U_1 + \cdots + \lambda_n(X)U_n$. For more details regarding this special matrix-valued functions, please refer to [12].

Theorem 4.13. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be an NCP-function. Suppose that $F(X)$ is the matrix-valued function induced by a function $f : \mathbb{R} \to \mathbb{R}$, that is, $F(X)$ is written as*

$$F(X) = f(\lambda_1(X))U_1 + \cdots + f(\lambda_n(X))U_n,$$

with $X = \lambda_1(X)U_1 + \cdots + \lambda_n(X)U_n$, where $\lambda_i(X)$, $i = 1, \ldots, n$ are eigenvalues of X and $\{U_i\}_{i=1}^n$ is a Jordan frame. For any $X \in \mathbb{S}^{n\times n}$, we define $\Phi^3 : \mathbb{S}^{n\times n} \to \mathbb{R}^{\frac{n(n+1)}{2}}$

by

$$\Phi^3(X) := \begin{pmatrix} \phi(\lambda_1(X), f(\lambda_1(X))) \\ \vdots \\ \phi(\lambda_n(X), f(\lambda_n(X))) \\ \mathbf{0} \end{pmatrix},$$

where the zero vector belongs to $\mathbb{R}^{\frac{n(n-1)}{2}}$. *Then the equation* $\Phi^3(X) = 0$ *solves the SDCP.*

Proof. Again, applying Lemma 2.5 yields

$$\begin{aligned} & X \succeq 0,\ F(X) \succeq 0,\ \langle X, F(X)\rangle = 0 \\ \Longleftrightarrow\ & X \succeq 0,\ F(X) \succeq 0,\ XF(X) = 0 \\ \Longleftrightarrow\ & \lambda_i(X) \geq 0,\ f(\lambda_i(X)) \geq 0,\ i = 1, \ldots, n, \text{ and } \sum_{i=1}^{n} \lambda_i(X) f(\lambda_i(X)) U_i = 0 \\ \Longleftrightarrow\ & \lambda_i(X) \geq 0,\ f(\lambda_i(X)) \geq 0,\ \lambda_i(X) f(\lambda_i(X)) = 0,\ i = 1, \ldots, n \\ \Longleftrightarrow\ & \phi(\lambda_i(X),\ f(\lambda_i(X))) = 0,\ i = 1, \ldots, n \\ \Longleftrightarrow\ & \Phi^3(X) = 0. \end{aligned}$$

This clearly proves that $\Phi^3(X) = 0$ solves the SDCP. $\square$

Similarly, there exists matrix forms for $\Phi^3(X)$ in Theorem 4.13, one of them is

$$\Phi^3(X) \equiv \begin{pmatrix} \phi(\lambda_1(X), f(\lambda_1(X))) & 0 & \ldots & 0 \\ 0 & \phi(\lambda_2(X), f(\lambda_2(X))) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \phi(\lambda_n(X), f(\lambda_n(X))) \end{pmatrix}.$$

5. Concluding Remarks

This paper proposes two methods of constructing C-functions for the SCCP. The first method is the traditional way by extending a class of NCP-functions to C-functions whose method was also studied by many researchers in the literature. An interesting point of this method is that we explore the composition of the existing C-functions with the function θ which provides many C-functions under three different assumptions of θ.

The important contribution of our work is the second method that using the real-valued NCP-functions defined on $\mathbb{R}^2$ to generate the vector-valued C-functions. In particular, we have constructed the general formulas of C-functions for the SCCP using given NCP-functions. In other words, there is an NCP-function, there is a C-function. Moreover, our C-functions recover the already existing known C-functions in the literature for the special case of two commutative operator. A remarkable point of the second method is that we have established the simple formulas of C-functions for second-order cone and positive semidefinite cone settings based on their explicit expressions of the inner product (Jordan product). We conclude that this novel idea opens up a new approach for solving the SCCP based on NCP-functions as we mentioned in Section 4. A study in detail leaves for future work.

Appendix

Proof of the well-defined of $\Phi(x, y)$ in Theorem 4.1

First, we recall that the degree of an element x is the number of its distinct eigenvalues. Here we only demonstrate one of the special case in which the degrees of x and y are $r-p$ and $r-q$ respectively. Similar arguments work for other cases of other degrees.

Suppose that

$$x=\sum_{i=1}^{r}\lambda_i(x)e_i \text{ and } x=\sum_{i=1}^{r}\lambda_i'(x)e_i'$$

satisfying $\lambda_1(x)=\cdots=\lambda_p(x)$, $1\le p\le r$, and other eigenvalues are distinct.

$$y=\sum_{j=1}^{r}\lambda_j(y)f_j \text{ and } y=\sum_{j=1}^{r}\lambda_j'(y)f_j'.$$

satisfying $\lambda_1(y)=\cdots=\lambda_q(y)$, $1\le q\le r$, and other eigenvalues are distinct.

From Theorem 5 in Baes's paper, we have $\lambda_i(x)=\lambda_i^{'}(x)$, $\lambda_j(y)=\lambda_j^{'}(y)$ and $e_i=e_i^{'}$, $f_j=f_j'$ for $p+1\le i\le r$, $q+1\le j\le r$. Moreover, $\lambda_1(x)=\cdots=\lambda_p(x)=\lambda_1'(x)=\cdots=\lambda_p'(x)$ and $\sum_{i=1}^{p}\lambda_1(x)e_i=\sum_{i=1}^{p}\lambda_1(x)e_i'$; $\lambda_1(y)=\cdots=\lambda_q(y)=\lambda_1'(y)=\cdots=\lambda_q'(y)$ and $\sum_{j=1}^{q}\lambda_1(y)f_j=\sum_{j=1}^{q}\lambda_1(y)f_j'$. Then, it follows that

$$x=\sum_{i=1}^{p}\lambda_1(x)e_i+\sum_{i=p+1}^{r}\lambda_i(x)e_i=\sum_{i=1}^{p}\lambda_1(x)e_i'+\sum_{i=p+1}^{r}\lambda_i(x)e_i,$$

$$y=\sum_{j=1}^{q}\lambda_1(y)f_j+\sum_{j=q+1}^{r}\lambda_j(y)f_j=\sum_{j=1}^{q}\lambda_1(y)f_j'+\sum_{j=q+1}^{r}\lambda_j(y)f_j.$$

Now we need to show that

$$\Phi(x,y)=\sum_{i,j=1}^{r}\phi^2(\lambda_i(x),\lambda_j(y))e_i\circ f_j=\sum_{i,j=1}^{r}\phi^2(\lambda_i'(x),\lambda_j'(y))e_i'\circ f_j'.$$

Indeed, there hold

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=1}^{q}\phi^2(\lambda_i(x),\lambda_j(y))e_i\circ f_j &= \phi^2(\lambda_1(x),\lambda_1(y))\sum_{i=1}^{p}\sum_{j=1}^{q}e_i\circ f_j\\ &= \phi^2(\lambda_1'(x),\lambda_1'(y))\sum_{i=1}^{p}\sum_{j=1}^{q}e_i'\circ f_j'\\ &= \sum_{i=1}^{p}\sum_{j=1}^{q}\phi^2(\lambda_i'(x),\lambda_i'(y))e_i'\circ f_j',\end{aligned}$$

$$\sum_{i=1}^{p}\sum_{j=q+1}^{r}\phi^2(\lambda_i(x),\lambda_j(y))e_i\circ f_j=\sum_{j=q+1}^{r}\phi^2(\lambda_1(x),\lambda_j'(y))\sum_{i=1}^{p}e_i\circ f_j'$$

$$
\begin{aligned}
&= \sum_{j=q+1}^{r} \phi^2(\lambda_1'(x), \lambda_j'(y)) \sum_{i=1}^{p} e_i' \circ f_j' \\
&= \sum_{i=1}^{p} \sum_{j=q+1}^{r} \phi^2(\lambda_i'(x), \lambda_j'(y)) e_i' \circ f_j',
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=p+1}^{r} \sum_{j=1}^{q} \phi^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j &= \sum_{i=p+1}^{r} \phi^2(\lambda_i(x), \lambda_1'(y)) \sum_{j=1}^{q} e_i' \circ f_j \\
&= \sum_{i=p+1}^{r} \phi^2(\lambda_i'(x), \lambda_1'(y)) \sum_{j=1}^{q} e_i' \circ f_j' \\
&= \sum_{i=p+1}^{r} \sum_{j=1}^{q} \phi^2(\lambda_i'(x), \lambda_j'(y)) e_i' \circ f_j',
\end{aligned}
$$

$$
\sum_{i=p+1}^{r} \sum_{j=q+1}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j = \sum_{i=p+1}^{r} \sum_{j=q+1}^{r} \phi^2(\lambda_i'(x), \lambda_j'(y)) e_i' \circ f_j'.
$$

From all above, it is easy to see that

$$
\sum_{i,j=1}^{r} \phi^2(\lambda_i(x), \lambda_j(y)) e_i \circ f_j = \sum_{i,j=1}^{r} \phi^2(\lambda_i'(x), \lambda_j'(y)) e_i' \circ f_j'.
$$

Thus, the proof is complete.

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References

[1] J. H. Alcantara, C.-H. Lee, C.T Nguyen, Y.-L. Chang and J.-S Chen, *On construction of new NCP functions*, Oper. Res. Lett. **48** (2020), 115–121.

[2] M. Baes, *Convexity and differentiability properties of spectral functions and spectral mappings on Euclidean Jordan algebras*, Lin. Alg. Appl. **422** (2007), 664–700.

[3] S.-J. Bi, S.-H. Pan and J.-S. Chen, *The same growth of FB and NR symmetric cone complementarity functions*, Oper. Res. Lett. **6** (2012) 153–162.

[4] Y.-L. Chang, J.-S. Chen and C.-Y. Yang, *Symmetrization of generalized natural residual function for NCP*, Oper. Res. Lett. **43** (2015), 354–358.

[5] B. Chen, X. Chen and C. Kanzow, *A penalized Fischer-Burmeister NCP-function*, Math. Program. **88** (2000), 211–216.

[6] J.-S. Chen, *On some NCP-functions based on the generalized Fischer-Burmeister function*, Asia-Pac. J. Oper. Res. **24** (2007), 401–420.

[7] J.-S. Chen, *SOC Functions and Their Applications*, Springer Optimization and Its Applications 143, Springer, Singapore 2019.

[8] J.-S. Chen, C.-H. Ko, X.-R. Wu, *What is the generalization of natural residual function for NCP?*, Pac. J. Optim. **12** (2016), 19–27.

[9] J.-S. Chen and S.-H. Pan, *A family of NCP functions and a descent method for the nonlinear complementarity problem*, Comput. Optim. Appl. **40** (2008): 389404.

[10] J.-S. Chen and P. Tseng, *An unconstrained smooth minimization reformulation of second-order cone complementarity problem*, Math. Program. **104** (2005), 293–327.
[11] X. D. Chen, D. Sun and J. Sun, *Complementarity functions and numerical experiments on some smoothing Newton methods for second-order cone complementarity problems*, Comput. Optim. Appl. **25** (2003), 39–56.
[12] X. Chen, H. D. Qi and P. Tseng, *Analysis of nonsmooth symmetric-matrix-valued functions with applications to semidefinite complementarity problems*, SIAM J. Optim. **13** (2003), 960–985
[13] A. Galantai, *Properties and construction of NCP functions*, Comput. Optim. Appl. **52** (2012), 805–824.
[14] M. S. Gowda and Y. Song, *On semidefinite linear complementarity problems*, Math. Program. **88** (2000), 575–587.
[15] M.S. Gowda, R. Sznaijder, J. Tao, *Some P-properties for linear transformations on Euclidean Jordan algebras*, Lin. Alg. Appl. **393** (2004) 203–232.
[16] J. Faraut and A. Korányi, *Analysis on Symmetric Cones, Oxford Mathematical Monographs, Oxford University Press*, New York, 1994.
[17] M. Fukushima, Z.-Q. Luo and P. Tseng, *Smoothing functions for second-order cone complementarity problems*, SIAM J. Optim. **12** (2002), 436–460.
[18] C.-H. Huang, J.-S. Chen and J. E. Martinez-Legaz, *Differentiability v.s. convexity for complementarity functions*, Optim. Lett. **11** (2017), 209–216.
[19] C. Kanzow, *Nonlinear complementarity as unconstrained optimization*, J. Optim. Theory Appl. **88** (1996), 139–155.
[20] C. Kanzow and H. Kleinmichel, *A Class of Netwton-type methods for equality and ineqality constrained optimization*, Optim. Method Software **5** (1995), 173–198.
[21] C. Kanzow, N. Yamashita and M. Fukushima, *New NCP-functions and their properties*, J. Optim. Theory Appl. **94** (1997), 115-135.
[22] L. C. Kong, L. Tuncel and N. H. Xiu, *Vector-valued implicit Lagrangian for symmetric cone complementarity problems*, Asia-Pac. J. Oper. Res. **26** (2009) 199–233.
[23] S. Kum and Y. Lim, *Penalized complementarity functions on symmetric cones*, J. Glob. Optim. **46** (2010), 475–485.
[24] Y. Liu, L.-W. Zhang and Y.-H. Wang, *Some properties of a class of merit functions for symmetric cone complementarity problems*, Asia-Pac. J. Oper. Res. **23** (2006), 473–495.
[25] Z.-Q. Luo and P. Tseng, *A new class of merit functions for the nonlinear complementarity problem*, in Complementarity and Variational Problems: State of the Art, edited by M. C. Ferris and J.-S. Pang, SIAM, 1997, pp. 204–225.
[26] P.-F. Ma, J.-S. Chen, C.-H. Huang and C.-H. Ko, *Discovery of new complementarity functions for NCP and SOCCP*, Comput. Appl. Math. **37** (2018) 5727–5749.
[27] O. L. Mangasarian, *Equivalence of the complementarity problem to a system of nonlinear equations*, SIAM J. Appl. Math. **31** (1976) 89–92.
[28] S. M. Miri and S. Effati, *On generalized convexity of nonlinear complementarity functions*, J. Optim. Theory Appl. **164** (2015) 723–730.
[29] S.-H. Pan and J.-S. Chen, *A one-parametric class of merit functions for the symmetric cone complementarity problem*, J. Optim. Anal. Appl. **355** (2009) 195–215.
[30] S.-H. Pan and J.-S. Chen, *A damped Gauss-Newton method for the second-order cone complementarity problem*, Appl. Math. Optim. **59** (2009) 293–318.
[31] S.-H. Pan, S. Kum, Y. Lim and J.-S. Chen, *On the generalized Fischer-Burmeister merit function for the second-order cone complementarity problem*, Math. Comput. **83** (2014) 1143–1171.
[32] M. V. Ramana, *An exact duality theory for semidefinite programming and its complexity implications*, Math. Program. **77** (1997) 129–162.
[33] D. Sun and J. Sun, *Löwner's operator and spectral functions in Euclidean Jordan algebras*, Math. Oper. Res. **33** (2008) 421–445.
[34] J. Tao and M.S. Gowda, *Some P-properties for the nonlinear transformations on Euclidean Jordan algebras*, Math. Oper. Res. **30** (2005) 985–1004.

[35] P. Tseng, *Merit functions for semi-definite complementarity problems*, Math. Program. **83** (1998) 159–185.

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