

## SOME USEFUL INEQUALITIES VIA TRACE FUNCTION METHOD IN EUCLIDEAN JORDAN ALGEBRAS

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ABSTRACT. In this paper, we establish convexity of some functions associated with symmetric cones, called SC trace functions. As illustrated in the paper, these functions play a key role in the development of penalty and barrier function methods for symmetric cone programs. With trace function method we offer much simpler proofs to these useful inequalities.

**1. Introduction.** The second-order cone (SOC) in  $\mathbb{R}^n$ , also called Lorentz cone, is the set defined as

$$\mathcal{K}^n := \left\{ (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|x_2\| \right\}, \quad (1)$$

where  $\|\cdot\|$  denotes the Euclidean norm. When  $n = 1$ ,  $\mathcal{K}^n$  reduces to the set of nonnegative real numbers  $\mathbb{R}_+$ . As shown in [14],  $\mathcal{K}^n$  is also a set composed of the squared elements from Jordan algebra  $(\mathbb{R}^n, \circ)$ , where the Jordan product “ $\circ$ ” is a binary operation defined by

$$x \circ y := (\langle x, y \rangle, x_1 y_2 + y_1 x_2) \quad (2)$$

for any  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ . Here for any  $x \in \mathbb{R}^n$ , we use  $x_1$  to denote the first component of  $x$ , and  $x_2$  to denote the vector consisting of the rest  $n - 1$  components.

From [13, 14], we recall that each  $x \in \mathbb{R}^n$  admits a spectral decomposition associated with  $\mathcal{K}^n$  of the following form

$$x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}, \quad (3)$$

where  $\lambda_i(x)$  and  $u_x^{(i)}$  for  $i = 1, 2$  are the spectral values and the associated spectral vectors of  $x$ , respectively, defined by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|, \quad u_x^{(i)} = \frac{1}{2} \left( 1, (-1)^i \bar{x}_2 \right), \quad (4)$$

with  $\bar{x}_2 = \frac{x_2}{\|x_2\|}$  if  $x_2 \neq 0$ , and otherwise  $\bar{x}_2$  being any vector in  $\mathbb{R}^{n-1}$  such that  $\|\bar{x}_2\| = 1$ . When  $x_2 \neq 0$ , the spectral factorization is unique. The determinant and trace of  $x$  are defined as  $\det(x) := \lambda_1(x)\lambda_2(x)$  and  $\text{tr}(x) := \lambda_1(x) + \lambda_2(x)$ , respectively.

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With the spectral decomposition above, for any given scalar function  $\phi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , we may define a vector-valued function  $\phi^{\text{soc}} : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$\phi^{\text{soc}}(x) := \phi(\lambda_1(x))u_x^{(1)} + \phi(\lambda_2(x))u_x^{(2)} \quad (5)$$

where  $J$  is an interval (finite or infinite, open or closed) of  $\mathbb{R}$ , and  $S$  is the domain of  $\phi^{\text{soc}}$  determined by  $\phi$ . Then, we can define the SOC trace function associated with  $\phi$

$$\phi^{\text{tr}}(x) := \phi(\lambda_1(x)) + \phi(\lambda_2(x)) = \text{tr}(\phi^{\text{soc}}(x)) \quad \forall x \in S. \quad (6)$$

Chen, Liao and Pan [12] give the following relation between  $\phi^{\text{tr}}$  and  $\phi^{\text{soc}}$

$$\nabla \phi^{\text{tr}}(x) = (\phi')^{\text{soc}}(x) \quad \text{and} \quad \nabla^2 \phi^{\text{tr}}(x) = \nabla(\phi')^{\text{soc}}(x) \quad \forall x \in \text{int}S. \quad (7)$$

By using Schur Complement Theorem, they establish the convexity of SOC trace functions and the compounds of SOC trace functions. Some of these functions are the key of penalty and barrier function methods for second-order cone programs (SOCPs), as well as the establishment of some important inequalities associated with SOCs, for which the proof of convexity of these functions is a necessity.

Some similar results associated with positive semi-definite cone are also investigated by Auslender in [1, 2]. Since both SOC and positive semi-definite cone are special cases of symmetric cone (SC for short). A natural question leads us to consider the more general case. To this end, we need to recall some concepts regarding Euclidean Jordan algebra. Let  $\mathbb{A} = (\mathbb{V}, \langle \cdot, \cdot \rangle, \circ)$  be an  $n$ -dimensional Euclidean Jordan algebra (see Section 2) and  $\mathcal{K}$  be the symmetric cone in  $\mathbb{V}$ . For any given scalar function  $\phi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , we define the associated function

$$\phi_{\mathbb{V}}^{\text{sc}}(x) := \phi(\lambda_1(x))c_1 + \cdots + \phi(\lambda_r(x))c_r, \quad (8)$$

and SC trace function

$$\phi_{\mathbb{V}}^{\text{tr}}(x) := \phi(\lambda_1(x)) + \cdots + \phi(\lambda_r(x)) = \text{tr}(\phi_{\mathbb{V}}^{\text{sc}}(x)) \quad \forall x \in S, \quad (9)$$

where  $x \in \mathbb{V}$  has the spectral decomposition

$$x = \lambda_1(x)c_1 + \cdots + \lambda_r(x)c_r.$$

In this paper we extend the aforementioned results to general symmetric cone setting where we establish the convexity of SC trace functions and the compounds of SC trace functions. Throughout this note, for any  $x, y \in \mathbb{V}$ , we write  $x \succeq_{\mathcal{K}} y$  if  $x - y \in \mathcal{K}$ ; and write  $x \succ_{\mathcal{K}} y$  if  $x - y \in \text{int}\mathcal{K}$ . For a real symmetric matrix  $A$ , we write  $A \succeq 0$  (respectively,  $A \succ 0$ ) if  $A$  is positive semi-definite (respectively, positive definite). For any  $\phi : J \rightarrow \mathbb{R}$ ,  $\phi'(t)$  and  $\phi''(t)$  denote the first derivative and second-order derivative of  $\phi$  at the differentiable point  $t \in J$ , respectively. Suppose  $F : S \subseteq \mathbb{V} \rightarrow \mathbb{R}$ ,  $\nabla F(x)$  and  $\nabla^2 F(x)$  denote the gradient and the Hessian matrix of  $F$  at the differentiable point  $x \in S$ , respectively.

**2. Preliminaries.** This section recalls some results on Euclidean Jordan algebras that will be used in subsequent analysis. More detailed expositions of Euclidean Jordan algebras can be found in Koecher's lecture notes [17] and the monograph by Faraut and Korányi [14].

Let  $\mathbb{V}$  be an  $n$ -dimensional vector space over the real field  $\mathbb{R}$ , endowed with a bilinear mapping  $(x, y) \mapsto x \circ y$  from  $\mathbb{V} \times \mathbb{V}$  into  $\mathbb{V}$ . The pair  $(\mathbb{V}, \circ)$  is called a *Jordan algebra* if

- (i):  $x \circ y = y \circ x$  for all  $x, y \in \mathbb{V}$ ,
- (ii):  $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$  for all  $x, y \in \mathbb{V}$ .

Note that a Jordan algebra is not necessarily associative, i.e.,  $x \circ (y \circ z) = (x \circ y) \circ z$  may not hold for all  $x, y, z \in \mathbb{V}$ . We call an element  $e \in \mathbb{V}$  the *identity* element if  $x \circ e = e \circ x = x$  for all  $x \in \mathbb{V}$ . A Jordan algebra  $(\mathbb{V}, \circ)$  with an identity element  $e$  is called a *Euclidean Jordan algebra* if there is an inner product  $\langle \cdot, \cdot \rangle_{\mathbb{V}}$  such that

$$\text{(iii): } \langle x \circ y, z \rangle_{\mathbb{V}} = \langle y, x \circ z \rangle_{\mathbb{V}} \text{ for all } x, y, z \in \mathbb{V}.$$

Given a Euclidean Jordan algebra  $\mathbb{A} = (\mathbb{V}, \circ, \langle \cdot, \cdot \rangle_{\mathbb{V}})$ , we denote the set of squares as

$$\mathcal{K} := \{x^2 \mid x \in \mathbb{V}\}.$$

From [14, Theorem III.2.1],  $\mathcal{K}$  is a symmetric cone which means that  $\mathcal{K}$  is a self-dual closed convex cone with nonempty interior and for any two elements  $x, y \in \mathbf{int}\mathcal{K}$ , there exists an invertible linear transformation  $\mathcal{T} : \mathbb{V} \rightarrow \mathbb{V}$  such that  $\mathcal{T}(\mathcal{K}) = \mathcal{K}$  and  $\mathcal{T}(x) = y$ .

For any given  $x \in \mathbb{A}$ , let  $\zeta(x)$  be the degree of the minimal polynomial of  $x$ , i.e.,

$$\zeta(x) := \min \{k : \{e, x, x^2, \dots, x^k\} \text{ are linearly dependent}\}.$$

Then the *rank* of  $\mathbb{A}$  is defined as  $\max\{\zeta(x) : x \in \mathbb{V}\}$ . In this paper, we use  $r$  to denote the rank of the underlying Euclidean Jordan algebra. Recall that an element  $c \in \mathbb{V}$  is *idempotent* if  $c^2 = c$ . Two idempotents  $c_i$  and  $c_j$  are said to be *orthogonal* if  $c_i \circ c_j = 0$ . One says that  $\{c_1, c_2, \dots, c_k\}$  is a *complete system of orthogonal idempotents* if

$$c_j^2 = c_j, \quad c_j \circ c_i = 0 \text{ if } j \neq i \text{ for all } j, i = 1, 2, \dots, k \quad \text{and} \quad \sum_{j=1}^k c_j = e.$$

An idempotent is *primitive* if it is nonzero and cannot be written as the sum of two other nonzero idempotents. We call a complete system of orthogonal primitive idempotents a *Jordan frame*. Now we state the second version of the spectral decomposition theorem.

**Theorem 2.1.** [14, Theorem III.1.2] *Suppose that  $\mathbb{A}$  is a Euclidean Jordan algebra with rank  $r$ . Then for any  $x \in \mathbb{V}$ , there exists a Jordan frame  $\{c_1, \dots, c_r\}$  and real numbers  $\lambda_1(x), \dots, \lambda_r(x)$ , arranged in the decreasing order  $\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_r(x)$ , such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \dots + \lambda_r(x)c_r.$$

*The numbers  $\lambda_j(x)$  (counting multiplicities), which are uniquely determined by  $x$ , are called the eigenvalues and  $\text{tr}(x) = \sum_{j=1}^r \lambda_j(x)$  the trace of  $x$ .*

Since, by [14, Proposition III.1.5], a Jordan algebra  $(\mathbb{V}, \circ)$  with an identity element  $e \in \mathbb{V}$  is Euclidean if and only if the symmetric bilinear form  $\text{tr}(x \circ y)$  is positive definite, we may define another inner product on  $\mathbb{V}$  by  $\langle x, y \rangle := \text{tr}(x \circ y)$  for any  $x, y \in \mathbb{V}$ . The inner product  $\langle \cdot, \cdot \rangle$  is associative by [14, Prop. II. 4.3], i.e.,  $\langle x, y \circ z \rangle = \langle y, x \circ z \rangle$  for any  $x, y, z \in \mathbb{V}$ . For any given  $x \in \mathbb{V}$ , let  $\mathcal{L}(x)$  be the linear operator of  $\mathbb{V}$  defined by

$$\mathcal{L}(x)y := x \circ y \quad \forall y \in \mathbb{V}.$$

Then,  $\mathcal{L}(x)$  is symmetric with respect to the inner product  $\langle \cdot, \cdot \rangle$  in the sense that

$$\langle \mathcal{L}(x)y, z \rangle = \langle y, \mathcal{L}(x)z \rangle \quad \forall y, z \in \mathbb{V}.$$

In the sequel, we let  $\|\cdot\|$  be the norm on  $\mathbb{V}$  induced by the inner product, namely,

$$\|x\| := \sqrt{\langle x, x \rangle} = \left( \sum_{j=1}^r \lambda_j^2(x) \right)^{1/2} \quad \forall x \in \mathbb{V}. \quad (10)$$

A Euclidean Jordan algebra is called *simple* if it cannot be written as a direct sum of the other two Euclidean Jordan algebras. It is known that every Euclidean Jordan

algebra is a direct sum of simple Euclidean Jordan algebras. Unless otherwise stated, in the rest of this paper, we assume that  $\mathbb{A} = (\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$  is a simple Euclidean Jordan algebra of rank  $r$ . Let  $\{c_1, c_2, \dots, c_r\}$  be a Jordan frame of  $\mathbb{A}$ . From [14, Lemma IV. 1.3], we know that the operators  $\mathcal{L}(c_j)$ ,  $j = 1, 2, \dots, r$  commute and admit a simultaneous diagonalization. For  $i, j \in \{1, 2, \dots, r\}$ , define the subspaces

$$\mathbb{V}_{ii} := \mathbb{R}c_i \quad \text{and} \quad \mathbb{V}_{ij} := \left\{ x \in \mathbb{V} \mid c_i \circ x = c_j \circ x = \frac{1}{2}x \right\} \quad \text{when } i \neq j.$$

Then, [14, Corollary IV.2.6] says

$$\dim(\mathbb{V}_{ij}) = \dim(\mathbb{V}_{st}) \quad \text{for any } i \neq j \in \{1, 2, \dots, r\} \text{ and } s \neq t \in \{1, 2, \dots, r\},$$

and  $n = r + \frac{d}{2}r(r-1)$  where  $d$  denotes this common dimension. Moreover, from [14, Theorem IV.2.1], we have the following conclusion.

**Theorem 2.2.** *The space  $\mathbb{V}$  is the orthogonal direct sum of subspaces  $\mathbb{V}_{ij}$  ( $1 \leq i \leq j \leq r$ ), i.e.,  $\mathbb{V} = \bigoplus_{i \leq j} \mathbb{V}_{ij}$ . Furthermore,*

$$\begin{aligned} \mathbb{V}_{ij} \circ \mathbb{V}_{ij} &\subset \mathbb{V}_{ii} + \mathbb{V}_{ij}, \\ \mathbb{V}_{ij} \circ \mathbb{V}_{jk} &\subset \mathbb{V}_{ik}, \text{ if } i \neq k, \\ \mathbb{V}_{ij} \circ \mathbb{V}_{kl} &= \{0\}, \text{ if } \{i, j\} \cap \{k, l\} = \emptyset. \end{aligned}$$

Let  $x \in \mathbb{V}$  have the spectral decomposition  $x = \sum_{j=1}^r \lambda_j(x)c_j$ , where  $\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_r(x)$  are the eigenvalues of  $x$  and  $\{c_1, c_2, \dots, c_r\}$  is the corresponding Jordan frame. For  $i, j \in \{1, 2, \dots, r\}$ , let  $\mathcal{C}_{ij}(x)$  be the orthogonal projection operator onto  $\mathbb{V}_{ij}$ . Then, from Theorem IV 2.1 of [14], it follows that for all  $i, j = 1, 2, \dots, r$ ,

$$\mathcal{C}_{jj}(x) = 2\mathcal{L}^2(c_j) - \mathcal{L}(c_j) \quad \text{and} \quad \mathcal{C}_{ij}(x) = 4\mathcal{L}(c_i)\mathcal{L}(c_j) = 4\mathcal{L}(c_j)\mathcal{L}(c_i) = \mathcal{C}_{ji}(x). \quad (11)$$

Moreover, the orthogonal projection operators  $\{\mathcal{C}_{ij}(x) : i, j = 1, 2, \dots, r\}$  satisfy

$$\mathcal{C}_{ij}(x) = \mathcal{C}_{ij}^*(x), \quad \mathcal{C}_{ij}^2(x) = \mathcal{C}_{ij}(x), \quad \mathcal{C}_{ij}(x)\mathcal{C}_{kl}(x) = 0 \text{ if } \{i, j\} \neq \{k, l\} \quad (12)$$

and

$$\sum_{1 \leq i \leq j \leq r} \mathcal{C}_{ij}(x) = \mathcal{I}. \quad (13)$$

Suppose  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a scalar valued function and we define the Löwner operator associated with  $\phi$  as

$$\phi_{\mathbb{V}}^{\text{sc}}(x) =: \sum_{j=1}^r \phi(\lambda_j(x))c_j,$$

where  $x \in \mathbb{V}$  has the spectral decomposition  $x = \sum_{j=1}^r \lambda_j(x)c_j$ . Korányi [16] (or see [21]) proves the following result, which generalizes Löwner result on symmetric matrices to Euclidean Jordan algebras.

**Theorem 2.3.** *Let  $x = \sum_{j=1}^r \lambda_j(x)c_j$  and  $(a, b)$  be an open interval in  $\mathbb{R}$  that contains  $\lambda_j(x)$ ,  $j = 1, 2, \dots, r$ . If  $\phi$  is continuously differentiable on  $(a, b)$ , then  $\phi_{\mathbb{V}}^{\text{sc}}$  is differentiable at  $x$  and its derivative, for any  $h \in \mathbb{V}$ , is given by*

$$(\nabla \phi_{\mathbb{V}}^{\text{sc}})(x)(h) = \sum_{i=1}^r \left( \phi^{[1]}(\lambda(x)) \right)_{ii} \mathcal{C}_{ii}(x)h + \sum_{1 \leq j < l \leq r} \left( \phi^{[1]}(\lambda(x)) \right)_{jl} \mathcal{C}_{ji}(x)h \quad (14)$$

where the coefficient is defined as

$$\phi^{[1]}(\lambda(x))_{jl} := \begin{cases} \phi'(\lambda_j) & \text{if } \lambda_j = \lambda_l, \\ \frac{\phi(\lambda_j) - \phi(\lambda_l)}{\lambda_j - \lambda_l} & \text{if } \lambda_j \neq \lambda_l. \end{cases} \quad (15)$$

Moreover based on this, Sun and Sun [21] show that  $\phi_v^{\text{sc}}$  is continuously differentiable at  $x$  if and only if  $\phi$  is continuously differentiable at  $\lambda_j(x)$ ,  $j = 1, 2, \dots, r$ . We will exploit this property to achieve Lemma 3.1 which paves a way to our main result.

**3. Main results.** At first we recall some technical results in [9].

**Lemma 3.1** ([9]). *For any given scalar function  $\phi: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , let  $\phi_v^{\text{sc}}: S \rightarrow \mathbb{V}$  and  $\phi_v^{\text{tr}}: S \rightarrow \mathbb{R}$  be given by (8) and (9), respectively. Assume that  $J$  is an open interval in  $\mathbb{R}$ . Then, the following results hold.*

- (a): *The domain  $S$  of  $\phi_v^{\text{sc}}$  and  $\phi_v^{\text{tr}}$  is open and convex.*
- (b): *If  $\phi$  is (continuously) differentiable, then  $\phi_v^{\text{tr}}$  is (continuously) differentiable on  $S$  with  $\nabla \phi_v^{\text{tr}}(x)(h) = \langle h, (\phi')_v^{\text{sc}}(x) \rangle$  for all  $h \in \mathbb{V}$ .*
- (c): *If  $\phi$  is twice (continuously) differentiable, then  $\phi_v^{\text{tr}}$  is twice (continuously) differentiable on  $S$  with  $\nabla^2 \phi_v^{\text{tr}}(x)(h, k) = \langle h, \nabla(\phi')_v^{\text{sc}}(x)k \rangle$  for all  $h, k \in \mathbb{V}$ .*

**Theorem 3.2** ([9]). *For any given  $\phi: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , let  $\phi_v: S \subseteq \mathbb{V} \rightarrow \mathbb{R}^n$  and  $\phi_v^{\text{tr}}: S \subseteq \mathbb{V} \rightarrow \mathbb{R}$  be given by (8) and (9), respectively. Assume that  $J$  is an open interval in  $\mathbb{R}$ . If  $\phi$  is twice differentiable on  $J$ , then*

- (a):  *$\phi''(t) \geq 0$  for any  $t \in J \iff \nabla^2 \phi_v^{\text{tr}}(x) \succeq 0$  for any  $x \in S \iff \phi_v^{\text{tr}}$  is convex in  $S$ .*
- (b):  *$\phi''(t) > 0$  for any  $t \in J \iff \nabla^2 \phi_v^{\text{tr}}(x) \succ 0 \forall x \in S \implies \phi_v^{\text{tr}}$  is strictly convex in  $S$ .*

Indeed, the fact that the strict convexity of  $\phi$  implies the strict convexity of  $\phi_v^{\text{tr}}$  was proved in [2, 8] via checking the definition of convex function. But, here our analysis is much simpler and we also give the relation between  $\nabla(\phi')_v^{\text{sc}}$  and  $\nabla^2 \phi_v^{\text{tr}}$  to achieve the convexity of SC trace functions. In addition, we note that the necessity involved in the first equivalence of Theorem 3.2(a) was given in [13] for SOC case via a different way. Next, we will illustrate the application of Theorem 3.2 with some SC trace functions.

**Theorem 3.3** ([9]). *The following functions associated with  $\mathcal{K}$  are all strictly convex.*

- (a):  $F_1(x) = -\ln(\det(x))$  for  $x \in \mathbf{int}\mathcal{K}$ .
- (b):  $F_2(x) = \text{tr}(x^{-1})$  for  $x \in \mathbf{int}\mathcal{K}$ .
- (c):  $F_3(x) = \text{tr}(h(x))$  for  $x \in \mathbf{int}\mathcal{K}$ , where

$$h(x) = \begin{cases} \frac{x^{p+1} - e}{p+1} + \frac{x^{1-q} - e}{q-1} & \text{if } p \in [0, 1], q > 1; \\ \frac{x^{p+1} - e}{p+1} - \ln x & \text{if } p \in [0, 1], q = 1. \end{cases}$$

- (d):  $F_4(x) = -\ln(\det(e - x))$  for  $x \prec_{\mathcal{K}} e$ .
- (e):  $F_5(x) = \text{tr}((e - x)^{-1} \circ x)$  for  $x \prec_{\mathcal{K}} e$ .
- (f):  $F_6(x) = \text{tr}(\exp(x))$  for  $x \in \mathbb{V}$ .
- (g):  $F_7(x) = \ln(\det(e + \exp(x)))$  for  $x \in \mathbb{V}$ .
- (h):  $F_8(x) = \text{tr}\left(\frac{x + (x^2 + 4e)^{1/2}}{2}\right)$  for  $x \in \mathbb{V}$ .

In the following we start to apply our new technique.

**Proposition 1.** *For any  $x \in \mathcal{K}$ , let  $F_9(x) := -[\det(x)]^p$  with  $0 < p < 1$ . Then,*

- (a):  $F_9$  is twice continuously differentiable in  $\mathbf{int}\mathcal{K}$ .
- (b):  $F_9$  is convex when  $p \leq \frac{1}{r}$ , and moreover, it is strictly convex when  $p < \frac{1}{r}$  in  $\mathbf{int}\mathcal{K}$  where  $r$  is the rank of the Euclidean Jordan Algebra.

*Proof.* (a) Note that  $\det(x) = \exp(\text{tr}(\ln x))$  for any  $x \in \mathbf{int}\mathcal{K}$ , and  $\text{tr}(\ln(x)) = \phi_v^{\text{tr}}(x)$  with  $\phi(t) = \ln(t)$  for  $t > 0$ . By Lemma 3.1(c),  $\text{tr}(\ln(x))$  is twice continuously differentiable in  $\mathbf{int}\mathcal{K}$ , and hence  $\det(x)$  is twice continuously differentiable in  $\mathbf{int}\mathcal{K}$ . Since  $t^{1/p}$  with  $p > 1$  is infinite continuously differentiable on  $(0, +\infty)$ , the desired result then follows.

(b) We now prove the convexity over  $\mathbf{int}\mathcal{K}$ . From our construction we have that  $F_9(x) = -\exp(p\phi_v^{\text{tr}}(x))$  with  $\phi(t) = \ln(t)$ . By direct computation we get

$$\nabla^2 F_9(x)(h, k) = -\exp(p\phi_v^{\text{tr}}(x)) [p^2 \nabla \phi_v^{\text{tr}}(x)(h) \nabla \phi_v^{\text{tr}}(x)(k) + p \nabla^2 \phi_v^{\text{tr}}(x)(h, k)].$$

From Lemma 3.1(c) and Theorem 2.3 we obtain

$$\nabla^2 \phi_v^{\text{tr}}(x) = \sum_{i=1}^r \frac{-1}{\lambda_i \lambda_i} C_{ii}(x) + \sum_{1 \leq j < l \leq r} \frac{-1}{\lambda_j \lambda_l} C_{jl}(x),$$

where  $\lambda_i$  ( $1 \leq i \leq r$ ) are the eigenvalues of  $x$ .

Define an  $r \times r$  matrix  $A$  as

$$A = \begin{bmatrix} \frac{1}{\lambda_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2^2} & 0 & \cdots & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & \cdots & 0 & & \frac{1}{\lambda_r^2} \end{bmatrix}.$$

Also define a diagonal matrix  $C$  as

$$C = \text{diag}(\overbrace{\frac{1}{\lambda_1 \lambda_2}, \dots, \frac{1}{\lambda_1 \lambda_2}}^{d's}, \overbrace{\frac{1}{\lambda_1 \lambda_3}, \dots, \frac{1}{\lambda_1 \lambda_3}}^{d's}, \dots, \overbrace{\frac{1}{\lambda_{(r-1)} \lambda_r}, \dots, \frac{1}{\lambda_{(r-1)} \lambda_r}}^{d's}).$$

We can extend the Jordan frame  $\{c_1, c_2, \dots, c_r\}$  to an orthonormal basis

$$\Xi = \{c_1, c_2, \dots, c_r, c_{12}^{(1)}, \dots, c_{12}^{(d)}, c_{13}^{(1)}, \dots, c_{13}^{(d)}, \dots, c_{(r-1)r}^{(1)}, \dots, c_{(r-1)r}^{(d)}\}$$

for  $\mathbb{V}$ . Under this basis  $\Xi$ , one is able to express  $-\nabla^2 \phi_v^{\text{tr}}(x)$  as an  $n \times n$  matrix

$$\begin{bmatrix} A & O \\ O & C \end{bmatrix}$$

which is positive definite. On the other hand note that  $\phi'(t) = \frac{1}{t}$ . From Lemma 3.1 (b) we get

$$\nabla \phi_v^{\text{tr}}(x)(h) = \langle h, (\phi')_v^{\text{sc}}(x) \rangle = \langle h, \sum_{i=1}^r \frac{1}{\lambda_i} c_i \rangle.$$

Denote  $H(h, k) = \nabla \phi_v^{\text{tr}}(x)(h) \nabla \phi_v^{\text{tr}}(x)(k)$ . Define an  $r \times r$  matrix  $B$  as

$$B = \begin{bmatrix} \frac{1}{\lambda_1^2} & \frac{1}{\lambda_1 \lambda_2} & \cdots & \frac{1}{\lambda_1 \lambda_r} \\ \frac{1}{\lambda_1 \lambda_2} & \frac{1}{\lambda_2^2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \frac{1}{\lambda_{r-1} \lambda_r} \\ \frac{1}{\lambda_1 \lambda_r} & \cdots & \frac{1}{\lambda_{(r-1)} \lambda_r} & \frac{1}{\lambda_r^2} \end{bmatrix}.$$

Under the basis  $\Xi$ , one is able to express  $H$  as an  $n \times n$  matrix

$$\begin{bmatrix} B & O \\ O & O \end{bmatrix}.$$

Thus we can conclude that  $\nabla^2 F_9(x)$  can be expressed as an  $n \times n$  matrix

$$\exp(p\phi_v^{\text{tr}}(x)) \begin{bmatrix} pA - p^2B & O \\ O & pC \end{bmatrix}$$

under the basis  $\Xi$ . In order to discuss positivity of  $\nabla^2 F_9(x)$  we pick up an arbitrary vector  $x = (x_1, \dots, x_r)^t$  in  $\Re^r$ . Apply  $x$  to  $pA - p^2B$  we obtain

$$x^t(pA - p^2B)x = p \sum_{i=1}^r \frac{x_i^2}{\lambda_i^2} - p^2 \left( \sum_{i=1}^r \frac{x_i}{\lambda_i} \right)^2.$$

By Cauchy-Schwartz inequality  $(\sum_{i=1}^r \frac{x_i}{\lambda_i})^2 \leq r \sum_{i=1}^r \frac{x_i^2}{\lambda_i^2}$ , we get

$$x^t(pA - p^2B)x \geq p(1 - rp) \sum_{i=1}^r \frac{x_i^2}{\lambda_i^2}.$$

Since  $x \in \mathbf{int}\mathcal{K}$ , all its  $\lambda_i(x) > 0$ . From above inequality and  $C$  is positive definite our work is done.  $\square$

It is worthwhile to point out that  $\det(x)$  is neither convex nor concave on  $\mathcal{K}$ , and it is difficult to argue the convexity of those compound functions involving  $\det(x)$  by the definition of convex function. But, as shown in Proposition 1, ours offers a simple way to prove their convexity.

To close section, we take a look at the application of some of convex functions above in establishing inequalities associated with symmetric cones. Some of these inequalities have been used to analyze the properties of SOC-function [11] and the convergence of interior point methods for SOCPs [2].

**Proposition 2.** *Suppose  $r$  is the rank of the Euclidean Jordan Algebra. For any  $x \succ_{\mathcal{K}} 0$  and  $y \succ_{\mathcal{K}} 0$ , the following inequalities hold.*

- (a):  $\det(\alpha x + (1 - \alpha)y) \geq (\det(x))^\alpha (\det(y))^{1-\alpha}$  for any  $0 < \alpha < 1$ .
- (b):  $\det(x + y)^{1/p} \geq 2^{\frac{r}{p}-1} (\det(x)^{1/p} + \det(y)^{1/p})$  for any  $p \geq r$ .
- (c):  $\det(x + y)^{1/r} \geq \det(x)^{1/r} + \det(y)^{1/r}$  and  $\det(x + y) \geq \det(x) + \det(y)$ .
- (d):  $\det(\alpha x + (1 - \alpha)y) \geq \alpha^r \det(x) + (1 - \alpha)^r \det(y)$  for any  $0 < \alpha < 1$ .
- (e):  $[\det(e + x)]^{1/r} \geq 1 + \det(x)^{1/r}$ .
- (f): If  $x \succ_{\mathcal{K}} y$ , then  $\det(x) > \det(y)$ .
- (g):  $\text{tr}(x \circ y) \geq r \det(x)^{1/r} \det(y)^{1/r}$ .
- (h):  $\det(x)^{1/r} = \min \left\{ \frac{1}{r} \text{tr}(x \circ y) : \det(y) = 1, y \succ_{\mathcal{K}} 0 \right\}$ .

*Proof.* (a) From Theorem 3.3(a), we know that  $\ln(\det(x))$  is strictly concave in  $\mathbf{int}\mathcal{K}$ . Thus,

$$\begin{aligned}\ln(\det(\alpha x + (1 - \alpha)y)) &\geq \alpha \ln(\det(x)) + (1 - \alpha) \ln(\det(y)) \\ &= \ln(\det(x)^\alpha) + \ln(\det(y)^{1-\alpha})\end{aligned}$$

for any  $0 < \alpha < 1$  and  $x, y \in \mathbf{int}\mathcal{K}$ . This, together with the increasing of  $\ln t$  ( $t > 0$ ) and the continuity of  $\det(x)$ , implies the desired result.

(b) By Prop. 1(b),  $\det(x)^{1/p}$  is concave over  $\mathbf{int}\mathcal{K}$  and observe that  $\det(\alpha x) = \alpha^r \det(x)$  for  $\alpha > 0$ . Thus we have that

$$\begin{aligned}\det\left(\frac{x+y}{2}\right)^{1/p} &\geq \frac{1}{2} [\det(x)^{1/p} + \det(y)^{1/p}] \\ \implies \det(x+y)^{1/p} \left(\frac{1}{2}\right)^{\frac{r}{p}} &\geq \frac{1}{2} [\det(x)^{1/p} + \det(y)^{1/p}] \\ \implies \det(x+y)^{1/p} &\geq 2^{\frac{r}{p}-1} (\det(x)^{1/p} + \det(y)^{1/p})\end{aligned}$$

which is the desired result.

(c) Using the inequality in part (b) with  $p = r$ , we have

$$\det(x+y)^{1/r} \geq \det(x)^{1/r} + \det(y)^{1/r}.$$

Taking  $r$ -th power both sides yields

$$\det(x+y) \geq \det(x) + \det(y) + \dots \geq \det(x) + \det(y),$$

where the last inequality is by the nonnegativity of  $\det(x)$  and  $\det(y)$  since  $x, y \in \mathbf{int}\mathcal{K}$ .

(d) The inequality is direct by part (c) and the fact  $\det(\alpha x) = \alpha^r \det(x)$ .

(e) The inequality follows from part (b) with  $p = r$  and the fact that  $\det(e) = 1$ .

(f) Using part (c) and noting that  $x \succ_{\kappa} y$ , it is easy to verify that

$$\det(x) = \det(y + x - y) > \det(y) + \det(x - y) > \det(y).$$

(g) Suppose  $x$  has the decomposition  $x = \sum_{i=1}^r \lambda_i c_i$ . With this Jordan frame  $\{c_1, c_2, \dots, c_r\}$ ,  $y$  can be written as  $y = \sum_{i=1}^r y_i c_i + \sum_{1 \leq j < l \leq r} y_{jl}$  which is called Peirce decomposition.

$$\begin{aligned}\mathrm{tr}(x \circ y) &= \langle x \circ y, e \rangle \\ &= \langle x, y \rangle \\ &= \sum_{i=1}^r \lambda_i \langle c_i, y \rangle \\ &\geq r \prod_{i=1}^r \lambda_i^{1/r} \prod_{i=1}^r \langle c_i, y \rangle^{1/r} \\ &= r \det(x)^{1/r} \prod_{i=1}^r y_i^{1/r} \\ &\geq r \det(x)^{1/r} \det(y)^{1/r}.\end{aligned}$$

Here we apply **Arithmetic-Geometric Mean Inequality** on first inequality, and apply inequality (13) in [20] on second inequality.

(h) Using part (g), and  $\det(y) = 1$ , we have the result, and the minimum value is attained when taking  $y^* = \det(x)^{1/r} x^{-1}$ . Thus, we complete the proof.  $\square$



Analogous to SOC case, e.g., [6, 7, 18, 19, 22], the functions  $F_1$ ,  $F_2$  and  $F_3$  can be adapted as barrier functions for symmetric cone programming (SCP) which play a key role in the development of interior point methods for SCPs. The function  $F_3$  covers a wide range of barrier functions, including the classical logarithmic barrier function, the self-regular functions and the non-self-regular functions; see [7] for details. The functions  $F_4$  and  $F_5$  are called shifted barrier functions [1, 2, 3] for SOCPs, and  $F_6$ - $F_8$  can be used as penalty functions for SCPs.

Besides the application in establishing convexity for SC trace functions, our establishment of convexity of some compound functions of SC trace functions and scalar-valued functions is much simpler, which is usually difficult to achieve by the definition of convex function.

In the end of this section we should point out that one of the referee offers another interesting direction to prove Proposition 1 (b) as the following. It is known that  $-\lceil \det(x) \rceil^p = \min_{t>0} \{t \log t - t - \text{trace}(tp(\log X))\}$ . So, if the function  $f(t; X) = t \log t - t - \text{trace}(tp(\log X))$  is shown to be joint convexity, then part(b) of Proposition 1 follows.

**4. Conclusions.** We establish convexity of SC-functions, especially for SC trace functions, which are the key of penalty and barrier function methods for symmetric cone programming and some important inequalities associated with symmetric cones. We believe that the results in this paper will be helpful towards establishing further properties of other SC functions. Here we have to thank the referee. They point out some mistakes on our first manuscript and offer their interesting view on this paper. Without their kind help and careful proofreading this work cannot be done.

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