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A novel corrector-predictor interior-point algorithm for $P_*(\kappa)$-weighted linear complementarity problems based on an AET function

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Abstract This paper considers $P_*(\kappa)$-Weighted Linear Complementarity Problems (WLCPs) and gives a novel corrector-predictor interior-point algorithm (IPA) based on an algebraic equivalent transformation function. The strict feasibility and convergence of our proposed method for $P_*(\kappa)$-WLCP are established. In particular, we demonstrate that the iteration bound of the algorithm enjoys a polynomial complexity bound, which is comparable to the best available one for such existing IPAs. Finally, the proposed corrector-predictor IPA is applied to a small set of numerical examples to support the viability and efficiency of the algorithm and illustrate potential for the efficient implementation.

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1 Introduction and Motivation

Linear Complementarity Problem (LCP) is a problem of finding vectors $x, s \in \mathbb{R}^n$ such that

$$\begin{aligned} xs &= 0, \\ s &= Mx + q, \\ x, s &\geq 0, \end{aligned} \tag{1}$$

for the given matrix $M \in \mathbb{R}^{n \times n}$ and vector $q \in \mathbb{R}^n$. We consider a Weighted Linear Complementarity Problem (WLCP) as a generalization of LCP (1) where zero on the right side of the complementarity equation is replaced by the positive vector of weights $\omega \in \mathbb{R}^n_{++}$,

$$\begin{aligned} xs &= \omega, \\ s &= Mx + q, \\ x, s &\geq 0. \end{aligned} \tag{2}$$

We further call the problem (2) a $P_*(\kappa)$-WLCP [22] if the matrix M in WLCP (2) is a $P_*(\kappa)$-matrix, i.e., it satisfies

$$\sum_{i \in T_+} x_i \left(Mx\right)_i + \sum_{i \in T_-} x_i \left(Mx\right)_i \geq -4\kappa \sum_{i \in T_+} x_i \left(Mx\right)_i, \quad \forall x \in \mathbb{R}^n,$$

where

$$T_+(x) = \left\{1 \leq i \leq n : x_i \left(Mx\right)_i > 0\right\}, \quad T_-(x) = \left\{1 \leq i \leq n : x_i \left(Mx\right)_i < 0\right\},$$

and $\kappa \geq 0$ is a nonnegative real number. The handicap of such matrix M [37] is defined by $\hat{\kappa} = \inf\{\kappa \mid M \in P_*(\kappa)\}$ with nonnegative real number. $P_*(\kappa)$-matrix is important in WLCP and LCP because it guarantees the existence of a unique solution when the problem satisfies the interior-point condition [34]. LCPs with general matrix M are NP-complete [6]. However, the characteristics of the matrix associated with the problem affect the solvability and computational complexity of WLCPs and LCPs. Specifically, interior-point algorithms (IPAs) for $P_*(\kappa)$-LCPs and WLCPs exhibit polynomial iteration complexity concerning the problem size, the accuracy parameter, the initial duality gap, the specific parameter $\kappa \geq 0$, and the vector of weights (for WLCPs).

It is important to mention that Väliaho [37] proved that the class of P_*-matrices, which is the union of all $P_*(\kappa)$-matrices with respect to $\kappa \geq 0$, coincides with the class of sufficient matrices [7]. A matrix is sufficient if it is both row and column sufficient. A matrix M is row sufficient if $x_i(Mx)_i \leq 0 \Rightarrow x_i(Mx)_i = 0$ for all i, and column sufficient if M^T is row sufficient. Here are some notable subclasses of P_*-matrices along with their definitions:

– Skew-symmetric matrices [22]:

$$x^T Mx = 0, \quad \forall x \in \mathbb{R}^n.$$

– Positive semidefinite matrices [22]:

$$x^T Mx \geq 0, \quad \forall x \in \mathbb{R}^n.$$

– P-Matrices [22]: matrices with all the principal minors positive or, equivalently,

$$\exists i \in \{1, ..., n\} \quad \text{such that } x_i(Mx)_i > 0, \quad \forall x (\neq 0) \in \mathbb{R}^n.$$

WLCPs were first introduced in [22]. Potra offered the definition of WLCPs in [33], in which the theoretical properties of WLCPs were studied and two IPAs for general WLCPs with a smooth central path were analyzed. Later, Potra [34] further gave the notion of sufficient WLCPs and established some fundamental results. A corrector-predictor IPA for sufficient WLCPs and the complexity of the algorithm were investigated therein.

The WLCPs arise in economic, scientific, and engineering problems, such as equilibrium problems, the Arrow-Debreu market equilibrium problem, and the Eisenberg-Gale markets [11, 20, 38]. During the study of WLCPs [8], various solution methods [21, 30, 36, 40] have been proposed, including IPAs that have gained attention recently. Modern study of IPAs for Linear Optimization (LO) started with the work of Karmarkar [24]. IPAs were successfully generalized to LCPs [1], convex quadratic optimization [2], semidefinite optimization [3, 25], and cone programming [31]. Due to the WLCPs being an extension of the LCPs, the IPAs designed for LCPs [9, 10, 17] have been also successfully generalized to WLCPs [4, 12, 23].

For IPAs one of the key features is to determine good search directions that would eventually contribute to obtain good iteration bounds. One way to determine search directions is due to Darvay [13] who proposed a method of an algebraic equivalent transformation (AET) on the nonlinear equation of the central path system. In [13], the centrality equation is built up by $\varphi(v^2) = \varphi(e)$, where variance vector is defined as $v = \sqrt{\frac{xs}{\mu}}$. Then, Newton's method is applied to the modified system to determine the search directions. Darvay [13] proposed function $\varphi(t) = \sqrt{t}$ in AET technique for LO. Since then, this AET technique using square root function has been extended to different types of optimization problems, including monotone LCP (MLCP) [32], $P_*(\kappa)$-LCP [5, 26], and monotone WLCP (MWLCP) [23], among others.

Instead of AET technique and function given in [13], Darvay and Takács [14] considered another way to determine search directions. This method involves the equivalent form $v^2 = v$ of the centrality equation. The authors applied the function $\varphi(t) = t^2$ to each side of equation $\varphi(v^2) = \varphi(v)$, then used Newton's method to identify new search directions. Kheirfam [27] directly generalized this technique to $P_*(\kappa)$-LCP. Subsequently, Darvay et al. [16] gave a new PC IPA for $P_*(\kappa)$-LCP, in which the search directions were derived from the positive-asymptotic kernel function in the type of AET functions [14]. Recently, a new class of AET functions [19] has been proposed to determine search directions, which differs from the previous class of AET functions [13, 14, 28, 35].

Predictor-corrector IPAs (PC IPAs) have proven their efficiency and effectiveness in practice. These algorithms combine predictor and corrector steps to obtain a solution at each iteration. The corrector-predictor IPAs [15] are distinct from the PC IPAs because they start with corrector steps followed by a predictor step.

	Function $\varphi(t)$ and variance vector v			
	Algorithm	centrality equation	θ	Complexity
Problem			$\varphi(t)=\sqrt{t}, \quad v=\sqrt{\frac{xs}{\mu}}$	
MLCP [32]	IPA	$\varphi(v^2)=\varphi(e)$	$\frac{1}{2\sqrt{n}}$	$O\left(\sqrt{n}\log\frac{n\mu^0}{\varepsilon}\right)$
$P_*(\kappa)$-LCP [5]	IPA	$\varphi(v^2)=\varphi(e)$	$\frac{1}{2(1+4\kappa)\sqrt{n}}$	$O\left((1+4\kappa)\sqrt{n}\log\frac{n\mu^0}{\varepsilon}\right)$
$P_*(\kappa)$-LCP [26]	PC IPA	$\varphi(v^2)=\varphi(e)$	$\frac{1}{4(1+2\kappa)\sqrt{n}}$	$O\left((1+2\kappa)\sqrt{n}\log\frac{n\mu^0}{\varepsilon}\right)$
MWLCP [23]	IPA	$\varphi(xs)=\varphi(\omega(\mu))$	$\frac{\min c}{3(\min c+\\|\omega-c\\|+1)}$	$O\left(\frac{6(\min c+\\|\omega-c\\|+1)}{\min c}\log\frac{\sqrt{\frac{\min c}{2}+\beta}}{\varepsilon}\right)$
Problem			$\varphi(t)=t^2, \quad v=\sqrt{\frac{xs}{\mu}}$	
LO [14]	IPA	$\varphi(v^2)=\varphi(v)$	$\frac{1}{12\sqrt{n}}$	$O\left(\sqrt{n}\log\frac{n+4}{\varepsilon}\right)$
$P_*(\kappa)$-LCP [27]	IPA	$\varphi(v^2)=\varphi(v)$	$\frac{1}{4(1+4\kappa)\sqrt{n}}$	$O\left((1+4\kappa)\sqrt{n}\log\frac{n+\frac{9}{16}}{\varepsilon}\right)$
$P_*(\kappa)$-LCP [16]	PC IPA	$\varphi(v^2)=\varphi(v)$	$\frac{1}{4(1+4\kappa)\sqrt{n}}$	$O\left((1+4\kappa)\sqrt{n}\log\frac{3n\mu^0}{4\varepsilon}\right)$
Problem			$\varphi(t)=t^2+\sqrt{t}, \quad v=\sqrt{\frac{xs}{\omega(\mu)}}$	
$P_*(\kappa)$-WLCP	PC IPA	$\varphi(v^2)=\varphi(e)$	$\frac{1+\mu}{12(2+\kappa')\sqrt{n}}$	$O\left((2+\kappa')\sqrt{n}\log\frac{\frac{1}{9(2+\kappa')}\max x_0s_0+\\|x_0s_0-\omega\\|}{\varepsilon}\right)$

Table 1: List of IPAs with various $\varphi(t)$ for different problems

In this paper, we propose a new corrector-predictor IPA for $P_*(\kappa)$-WLCP that retains the best-known polynomial iteration complexity by incorporating a function belonging to a novel class of AET functions introduced in [19], particularly for $P_*(\kappa)$-LCPs. We focus on a special case, namely $\varphi(t)=t^2+\sqrt{t}$, which is a member of the aforementioned class of AET functions. We also build up a few technical lemmas to analyze our algorithm's complexity and derive its iteration bound. Table 1 compares different AET functions as well as complexity of existing IPAs and our algorithm. The analysis of our algorithm is not just mimicking the $P_*(\kappa)$-LCPs case, it is indeed more complicated and subtle, due to the nonnegative weight vector in the $P_*(\kappa)$-WLCPs setting. As far as we know, this is the first corrector-predictor IPA for solving $P_*(\kappa)$-WLCP based on the above-mentioned AET function $\varphi(t)=t^2+\sqrt{t}$.

2 A Corrector-Predictor IPA for $P_*(\kappa)$-WLCP

2.1 Central Path of $P_*(\kappa)$-WLCP

The set of feasible region and the strictly feasible region of $P_*(\kappa)$-WLCP are denoted respectively as follows

$$\begin{aligned}\mathcal{F} &= \{(x,s)|s = Mx + q,\ x \geq 0,\ s \geq 0\},\\ \mathcal{F}^0 &= \{(x,s) \in \mathcal{F} \,|\, x > 0,\ s > 0\}\,.\end{aligned}$$

Since we assume interior-point condition (IPC) holds [35], it means that the strictly feasible region $\mathcal{F}^0$ is non-empty. We define

$$\omega(\mu) = \mu x_0 s_0 + (1-\mu)\omega, \tag{3}$$

where $\mu \in (0,1]$ and $(x_0, s_0) \in \mathcal{F}^0$. The central path problem for $P_*(\kappa)$-WLCP (2) is then described by

$$\begin{aligned} xs &= \omega(\mu),\\ s &= Mx + q,\\ x, s &\geq 0. \end{aligned} \tag{4}$$

If matrix M is a $P_*(\kappa)$-matrix and $\mathcal{F}^0 \neq \emptyset$, the system (4) has a solution $(x(\mu), s(\mu))$ for any $\mu \in (0,1]$, see [34]. When $\mu \in (0,1]$, the sequence of solutions $\{(x(\mu), s(\mu)) \,|\, \mu \in (0,1]\}$ forms the central path of $P_*(\kappa)$-WLCP (2) and approaches to the solution (x,s) of $P_*(\kappa)$-WLCP (2). Of course, due to weighted complementarity condition $xs = \omega(\mu)$, the system (4) is not easy to solve directly. Nonetheless, one can find approximate solutions by applying Newton's method.

2.2 Algebraic Equivalent Transformation Technique

Consider a continuously differentiable, monotone increasing, and invertible function $\varphi : (\xi, \infty) \to \mathbb{R}$, where $0 \leq \xi < 1$. Using this function, we recast the system (4) as

$$\begin{aligned} \varphi\left(\frac{xs}{\omega(\mu)}\right) &= \varphi(e),\\ s &= Mx + q,\\ x, s &\geq 0. \end{aligned} \tag{5}$$

For a pair $(x,s) \in \mathcal{F}^0$, we apply Newton's method to the system (5) to find search directions Δx and Δs yielding the system

$$\begin{aligned} \varphi\left(\frac{xs + x\Delta s + s\Delta x + \Delta x \Delta s}{\omega(\mu)}\right) &= \varphi(e),\\ s + \Delta s &= M\left(x + \Delta x\right) + q. \end{aligned} \tag{6}$$

Neglecting the quadratic term $\Delta x \Delta s$, system (6) can be reformulated as follows:

$$\begin{aligned} s\Delta x + x\Delta s &= a_\varphi, \\ -M\Delta x + \Delta s &= 0, \end{aligned} \tag{7}$$

where a_φ represents

$$a_\varphi = \omega(\mu)\,\frac{\varphi(e) - \varphi\left(\frac{xs}{\omega(\mu)}\right)}{\varphi'\left(\frac{xs}{\omega(\mu)}\right)}. \tag{8}$$

To proceed, we denote

$$v := \sqrt{\frac{xs}{\omega(\mu)}}, \quad d := \sqrt{\frac{x}{s}}, \quad d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}. \tag{9}$$

Then, substitution of terms defined in (9) into the system (7) gives

$$\begin{aligned} d_x + d_s &= p_v, \\ -\overline{M} d_x + d_s &= 0, \end{aligned} \tag{10}$$

where $D = \operatorname{diag}(d)$, $W(\mu) = \operatorname{diag}(\omega(\mu))$, $\overline{M} = \sqrt{W^{-1}(\mu)}DMD\sqrt{W(\mu)}$ and

$$p_v = \frac{\varphi(e) - \varphi\left(v^2\right)}{v\varphi'\left(v^2\right)}. \tag{11}$$

Furthermore, it can be seen from (11) that we have different vectors p_v for various functions φ. Below, we list a few instances of φ along with p_v that have been used in the literature.

- $\varphi(t) = t$ yields $p_v = v^{-1} - v$ [35], used for $P_*(\kappa)$-LCP in [39].
- $\varphi(t) = \sqrt{t}$ yields $p_v = 2(e - v)$ [13], used for $P_*(\kappa)$-LCP [5, 26], monotone LCP [32] and WLCP [23].
- $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ yields $p_v = e - v^2$, used for $P_*(\kappa)$-LCP in [28].

AET method consists of using a continuously differentiable, invertible and monotone increasing function $\varphi : (\xi, \infty) \to \mathbb{R}$, where $0 \le \xi < 1$, on the nonlinear equation of the central path system. After transformation, Newton's method is applied to obtain new search directions.

In this paper the calculation of the search directions is based on a specific AET function from the new type of AET functions for $P_*(\kappa)$-LCP [19]. These search directions are then used to design a new corrector-predictor IPA for $P_*(\kappa)$-WLCP.

Recall that a new class of AET functions was introduced for solving $P_*(\kappa)$-LCP in [19, Definition 2.4]. For subsequent needs, we present the definition below.

Definition 1 ([19, Definition 2.4]) Let $\varphi : (\xi, \infty) \to \mathbb{R}$ be a continuously differentiable and invertible function such that $\varphi'(t) > 0$ for $\forall t > \xi$, where $0 \le \xi < 1$. All functions φ satisfying the below conditions belong to the new class of AET functions.

AET(1). There exists $c_1 \in \mathbb{R}_+$ such that

$$\left| \frac{\varphi(1) - \varphi\left(t^2\right)}{2t\left(1-t^2\right)\varphi'\left(t^2\right)} \right| \le c_1,$$

for all $t > \xi$.

AET(2). There exists $c_2 \in \mathbb{R}_+$ such that

$$\left| \frac{4t^2\varphi'\left(t^2\right)\left[\left(1-t^2\right)\varphi'\left(t^2\right) - \varphi(1) + \varphi\left(t^2\right)\right]}{\left(\varphi(1) - \varphi\left(t^2\right)\right)^2} \right| \le c_2,$$

for all $t > \xi$.

AET(3). There exists $c_3 \in \mathbb{R}_+$ such that the inequality

$$\begin{aligned} &4t^2\left(\varphi(1) - \varphi\left(t^2\right)\right)\varphi'\left(t^2\right) - c_3\left(\varphi(1) - \varphi\left(t^2\right)\right)^2 \\ &\le 4t^2\left(1-t^2\right)\left(\varphi'\left(t^2\right)\right)^2 \\ &\le 4t^2\left(\varphi(1) - \varphi\left(t^2\right)\right)\varphi'\left(t^2\right) + \left(\varphi(1) - \varphi\left(t^2\right)\right)^2 \end{aligned}$$

holds for all $t > \xi$.

Definitions of functions $f, g, h : (\xi, \infty) \to \mathbb{R}$ are also needed

$$\begin{aligned} f(t) &= \frac{\varphi(1) - \varphi\left(t^2\right)}{t\varphi'\left(t^2\right)}, \\ g(t) &= \frac{f(t)}{2\left(1-t^2\right)}, \\ h(t) &= \frac{4\left(1 - t^2 - tf(t)\right)}{f(t)^2} = \frac{1 - 2tg(t)}{\left(1-t^2\right)g(t)^2}. \end{aligned} \tag{12}$$

In fact, Illés et. al. [19] also provide equivalent conditions in light of the aforementioned functions $f(t)$, $g(t)$ and $h(t)$, which are stated in the proposition below.

Proposition 1 *(Proposition 2.1 in [19]) Let $\varphi : (\xi, \infty) \to \mathbb{R}$ be a continuously differentiable and invertible function such that $\varphi'(t) > 0$ for all $t > \xi$, where $0 \le \xi < 1$. Consider the functions f, g, h in (12). The conditions given in Definition 1 can be formulated in the following equivalent form:*

(a) $\exists\, c_1 \in \mathbb{R}_+$ such that $g(t) = \dfrac{f(t)}{2\left(1-t^2\right)}$ and $|g(t)| \le c_1$ holds for all $t > \xi$.

(b) $\exists\, c_2 \in \mathbb{R}_+$ such that $h(t) = \dfrac{4\left(1 - t^2 - tf(t)\right)}{f(t)^2} = \dfrac{1 - 2tg(t)}{\left(1-t^2\right)g(t)^2}$ and $|h(t)| \le c_2$ holds for all $t > \xi$.

(c) $\exists\ c_3 \in \mathbb{R}_+$ *such that the inequality*

$$tf(t) - c_3\frac{f(t)^2}{4} \le 1 - t^2 \le tf(t) + \frac{f(t)^2}{4}$$

holds for all $t > \xi$.

In this paper, we focus on the function $\varphi : (0, \infty) \to \mathbb{R}$,

$$\varphi(t) = t^2 + \sqrt{t}$$

for $P_*(\kappa)$-WLCP, which belongs to the class of AET functions given in Definition 1 for $P_*(\kappa)$-LCP. In particular, from (8), (11) and (12), we have

$$\begin{aligned} a_\varphi &= \frac{2\sqrt{xs}\left(2\omega(\mu)^2 - \omega(\mu)\sqrt{xs}\sqrt{\omega(\mu)} - (xs)^2\right)}{4xs\sqrt{xs} + \omega(\mu)\sqrt{\omega(\mu)}} \\ &= \frac{8\omega(\mu)^2\sqrt{xs} - 3xs\omega(\mu)\sqrt{\omega(\mu)}}{2\left(4xs\sqrt{xs} + \omega(\mu)\sqrt{\omega(\mu)}\right)} - \frac{xs}{2}, \end{aligned}$$

$$p_v = \frac{2\left(2e - v^4 - v\right)}{4v^3 + e} = \frac{2(e - v)\left(v^3 + v^2 + v + 2e\right)}{4v^3 + e}, \tag{13}$$

and

$$f(t) = \frac{4 - 2t^4 - 2t}{4t^3 + 1}. \tag{14}$$

Next, we define norm-based proximity measure as follows

$$\delta = \delta(x, s; \omega(\mu)) = \frac{\|p_v\|}{2} = \left\| \frac{(e - v)\left(v^3 + v^2 + v + 2e\right)}{4v^3 + e} \right\|. \tag{15}$$

For $(x, s) \in \mathcal{F}^0$, we have

$$\delta = 0 \quad \Longleftrightarrow \quad e = v \quad \Longleftrightarrow \quad xs = \omega(\mu).$$

In Table 1, it can be observed from the expressions of the centrality equations $\varphi(v^2) = \varphi(e)$ and $\varphi(v^2) = \varphi(v)$ that there are currently two AET techniques in the literature. In this paper, the form of the centrality equation $\varphi(v^2) = \varphi(e)$ is adopted and an AET function $\varphi(t) = t^2 + \sqrt{t}$ is used to determine the search directions. AET functions $\varphi(t) = \sqrt{t}$ and $\varphi(t) = t - \sqrt{t}$ belong to this new class of AET functions. However, the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ does not satisfy condition AET(3) and therefore is excluded from the new class of AET functions. Particular attention should be paid to the domains of the functions φ in this approach. For example, $\varphi(t) = t$ is an AET function only if it is defined on (ξ, ∞) with $\xi > 0$. If ξ is zero, condition AET(1) would no longer hold.

3 Corrector-Predictor Algorithm

In this section we describe one iteration of the algorithm that consists of a corrector step and a predictor step, with search directions constituting a key component as elaborated below.

3.1 Search Directions for $P_*(\kappa)$-WLCP

Given the iteration (x, s), the $\tau\mu$-neighborhood of the central path is

$$N(\tau\mu) = \{(x, s) : s = Mx + q, \quad x > 0, \quad s > 0, \quad \delta(x, s; \omega(\mu)) \leq \tau\mu\},$$

where $0 < \tau < 1$ and $0 < \mu \leq 1$. The corrector-predictor IPA begins with a specified strictly feasible point $(x_0, s_0) \in N(\tau\mu_0)$, which satisfies the condition $x_0 s_0 \geq \omega$. If $\|xs-\omega\| > \varepsilon$ holds for the current iterate (x, s), then the algorithm generates a new iterate by executing corrector and predictor steps.

We first consider a corrector step. From (10) and (13), the scaled Newton system becomes

$$\begin{aligned} -\overline{M}d_x + d_s &= 0, \\ d_x + d_s &= \frac{2(e-v)\left(v^3+v^2+v+2e\right)}{4v^3+e}. \end{aligned} \tag{16}$$

In a corrector step, the search direction (d_x, d_s) is obtained by solving (16). Then, using (9) we obtain $(\Delta x, \Delta s)$. The corrector iterate is calculated by considering a full-Newton step as follows

$$(x_+, s_+) = (x + \Delta x, s + \Delta s).$$

We introduce the following notations

$$v_+ = \sqrt{\frac{x_+ s_+}{\omega(\mu)}}, \quad d_+ = \sqrt{\frac{x_+}{s_+}}, \quad D_+ = \operatorname{diag}(d_+),$$
$$\overline{M}_+ = \sqrt{W^{-1}(\mu)}D_+ M D_+ \sqrt{W(\mu)}. \tag{17}$$

In the predictor step, we calculate the search direction (d_x^p, d_s^p) by solving the scaled Newton system

$$\begin{aligned} -\overline{M}_+ d_x^p + d_s^p &= 0, \\ d_x^p + d_s^p &= -\frac{1}{2}v_+. \end{aligned} \tag{18}$$

Then, we compute the predictor direction $(\Delta^p x, \Delta^p s)$ as

$$\Delta^p x = \frac{x_+}{v_+} d_x^p, \quad \Delta^p s = \frac{s_+}{v_+} d_s^p, \tag{19}$$

and the predictor iterate as

$$(x_p, s_p) = (x_+,\ s_+) + \theta\mu\ (\Delta^p x,\ \Delta^p s),$$

where θ is a barrier parameter. We decrease the parameter μ using the parameter θ, specifically defining $\mu_p := (1-2\theta)\mu$. Next, we identify a new iterate $(x_p, s_p) \in \mathcal{F}^0$ and substitute μ with μ_p. Additionally, we set $\omega(\mu_p) := (1-\frac{1}{2}\theta\mu)\omega(\mu)$, ensuring that $\delta(x_p, s_p; \omega(\mu_p)) \leq \tau\mu_p$.

3.2 Corrector-Predictor Interior-Point Algorithm

To conclude Section 3, we outline the corrector-predictor IPA. Since the $P_*(\kappa)$-WLCP is strictly feasible, it guarantees that the WLCP has a solution [34]. To start the algorithm, we choose an iterate $(x_0, s_0) \in \mathcal{F}^0$ satisfying $(x_0, s_0) > 0$, $x_0 s_0 \geq \omega$ and $\delta(x_0, s_0; \omega(\mu_0)) = 0 \leq \frac{\mu_0}{16(2+\kappa')}$ with $\mu_0 = 1$. We would like to determine the values of δ and θ in such a way that corrector step and predictor step are strictly feasible. Moreover, we want that $\delta(x_p, s_p; \omega(\mu_p)) \leq \frac{\mu_p}{16(2+\kappa')}$ holds. Then, the algorithm repeats corrector and predictor steps alternatively until $\|x_k s_k - \omega\| \leq \varepsilon$ is satisfied. A detailed description of the algorithm is given in Table 2.

Algorithm 1. A corrector-predictor IPA for $P_*(\kappa)$-WLCP

Input
Let $(x_0, s_0) \in \mathcal{F}^0$ such that $x_0 s_0 \geq \omega$,
where $\delta(x_0, s_0; \omega(\mu_0)) \leq \frac{\mu_0}{16(2+\kappa')}$ with $\mu_0 = 1$;
Accuracy parameter $\varepsilon > 0$;
Barrier update parameter $\theta \in (0, 1)$;
begin
$k := 0$;
  while $\|x_k s_k - \omega\| > \varepsilon$ **do**
  begin
    (corrector step)
    calculate $(\Delta x, \Delta s)$ from (16) and (9);
    update $x_+ := x + \Delta x$, $s_+ := s + \Delta s$;
    (predictor step)
    calculate $(\Delta^p x, \Delta^p s)$ from (18) and (19);
    update $x_p := x_+ + \theta\mu\Delta^p x$, $s_p := s_+ + \theta\mu\Delta^p s$;
    set $\omega(\mu_p) := (1-\frac{1}{2}\theta\mu)\omega(\mu)$, $\mu_p := (1-2\theta)\mu$;
    $x_k := x_p$, $s_k := s_p$; $\omega(\mu) = \omega(\mu_p)$; $\mu := \mu_p$;
  end
end

Table 2: The corrector-predictor IPA

4 Analysis of the algorithm

In this section we provide the analysis of the corrector-predictor IPA. We show that the algorithm is well-defined and globally convergent under strict feasibility condition.

4.1 The analysis of the corrector step

First, let us deduce the following technical lemma.

Lemma 1 *Let $x_0 s_0 \geq \omega$ and (d_x, d_s) be a solution to system (16) with $\delta := \delta(x, s; \omega(\mu))$. One has*

$$\begin{aligned} \|d_x d_s\|_\infty &\leq (1 + 4\kappa')\, \delta^2, \\ \|d_x d_s\|_1 &\leq 2\,(1 + 2\kappa')\, \delta^2, \\ \|d_x d_s\| &\leq \sqrt{1 + (1 + 4\kappa')^2}\delta^2, \end{aligned}$$

where the handicaps κ and κ' of matrices M and $\overline{M}$ satisfy $\dfrac{1+4\kappa'}{1+4\kappa} = \dfrac{\max x^0 s^0}{\min \omega}$.

Proof. Please refer to [22, Theorem 3.5], [18, Corollary 7 and Lemma 9], and [39, Lemma 3.2 and Lemma 3.3] for detailed arguments. □

The next lemma proceeds to prove strict feasibility of the corrector iterate.

Lemma 2 *Suppose that $x_0 s_0 \geq \omega$ and $\delta < \dfrac{1}{\sqrt{2(1+2\kappa')}}$. With $\varphi(t) = t^2 + \sqrt{t}$, the iterate (x_+, s_+) in Algorithm 1 is positive.*

Proof. Let $\alpha \in [0, 1]$ and denote $x(\alpha) = x + \alpha \Delta x$, $s(\alpha) = s + \alpha \Delta s$. From (9) and (10), we have

$$\begin{aligned} \frac{x(\alpha)s(\alpha)}{\omega(\mu)} &= \frac{xs}{\omega(\mu)} + \frac{\alpha(s\Delta x + x\Delta s)}{\omega(\mu)} + \frac{\alpha^2 \Delta x \Delta s}{\omega(\mu)} \\ &= v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s \\ &= (1-\alpha)\, v^2 + \alpha \left(v^2 + v p_v + \alpha d_x d_s\right). \end{aligned} \tag{20}$$

Next, applying the second inequality of AET(3) in Definition 1 and (11), we obtain

$$\begin{aligned} & 4t^2\left(1 - t^2\right)\left(\varphi'\left(t^2\right)\right)^2 \leq 4t^2\left(\varphi(1) - \varphi\left(t^2\right)\right)\varphi'\left(t^2\right) + \left(\varphi(1) - \varphi(t^2)\right)^2 \\ \Longleftrightarrow\ & v^2 + v p_v \geq e - \frac{p_v^2}{4} \end{aligned} \tag{21}$$

for $t > \xi$. Then, combining (15), (20), (21) and Lemma 1 yields

$$\begin{aligned}\frac{x(\alpha)s(\alpha)}{\omega(\mu)} &\geq (1-\alpha)\,v^2 + \alpha\left(e - \frac{p_v^2}{4} + \alpha d_x d_s\right)\\ &\geq (1-\alpha)\,v^2 + \alpha\left(1 - \frac{\|p_v^2\|_\infty}{4} - \|d_x d_s\|_\infty\right)e\\ &\geq (1-\alpha)\,v^2 + \alpha\left(1 - \frac{\|p_v\|^2}{4} - \|d_x d_s\|_\infty\right)e\\ &\geq (1-\alpha)\,v^2 + \alpha\left(1 - \delta^2 - (1+4\kappa')\,\delta^2\right)e\\ &= (1-\alpha)\,v^2 + \alpha\left(1 - 2\,(1+2\kappa')\,\delta^2\right)e.\end{aligned} \tag{22}$$

Furthermore, if $\delta < \frac{1}{\sqrt{2(1+2\kappa')}}$, we know $1-2\,(1+2\kappa')\,\delta^2 > 0$, which implies

$$(1-\alpha)\,v^2 + \alpha\left(1 - 2\,(1+2\kappa')\,\delta^2\right)e > 0$$

for all $\alpha \in [0,1]$. This together with (22) indicates that $x(\alpha)s(\alpha) > 0$ for any $\alpha \in [0,1]$. Since $x(0) = x_0 > 0$ and $s(0) = s_0 > 0$, by continuity, we have $x(\alpha) > 0$ and $s(\alpha) > 0$. Thus, $x_+ = x(1) > 0$ and $s_+ = s(1) > 0$. □

It is useful to give a minimum lower bound for the value of $\min v_+$.

Lemma 3 *Suppose that* $v_+ = \sqrt{\frac{x_+ s_+}{\omega(\mu)}}$ *and* $x_0 s_0 \geq \omega$. *If* $\delta < \frac{1}{\sqrt{2(1+2\kappa')}}$, *then*

$$\min v_+ \geq \sqrt{1 - 2\,(1+2\kappa')\,\delta^2}.$$

Proof. From (22) with $\alpha = 1$, we get

$$v_+^2 = \frac{x_+ s_+}{\omega(\mu)} \geq \left(1 - 2\,(1+2\kappa')\,\delta^2\right)e.$$

Then, it easily follows $\min v_+ \geq \sqrt{1 - 2\,(1+2\kappa')\,\delta^2}$. □

Lemma 4 *Suppose that* $x_0 s_0 \geq \omega$ *and* $\delta < \frac{1}{\sqrt{2(1+2\kappa')}}$. *Let* φ *satisfy conditions AET(1)-(3) in Definition 1 with constants* c_1, c_2 *and* c_3. *Then, after a corrector step, we have* $v_+ > 0$ *and*

$$\delta\left(x_+, s_+; \omega(\mu)\right) \leq c_1\left(c_2 + \sqrt{1 + (1+4\kappa')^2}\right)\delta^2.$$

Proof. From Lemma 3 and $\delta < \dfrac{1}{\sqrt{2(1+2\kappa')}}$, it follows that $v_+ > 0$. In light of (15) and AET condition (a) in Proposition 1, we have

$$\delta(x_+, s_+; \omega(\mu)) = \frac{\|p_{v_+}\|}{2} = \left\|\left(e - v_+^2\right) g(v_+)\right\| \leq c_1 \left\|e - v_+^2\right\|. \tag{23}$$

By applying AET condition (b) in Proposition 1, we obtain

$$e - \left(v^2 + vp_v\right) = h(v)\frac{p_v^2}{4}, \quad |h(v_i)| \leq c_2. \tag{24}$$

Using (15), (20), (24) and Lemma 1, we get

$$\begin{aligned}
\left\|e - v_+^2\right\| &= \left\|e - \frac{x_+ s_+}{\omega(\mu)}\right\| \\
&= \left\|e - \left(v^2 + vp_v + d_x d_s\right)\right\| \\
&\leq \left\|h(v)\frac{p_v^2}{4}\right\| + \|d_x d_s\| \\
&\leq \|h(v)\|_\infty \frac{\|p_v\|^2}{4} + \|d_x d_s\| \\
&\leq c_2\delta^2 + \sqrt{1 + (1+4\kappa')^2}\,\delta^2 \\
&= \left(c_2 + \sqrt{1 + (1+4\kappa')^2}\right)\delta^2.
\end{aligned} \tag{25}$$

Then, combining (23) and (25) gives

$$\delta(x_+, s_+; \omega(\mu)) \leq c_1 \left\|e - v_+^2\right\| \leq c_1 \left(c_2 + \sqrt{1 + (1+4\kappa')^2}\right)\delta^2,$$

which is the desired result. □

Lemma 4 shows local quadratic convergence of the norm-based proximity measure.

Since M is a $P_*(\kappa)$-matrix and $-M\Delta^p x + \Delta^p s = 0$, we have the following inequality

$$\sum_{i\in T_-} \Delta^p x_i \Delta^p s_i + \sum_{i\in T_+} \Delta^p x_i \Delta^p s_i \geq -4\kappa \sum_{i\in T_+} \Delta^p x_i \Delta^p s_i, \tag{26}$$

where $T_- = \{i \in T : \Delta^p x_i \Delta^p s_i < 0\}$, $T_+ = \{i \in T : \Delta^p x_i \Delta^p s_i > 0\}$, and $T = \{1, 2, \ldots, n\}$. Then, applying [22, Theorem 3.5], Lemma 1, (17) and (18), we get

$$\sum_{i\in T_-} d_{x_i}^p d_{s_i}^p + (1+4\kappa') \sum_{i\in T_+} d_{x_i}^p d_{s_i}^p \geq 0, \quad \forall x \in \mathbb{R}^n. \tag{27}$$

Note that κ in (26) and κ' in (27) satisfy the relationship $\dfrac{1+4\kappa'}{1+4\kappa} = \dfrac{\max x^0 s^0}{\min \omega}$ in Lemma 1.

The results established in (26) and (27) play a crucial role in the following predictor step analysis.

4.2 The analysis of the predictor step

Lemma 5 *Suppose that $x_0 s_0 \geq \omega$. Then the following inequality holds*

$$\|v_+\|^2 \leq n + \left[2\left(1+2\kappa'\right) + c_3\right]\delta^2.$$

Proof. Substituting $\alpha = 1$ into (20) yields

$$v_+^2 = \frac{x_+ s_+}{\omega(\mu)} = v^2 + v p_v + d_x d_s. \tag{28}$$

Using the first inequality of AET(3) in Definition 1 and (11), we obtain

$$\begin{aligned} & 4t^2\left(\varphi(1) - \varphi\left(t^2\right)\right)\varphi'\left(t^2\right) - c_3\left(\varphi(1) - \varphi\left(t^2\right)\right)^2 \leq 4t^2\left(1-t^2\right)\left(\varphi'\left(t^2\right)\right)^2 \\ \Longleftrightarrow\ & v^2 + v p_v \leq e + c_3 \frac{p_v^2}{4}. \end{aligned} \tag{29}$$

Then, applying (15), (28), (29) and Lemma 1, we derive

$$\begin{aligned} \|v_+\|^2 &= \sum_{i=1}^{n} (v_+)_i^2 \leq \sum_{i=1}^{n} \left(e + c_3 \frac{p_v^2}{4} + d_x d_s\right)_i \\ &\leq e^T\left(e + c_3 \frac{p_v^2}{4}\right) + \|d_x d_s\|_1 \\ &\leq n + \left[2\left(1+2\kappa'\right) + c_3\right]\delta^2. \end{aligned}$$

Thus, the proof is complete. □

Lemma 6 *Suppose that $x^0 s^0 \geq \omega$. Then, the following inequality holds:*

$$\|d_x^p d_s^p\| \leq \frac{\left(2+\kappa'\right)\left[n + \left(2(1+2\kappa') + c_3\right)\delta^2\right]}{4}.$$

Proof. From the scaled Newton system (18), we obtain

$$4 \sum_{i \in T_+} d_{x_i}^p d_{s_i}^p \leq \|d_x^p + d_s^p\|^2 = \frac{\|v_+\|^2}{4}.$$

This relation together with (27) implies

$$\begin{aligned}\frac{\|v_+\|^2}{4} = \|d_x^p + d_s^p\|^2 &= \|d_x^p\|^2 + \|d_s^p\|^2 + 2\left(\sum_{i\in T_+} d_{x_i}^p d_{s_i}^p + \sum_{i\in T_-} d_{x_i}^p d_{s_i}^p\right)\\ &\geq \|d_x^p\|^2 + \|d_s^p\|^2 - 8\kappa' \sum_{i\in T_+} d_{x_i}^p d_{s_i}^p\\ &\geq \|d_x^p\|^2 + \|d_s^p\|^2 - \frac{1}{2}\kappa' \|v_+\|^2 .\end{aligned}$$

Hence,

$$\|d_x^p\|^2 + \|d_s^p\|^2 \leq \left(\frac{1}{4} + \frac{1}{2}\kappa'\right)\|v_+\|^2 \leq \left(1 + \frac{1}{2}\kappa'\right)\|v_+\|^2 . \tag{30}$$

Using (30) and Lemma 5, we have

$$\begin{aligned}\|d_x^p d_s^p\| &\leq \frac{1}{2}\left(\|d_x^p\|^2 + \|d_s^p\|^2\right) \leq \frac{1}{2}\left(1 + \frac{1}{2}\kappa'\right)\|v_+\|^2\\ &\leq \frac{(2+\kappa')\left[n + (2(1+2\kappa') + c_3)\,\delta^2\right]}{4}.\end{aligned}$$

Then, the proof is complete. □

It is worth mentioning that Lemma 5 and Lemma 6 are technical lemmas used in the proof of strict feasibility in Lemma 7.

For notational simplicity and subsequent analysis, we define

$$b(\delta,\theta,n) = 1 - 2\,(1+2\kappa')\,\delta^2 - \frac{(2+\kappa')\,\theta^2\mu^2}{2(2-\theta\mu)}\left[n + (2\,(1+2\kappa') + c_3)\,\delta^2\right].$$

Lemma 7 *Suppose that* $x_0 s_0 \geq \omega$, $(x_+, s_+) > 0$, $c_3 = 8$ *and* $\theta < \dfrac{\sqrt{2}}{\sqrt{3n\,(2+\kappa')}}$. *If* $\delta < \dfrac{1}{\sqrt{6(1+2\kappa')}}$, *then the predictor iterate*

$$(x_p, s_p) = (x_+ + \theta\mu\Delta^p x, s_+ + \theta\mu\Delta^p s)$$

in Algorithm 1 is strictly feasible.

Proof. Let $\alpha \in [0,1]$,

$$x_p(\alpha) = x_+ + \alpha\theta\mu\Delta^p x,\;\; s_p(\alpha) = s_+ + \alpha\theta\mu\Delta^p s.$$

Using (18) and (19), we have

$$\begin{aligned}x_p(\alpha)s_p(\alpha) &= x_+s_+ + \alpha\theta\mu\,(x_+\Delta^p s + s_+\Delta^p x) + \alpha^2\theta^2\mu^2\Delta^p x\Delta^p s\\ &= \omega(\mu)\left[v_+^2 + \alpha\theta\mu v_+\,(d_x^p + d_s^p) + \alpha^2\theta^2\mu^2 d_x^p d_s^p\right]\\ &= \omega(\mu)\left[\left(1 - \frac{1}{2}\alpha\theta\mu\right)v_+^2 + \alpha^2\theta^2\mu^2 d_x^p d_s^p\right].\end{aligned} \tag{31}$$

From Lemma 3 and Lemma 6, we obtain

$$
\begin{aligned}
&\min\left(\frac{x_p(\alpha)s_p(\alpha)}{\omega(\mu)(1-\frac{1}{2}\alpha\theta\mu)}\right)=\min\left(v_+^2+\frac{\alpha^2\theta^2\mu^2}{1-\frac{1}{2}\alpha\theta\mu}d_x^p d_s^p\right)\\
&\geq \min v_+^2-\frac{\alpha^2\theta^2\mu^2}{1-\frac{1}{2}\alpha\theta\mu}\,\|d_x^p d_s^p\|_\infty\\
&\geq \min v_+^2-\frac{\theta^2\mu^2}{1-\frac{1}{2}\theta\mu}\,\|d_x^p d_s^p\|\\
&\geq 1-2\left(1+2\kappa'\right)\delta^2-\frac{\left(2+\kappa'\right)\theta^2\mu^2}{2(2-\theta\mu)}\left[n+\left(2(1+2\kappa')+c_3\right)\delta^2\right]\\
&=b(\delta,\theta,n).
\end{aligned}
\tag{32}
$$

Since $c_3=8$, $\delta<\dfrac{1}{\sqrt{6(1+2\kappa')}}$ and $\theta<\dfrac{\sqrt{2}}{\sqrt{3n(2+\kappa')}}$, we estimate that

$$
\begin{aligned}
b\left(\delta,\theta,n\right)&>1-\frac{1}{3}-\frac{\left(2+\kappa'\right)\theta^2\mu^2}{2(2-\theta\mu)}\left[n+\left(2(1+2\kappa')+8\right)\delta^2\right]\\
&>\frac{2}{3}-\frac{1}{3n(2-\theta)}\left[n+\frac{2(5+2\kappa')}{6(1+2\kappa')}\right]\\
&>\frac{2}{3}-\frac{1}{3n}\times\frac{\sqrt{3n(2+\kappa')}}{2\sqrt{3n(2+\kappa')}-\sqrt{2}}\times\left[n+\frac{5+2\kappa'}{3(1+2\kappa')}\right]\\
&\geq\frac{2}{3}-\frac{1}{3}\times\frac{\sqrt{3}}{2\sqrt{3}-1}\times\frac{8}{3}\\
&>0.04\\
&>0.
\end{aligned}
$$

Now, we obtain that $x_p(\alpha)s_p(\alpha)>0$ for all $\alpha\in[0,1]$ from (31) and (32). Since $x_p(0)>0$ and $s_p(0)>0$, it follows from continuity that $x_p(\alpha)>0$ and $s_p(\alpha)>0$ for any such α. Especially for $\alpha=1$, this completes the proof of the lemma. □

To proceed, we further denote

$$
v^p=\sqrt{\frac{x_p s_p}{\omega(\mu_p)}},\quad \omega(\mu_p)=\left(1-\frac{1}{2}\theta\mu\right)\omega(\mu),\quad \mu_p=(1-2\theta)\mu. \tag{33}
$$

From (31) and (32) with $\alpha=1$, it is clear that

$$
\left(v^p\right)^2=v_+^2+\frac{\theta^2\mu^2}{1-\frac{1}{2}\theta\mu}d_x^p d_s^p, \tag{34}
$$

and

$$
\min v^p\geq\sqrt{b(\delta,\theta,n)}.
$$

Next, we investigate the effect of a predictor step and the update of μ on the proximity measure $\delta(v^p):=\delta(x_p,s_p;\omega(\mu_p))$.

Lemma 8 *Suppose that* $x_0 s_0 \geq \omega$, $\delta < \dfrac{1}{\sqrt{6(1+2\kappa')}}$, *and* $\theta < \dfrac{\sqrt{2}}{\sqrt{3n(2+\kappa')}}$. *Then, the following inequality holds*

$$\delta(v^p) \leq 2\left(c_2 + \sqrt{1+(1+4\kappa')^2}\right)\delta^2 + \frac{(2+\kappa')\theta^2\mu^2\left[n+\left(2\left(1+2\kappa'\right)+c_3\right)\delta^2\right]}{2-\theta\mu}.$$

Proof. According to (15), we have

$$\begin{aligned}
\delta(v^p) &= \left\| \frac{\left((v^p)^3+(v^p)^2+v^p+2e\right)(e-v^p)}{4(v^p)^3+e} \right\| \\
&= \left\| \frac{(v^p)^3+(v^p)^2+v^p+2e}{(v^p+e)\left(4(v^p)^3+e\right)}\left(e-(v^p)^2\right) \right\| \\
&= \left\| g(v)\left(e-(v^p)^2\right) \right\| \\
&\leq 2\left\| e-(v^p)^2 \right\|,
\end{aligned} \tag{35}$$

where the function $g(t) = \dfrac{t^3+t^2+t+2}{(t+1)\left(4t^3+1\right)}$ satisfies $|g(t)| \leq 2$ for all $t > 0$. We need an upper bound for $\left\|e-(v^p)^2\right\|$. Applying (25), (34) and Lemma 6 to (35), we obtain

$$\begin{aligned}
&\left\|e-(v^p)^2\right\| \\
&= \left\| e - v_+^2 - \frac{\theta^2\mu^2}{1-\frac{1}{2}\theta\mu} d_x^p d_s^p \right\| \\
&\leq \left\| e - v_+^2 \right\| + \frac{\theta^2\mu^2}{1-\frac{1}{2}\theta\mu}\left\| d_x^p d_s^p \right\| \\
&\leq \left(c_2 + \sqrt{1+(1+4\kappa')^2}\right)\delta^2 \\
&\quad + \frac{2\theta^2\mu^2}{2-\theta\mu}\cdot\frac{(2+\kappa')\left[n+\left(2(1+2\kappa')+c_3\right)\delta^2\right]}{4} \\
&= \left(c_2 + \sqrt{1+(1+4\kappa')^2}\right)\delta^2 \\
&\quad + \frac{(2+\kappa')\theta^2\mu^2\left[n+\left(2\left(1+2\kappa'\right)+c_3\right)\delta^2\right]}{2(2-\theta\mu)}.
\end{aligned} \tag{36}$$

From (35) and (36), we get

$$\delta(v^p) \leq 2\left(c_2 + \sqrt{1+(1+4\kappa')^2}\right)\delta^2 + \frac{(2+\kappa')\theta^2\mu^2\left[n+\left(2\left(1+2\kappa'\right)+c_3\right)\delta^2\right]}{2-\theta\mu},$$

which is the desired result. □

Lemma 9 *Suppose that* $x_0 s_0 \geq \omega$, $c_2 = 6$, $c_3 = 8$, *and* $\theta \leq \frac{1+\mu}{12(2+\kappa')\sqrt{n}}$. *If* $\delta \leq \frac{\mu}{16(2+\kappa')}$, *one has* $\delta(v^p) \leq \frac{\mu_p}{16(2+\kappa')}$.

Proof. From Lemma 8, it follows that $\delta(v^p) \leq \frac{\mu_p}{16(2+\kappa')}$ holds if

$$2\left(6+\sqrt{1+(1+4\kappa')^2}\right)\delta^2 + \frac{(2+\kappa')\theta^2\mu^2\left[n+(2(1+2\kappa')+8)\delta^2\right]}{2-\theta\mu}$$
$$\leq \frac{(1-2\theta)\mu}{16(2+\kappa')}.$$

Since $\delta \leq \frac{\mu}{16(2+\kappa')}$ and $\sqrt{1+(1+4\kappa')^2} < 2(1+2\kappa')$, we have

$$\frac{\mu}{32(2+\kappa')} - \frac{1-2\theta}{16(2+\kappa')} + \frac{(2+\kappa')\theta^2\mu}{2-\theta\mu}\left[n+\frac{(5+2\kappa')\mu^2}{128(2+\kappa')^2}\right] \leq 0. \quad (37)$$

Multiplying the inequality (37) by $32(2+\kappa')$ yields

$$-1+4\theta+\frac{32}{2-\theta}(2+\kappa')^2\theta^2\left(n+\frac{5}{512}\right)$$
$$\leq -1+\frac{1}{3}+\frac{12\times 32}{23}\times\frac{4(2+\kappa')^2}{144(2+\kappa')^2 n}\times\left(n+\frac{5}{512}\right)$$
$$< -0.198,$$

where the first inequality follows from the fact that $\theta \leq \frac{1+\mu}{12(2+\kappa')\sqrt{n}}$ and $\theta_{\max} = \frac{1}{12}$. Then, combining $\delta \leq \frac{\mu}{16(2+\kappa')}$ and $\theta \leq \frac{1+\mu}{12(2+\kappa')\sqrt{n}}$, we reach the conclusion that $\delta(v^p) \leq \frac{\mu_p}{16(2+\kappa')}$. □

Algorithm 1 begins at a strictly feasible point $(x_0, s_0) \in N(\tau\mu_0)$. Let

$$\tau = \frac{1}{16(2+\kappa')}, \quad \text{and} \quad \theta \leq \frac{1+\mu}{12(2+\kappa')\sqrt{\mathrm{n}}}.$$

From (3), (9) and (15), it follows that

$$\delta(x_0, s_0; \omega(\mu_0)) = 0 \leq \frac{\mu_0}{16(2+\kappa')}.$$

Lemma 1, Lemma 7 and Lemma 9 guarantee that the new iterate (x_p, s_p) produced after a main iteration of the algorithm remains strictly feasible, satisfying

$$\delta(x_p, s_p; \omega(\mu_p)) \leq \frac{\mu_p}{16(2+\kappa')}.$$

This indicates the algorithm's validity.

5 Iteration bound of the algorithm

In this section, we provide an upper bound for $\|x_p s_p - \omega\|$. From Lemma 9 and the fact that $\theta \leq \dfrac{1+\mu}{12\,(2+\kappa')\sqrt{n}}$ with $0 < \mu \leq 1$, we have

$$\theta_{\max} = \frac{1}{6\,(2+\kappa')\sqrt{n}} \leq \frac{1}{12}. \tag{38}$$

Lemma 10 *Suppose that $x_0 s_0 \geq \omega$, $c_2 = 6$ and $c_3 = 8$. If $\delta \leq \dfrac{\mu}{16\,(2+\kappa')}$ and $\theta \leq \dfrac{1+\mu}{12\,(2+\kappa')\sqrt{n}}$, then*

$$\|x_p s_p - \omega\| \leq \left[\frac{1}{9\,(2+\kappa')}\beta + \|x_0 s_0 - \omega\|\right]\mu,$$

where $\beta = \max x_0 s_0$.

Proof. See Appendix. □

The next theorem provides an upper bound for the number of iterations of the algorithm.

Theorem 1 *Consider $c_2 = 6$ and $c_3 = 8$. Suppose that $x_0 s_0 \geq \omega$, $(x_0, s_0) \in \mathcal{F}^0$ and $\theta = \dfrac{1+\mu}{12\,(2+\kappa')\sqrt{n}}$. Let (x_k, s_k) be the kth iteration point generated by Algorithm 1. Then, we have $\|x_k s_k - \omega\| \leq \varepsilon$ for*

$$k \geq \left\lceil (2+\kappa')\sqrt{n}\,\log\frac{\frac{1}{9(2+\kappa')}\max x_0 s_0 + \|x_0 s_0 - \omega\|}{\varepsilon}\right\rceil + 1.$$

Proof. After k iterations, from Lemma 10, it follows that

$$\begin{aligned}\|x_k s_k - \omega\| &\leq \left[\frac{1}{9\,(2+\kappa')}\beta + \|x_0 s_0 - \omega\|\right]\mu_{k-1}\\ &\leq \left[\frac{1}{9\,(2+\kappa')}\beta + \|x_0 s_0 - \omega\|\right](1-2\theta_{\min})^{k-1}.\end{aligned}$$

It is clear that the condition $\|x_k s_k - \omega\| \leq \varepsilon$ holds if

$$\left[\frac{1}{9\,(2+\kappa')}\beta + \|x_0 s_0 - \omega\|\right](1-2\theta_{\min})^{k-1} \leq \varepsilon. \tag{39}$$

Hence, we look into (39). Taking the logarithm on both sides of (39) and using $-\log(1-\theta) \geq \theta$ with $\theta \in (0,1)$, we see that the above inequality holds if

$$k \geq \frac{1}{2\theta_{\min}}\log\frac{\frac{1}{9(2+\kappa')}\beta + \|x_0 s_0 - \omega\|}{\varepsilon} + 1.$$

Since $\theta \leq \dfrac{1+\mu}{12\,(2+\kappa')\,\sqrt{n}}$ with $0 < \mu \leq 1$ and $\beta = \max x_0 s_0$, Algorithm 1 requires no more than

$$\left\lceil (2+\kappa')\,\sqrt{n}\log \frac{\frac{1}{9(2+\kappa')}\max x_0 s_0 + \|x_0 s_0 - \omega\|}{\varepsilon} \right\rceil + 1$$

iterations to find an ε-approximate solution (x_k, s_k) satisfying $\|x_k s_k - \omega\| \leq \varepsilon$. Then, the proof is complete. $\square$

6 Numerical results

To demonstrate the efficiency of Algorithm 1, we implement it on a desktop computer with MATLAB (R2016a) Intel(R) Core(TM) i7-8565U CPU @1.80 GHz 8.00GB, Windows 11. In all problems, we set accuracy as $\varepsilon = 10^{-5}$. Note that the gap (Gap) and proximity measure ($\delta(v)$) are the values of $\|xs - \omega\|$ and $\left\| \frac{(e-v)\left(v^3+v^2+v+2e\right)}{4v^3+e} \right\|$, respectively.

Problem 1 Consider the $P_*(\kappa)$-WLCP (2), where M and ω are given by

$$M = \begin{pmatrix} 1 & 0 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega = rand(3,1).$$

It is pointed out that matrix M is a $P_*(\kappa)$-matrix with $\kappa = 6$ in the literature [29]. We select a strictly feasible initial point $x_0 = s_0 = e$ and set the parameter $\theta = \frac{1}{12}$. We proceed to generate test problems using the vector q defined as

$$q = -Me + e.$$

According to Lemma 1, we have $1 + 4\kappa' = \dfrac{1+4\kappa}{\min \omega}$. Clearly, this initial point meets the criteria outlined in the first two lines of Algorithm 1, specifically, $(x_0, s_0) \in \mathcal{F}^0$, $\delta(x_0, s_0; \omega(\mu_0)) = 0$ and $x_0 s_0 \geq \omega$.

After 64 iterations and 0.0172s, we achieve the unique solution of the WLCP, i.e.,

$$x_* = (7.0070578 \quad 0.6323624 \quad 0.3123292)^T,$$
$$s_* = (0.1303508 \quad 1.0000000 \quad 0.3123292)^T.$$

In Figure 1, we plot the gap (Gap) and proximity measure ($\delta(v)$) for Problem 1.

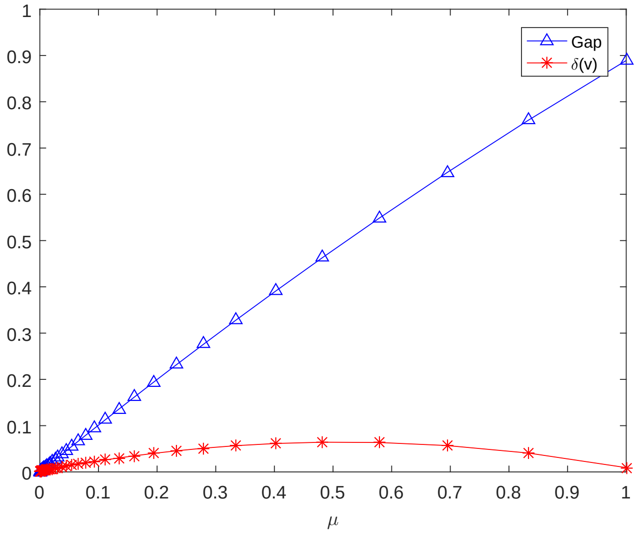

Fig. 1: The Gap and $\delta(v)$ for Problem 1

Problem 2 de Klerk and Nagy [16] demonstrated that the handicap associated with the matrix can grow exponentially in relation to the problem size. They examined a specific matrix proposed by Csizmadia, denoted as

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n\times n},$$

and showed that its handicap satisfies the inequality $\kappa(M) \geq 2^{2n-8} - 0.25$.

Let $\omega = rand(n,1) \in \mathbb{R}^n_+$. We select the starting points as $(x_0, s_0) = \left(\frac{e}{20}, 30e\right)$ and $(x_0, s_0) = \left(\frac{e}{20}, 50e\right)$. Define

$$q = 30e - \frac{e}{20}M \quad \text{and} \quad q = 50e - \frac{e}{20}M.$$

Obviously, the initial points satisfy the conditions stated in the first two lines of Algorithm 1, i.e., $(x_0, s_0) \in \mathcal{F}^0$, $x_0 s_0 \geq \omega$ and $\delta(x_0, s_0; \omega(\mu_0)) = 0$.

The barrier parameter is set as $\theta \in \left\{\frac{1}{12}, 0.1, \frac{1}{6}, 0.2\right\}$. Then the $P_*(\kappa)$-WLCPs with dimension $n \in \{40, 80, 150, 210, 300, 450, 650, 900, 1300, 1500\}$ are generated, respectively. The average CPU times (CPU) in seconds, number of iterations (Iter), the gap (Gap) and proximity measure ($\delta(v)$) in each case are reported in Table 3.

Table 3: Numerical results for Problem 2

	$x_0 = \frac{e}{20},\ s_0 = 30e$				$x_0 = \frac{e}{20},\ s_0 = 50e$			
	CPU	Iter	Gap	$\delta(v)$	CPU	Iter	Gap	$\delta(v)$
n	$\theta = \frac{1}{12}$							
40	0.0228	75	9.05965e-06	1.39950e-09	0.0243	78	8.87651e-06	1.16861e-08
80	0.0510	77	8.94984e-06	3.31680e-09	0.0621	80	9.34607e-06	1.55539e-09
150	0.1727	78	9.41366e-06	1.93969e-08	0.1878	82	9.31781e-06	8.45864e-09
210	0.6086	79	9.56816e-06	1.16395e-08	0.5644	83	9.19977e-06	1.34811e-08
300	0.9689	80	9.47580e-06	8.17032e-09	1.1721	84	9.22664e-06	4.65633e-08
450	2.6852	81	9.74702e-06	2.29743e-09	3.2071	85	9.41091e-06	9.60065e-08
650	8.1800	82	9.72736e-06	1.13346e-08	8.3442	86	9.40167e-06	7.82204e-08
900	18.9519	83	9.68937e-06	9.10354e-09	19.8234	87	9.23915e-06	2.46749e-09
1300	48.6846	84	9.77329e-06	8.90819e-07	47.3827	88	9.23272e-06	4.87706e-08
1500	67.4732	85	8.70625e-06	1.22049e-07	70.3104	89	9.76339e-06	5.87582e-07
n	$\theta = 0.1$							
40	0.0206	61	9.26549e-06	1.56693e-09	0.0202	64	9.74328e-06	1.70549e-09
80	0.0434	63	8.89723e-06	1.39470e-08	0.0541	66	8.90782e-06	2.74221e-10
150	0.1366	64	9.56043e-06	4.64302e-09	0.1558	67	9.64541e-06	4.01279e-09
210	0.3915	65	9.13906e-06	8.22429e-09	0.4553	68	9.17432e-06	1.25113e-08
300	0.7998	66	8.71434e-06	1.39089e-08	0.9011	69	8.77887e-06	3.02676e-08
450	2.3536	67	8.53053e-06	5.46674e-09	2.5081	70	8.64243e-06	7.17604e-08
650	6.1487	68	8.36364e-06	6.93297e-07	6.9175	71	8.27228e-06	3.85241e-08
900	16.0514	68	9.62662e-06	5.71179e-06	18.2969	71	9.73965e-06	2.55455e-08
1300	37.7373	69	9.27350e-06	1.99895e-07	39.4361	72	9.37307e-06	1.40003e-08
1500	59.3892	70	9.16933e-06	2.58590e-06	61.1886	73	8.07251e-06	1.22488e-07
n	$\theta = \frac{1}{6}$							
40	0.0103	34	9.11122e-06	4.86845e-08	0.0112	36	8.48481e-06	2.16448e-08
80	0.0256	35	8.86239e-06	3.94551e-07	0.0292	37	8.04065e-06	2.55325e-08
150	0.0759	36	8.13569e-06	6.55188e-05	0.0864	38	7.22278e-06	1.53472e-08
210	0.2222	36	9.82349e-06	1.94498e-08	0.2687	38	8.64857e-06	5.41101e-09
300	0.4517	37	7.79039e-06	1.55985e-06	0.5427	39	6.85383e-06	6.36951e-09
450	1.3332	37	9.52476e-06	4.91082e-08	1.5671	39	8.40857e-06	5.80645e-08
650	3.5448	38	7.61658e-06	1.02007e-07	4.4389	40	6.72598e-06	4.96949e-07
900	8.1720	38	8.91531e-06	1.92452e-08	10.3095	40	7.93934e-06	4.11639e-08
1300	21.1970	39	7.16246e-06	1.56030e-07	22.6029	40	9.49471e-06	4.11947e-06
1500	90.7077	39	7.70455e-06	5.98194e-08	34.6244	41	6.81211e-06	4.70366e-07
n	$\theta = 0.2$							
40	0.0115	27.9	6.66927e-06	1.29565e-09	0.0089	29	7.62983e-06	1.01351e-07
80	0.0197	28	8.86095e-06	8.15885e-08	0.0251	30	6.34954e-06	4.85987e-10
150	0.0710	29	7.16239e-06	8.50407e-08	0.0721	30	8.69082e-06	8.58543e-09
210	0.1947	29	8.49164e-06	1.37684e-07	0.2193	31	6.15781e-06	2.39476e-08
300	0.3707	30	6.18976e-06	8.03816e-08	0.4536	31	7.39424e-06	1.71547e-07
450	1.0017	30	7.56802e-06	1.34730e-06	1.1304	31	9.05229e-06	1.08748e-06
650	2.9224	30	9.06073e-06	1.22848e-06	3.0824	32	6.53661e-06	2.84435e-08
900	7.2172	31	6.41781e-06	1.94017e-08	7.3727	32	7.68607e-06	5.39601e-08
1300	17.6244	31	7.67200e-06	5.77009e-07	16.8077	32	9.23467e-06	3.34489e-08

Problem 3 Consider $P_*(0)$-WLCPs, where matrices M are positive semidefinite matrices. We randomly generate $P_*(0)$-WLCP test problems with dimensions $n \in \{50, 100, 200, 300, 400, 600, 900, 950, 1300, 1500, 1800, 2100\}$. By taking $A = rand(n, n)$, $M = AA^T$, $x_0 = s_0 = 2e$ and $\omega = \frac{x_0 s_0}{2}$, it follows from Lemma 1 that $\kappa' = \dfrac{1}{4}$. Let $q = s_0 - Mx_0$. We can easily verify that $x_0,\ s_0$ are strictly feasible and their proximity measure satisfies $\delta(x_0, s_0; \omega(\mu_0)) = 0 \leq \frac{\mu_0}{16(2+\kappa')}$ with $\mu_0 = 1$.

Then, we compare the performance of Algorithm 1 with two existing full-Newton step feasible IPAs based on the search directions of Wang [39] and Kheirfam [28]. Denote the two full-Newton step feasible IPAs as Wang's Algorithm and Kheirfam's Algorithm, respectively. In all the test problems, we select $\theta = \frac{1}{6}$. For Problem 2 and Problem 3, Algorithm 1 along with two other algorithms are used to solve $P_*(\kappa)$-WLCPs. But in Problem 2, we use the following two sets of initial points

$$(x_0, s_0) = \left(\frac{e}{20}, 30e\right) \quad \text{and} \quad (x_0, s_0) = \left(\frac{e}{20}, 20e\right),$$

so q is defined as

$$q = 30e - \frac{e}{20}M \quad \text{and} \quad q = 20e - \frac{e}{20}M.$$

In Table 4, we present the dimension n of the test problems, the number of iterations (Iter), CPU time, and the gap (Gap) of the test problems. As shown in Table 4, Algorithm 1 has fewer number of total iterations (Total Iter) which includes one corrector step and one predictor step at each main iteration, compared to Wang's Algorithm and Kheirfam's Algorithm. Accordingly, Algorithm 1 can reach optimal solutions faster than other two algorithms.

The observation that Wang's and Kheirfam's algorithms exhibit almost the same number of iterations is indeed intriguing, which could be attributed to their algorithmic design.

- It follows from [39] and [28] that the main difference between Wang's Algorithm and Kheirfam's Algorithm is the calculation of the search direction. The scaling search direction $p_v = v^{-1} - v = \frac{e - v^2}{v}$ in Wang's Algorithm is derived from function $\varphi(t) = t$. In contrast, the scaling search direction $p_v = e - v^2$ in Kheirfam's Algorithm is based on function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$. By looking at the scaling search directions of Wang's Algorithm and Kheirfam's Algorithm, we can see that the same numerator of the two search directions is $e - v^2$, and the denominator v in Wang's Algorithm is always close and finally tends to the denominator e in Kheirfam's Algorithm. Therefore Wang's Algorithm and Kheirfam's Algorithm have very close search directions at each iteration, which leads to almost the same number of iterations for solving some problems.
- It could be seen from Table 4 that Wang's Algorithm and Kheirfam's Algorithm have different number of iterations in some instances of Problem 2.

Table 4: Numerical results for Algorithm 1, Kheirfam's Algorithm and Wang's Algorithm

$\theta = \frac{1}{6}$	Algorithm 1			Kheirfam's Algorithm			Wang's Algorithm		
	Total Iter	CPU	Gap	Iter	CPU	Gap	Iter	CPU	Gap
n	Problem 2 with $(x_0, s_0) = \left(\frac{e}{20}, 30e\right)$								
50	70	0.0205	7.2803e-06	76	0.0258	8.6572e-06	75	0.0193	9.7981e-06
100	70	0.0511	8.5461e-06	78	0.0461	8.3706e-06	77	0.0457	9.9888e-06
200	72	0.2118	9.3977e-06	80	0.2445	9.8043e-06	79	0.2425	8.3481e-06
300	74	0.5521	7.7739e-06	81	0.6291	9.7845e-06	80	0.5457	8.3712e-06
400	74	1.0229	8.9402e-06	81	1.2182	9.3854e-06	81	1.1758	9.5813e-06
600	76	3.0673	7.3261e-06	82	3.4891	9.8567e-06	82	3.2852	9.7951e-06
900	76	8.4744	8.9524e-06	84	9.8569	8.4805e-06	83	9.0762	9.9979e-06
1300	78	22.9733	7.1449e-06	85	70.0155	8.4373e-06	85	22.2051	8.4183e-06
1500	78	97.1789	7.7221e-06	85	89.8044	9.0378e-06	85	34.4018	9.0615e-06
n	Problem 2 with $(x_0, s_0) = \left(\frac{e}{20}, 20e\right)$								
50	66	0.0143	8.8246e-06	72	0.0145	8.9275e-06	73	0.0138	8.8316e-06
100	68	0.0395	8.4095e-06	74	0.0481	9.9005e-06	75	0.0553	8.6238e-06
200	70	0.2045	7.8991e-06	76	0.2564	9.1059e-06	76	0.2485	9.7806e-06
300	70	0.4671	9.1994e-06	77	0.5414	9.2012e-06	77	0.4906	9.6436e-06
400	72	0.9629	7.3325e-06	78	1.0674	9.3612e-06	78	1.0273	9.1627e-06
600	72	2.7397	8.9899e-06	79	2.9637	9.2054e-06	79	2.9681	9.3124e-06
900	74	8.5499	7.3952e-06	80	9.5858	9.6581e-06	80	9.2971	9.8017e-06
950	74	10.6652	7.5891e-06	80	25.9293	9.9276e-06	80	10.8754	9.7928e-06
n	Problem 3 with $(x_0, s_0) = (2e, 2e)$								
100	74	0.4156	7.6307e-06	81	0.0570	9.2583e-06	81	0.0585	9.2583e-06
300	76	0.6024	8.8112e-06	84	0.6482	9.2800e-06	84	0.6461	9.2800e-06
600	78	3.2062	8.3072e-06	86	3.4889	9.1138e-06	86	3.4889	9.1138e-06
900	80	9.6649	6.7828e-06	87	9.5891	9.3017e-06	87	9.7213	9.3017e-06
1300	80	20.2848	8.1519e-06	88	22.0968	9.3161e-06	88	21.9320	9.3161e-06
1500	80	33.7087	8.7566e-06	89	33.2532	8.3392e-06	89	34.5044	8.3392e-06
1800	80	47.1488	9.5924e-06	89	56.2288	9.1352e-06	89	58.3538	9.1352e-06
2100	82	78.3520	6.9073e-06	89	83.7749	9.8671e-06	89	78.4430	9.8671e-06

We conclude this section by summarizing the numerical findings from Figure 1, Table 3 and Table 4.

– The duality gap and proximity measure gradually decrease to 0 as t tends to 0.
– It is general fact that in IPMs smaller θ leads to higher number of iterations and Algorithm 1 is no exception.
– Numerical performance indicates that the proposed algorithm is reliable and promising.

7 Conclusion

In this paper, we present a new IPA for solving $P_*(\kappa)$-WLCP. The novelty of the paper is that we use function $\varphi(t) = t^2 + \sqrt{t}$, which belongs to a new class of AET functions [19] for $P_*(\kappa)$-LCP, to determine the search directions, and derive the iteration bound for $P_*(\kappa)$-WLCP. However, the analysis of our proposed algorithm is more complicated than in the LCP case mainly because of the existence of the nonnegative weight vector in the $P_*(\kappa)$-WLCP. Some preliminary numerical results are provided to demonstrate the potential for the proposed algorithm to be efficient. Several future developments are possible. For instance, the corrector-predictor IPAs could be extended to the WLCP over symmetric cones. Proposing the algorithm of WLCPs based on other different kernel functions is another possible future development. Finally, the wide neighborhood methods could be further analyzed and expanded.

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8 Appendix

8.1 Important comments

Remark 1. From (12) and (14), for any $t > 0$, we compute

$$g(t) = \frac{f(t)}{2(1-t^2)} = \frac{-t^4 - t + 2}{(1-t^2)\,(4t^3+1)}$$

and

$$\begin{aligned} h(t) &= \frac{4(1-t^2-tf(t))}{f(t)^2} = \frac{1-2tg(t)}{(1-t^2)g(t)^2} \\ &= \frac{(-2t^5+4t^3+t^2-4t+1)(4t^3+1)}{(-t^4-t+2)^2}. \end{aligned}$$

It follows from [19] that

$$|g(t)| \le 2, \quad |h(t)| \le 6$$

for any $t \in (0, 2.5]$. In other words, the conditions AET(a) and AET(b) in Proposition 1 are satisfied with $c_1 = 2$, $c_2 = 6$. Therefore, we assume $c_1 = 2$, $c_2 = 6$ in this paper. Figure 2 and Figure 3 depict graphs of $g(t)$ and $h(t)$, respectively.

Remark 2. In order to evaluate the value of c_3 mentioned in Definition 1, it is crucial to look into (29) in Lemma 5. After simple calculations, it gives

$$
\begin{aligned}
&4t^2\left(\varphi(1)-\varphi\left(t^2\right)\right)\varphi'\left(t^2\right)-c_3\left(\varphi(1)-\varphi\left(t^2\right)\right)^2\leq 4t^2\left(1-t^2\right)\left(\varphi'\left(t^2\right)\right)^2\\
\Longleftrightarrow\ & vp_v-c_3\frac{p_v^2}{4}\leq e-v^2\\
\Longleftrightarrow\ & \frac{4v^2\left(\varphi(e)-\varphi(v^2)\right)\varphi^{'}(v^2)-c_3\left(\varphi(e)-\varphi(v^2)\right)^2}{4v^2\left(\varphi'(v^2)\right)^2}\leq e-v^2\\
\Longleftrightarrow\ & \frac{2v\left(-v^4-v+2e\right)}{4v^3+e}-e+v^2\leq c_3\frac{\left(-v^4-v+2e\right)^2}{\left(4v^3+e\right)^2}\\
\Longleftrightarrow\ & \frac{\left(4v^3+e\right)^2}{\left(-v^4-v+2e\right)^2}\left[\frac{2v\left(-v^4-v+2e\right)}{4v^3+e}-e+v^2\right]\leq c_3e\\
\Longleftrightarrow\ & J(v):=\frac{\left(4v^3+e\right)^2}{\left(-v^4-v+2e\right)^2}\left(\frac{2v^5-4v^3-v^2+4v-e}{4v^3+e}\right)\leq c_3e.
\end{aligned}
$$

Then as $v\to\infty$, we have

$$\lim_{v\to\infty}J(v)=8.$$

It also can be seen from Figure 4 that $J(t)\leq 8$. Hence, it is assumed that $c_3=8$ in this paper.

8.2 A Proof of Lemma 10

Proof. First, from (3) and $x_0s_0\geq\omega$, we obtain

$$\omega(\mu)=(1-\mu)\omega+\mu x_0s_0\leq\max\{\omega,x_0s_0\}e=\beta e. \tag{40}$$

Combining (3) and (33) yields

$$
\begin{aligned}
\|x_ps_p-\omega\|&\leq\|x_ps_p-\omega(\mu_p)\|+\|\omega(\mu_p)-\omega(\mu)\|+\|\omega(\mu)-\omega\|\\
&\leq\left\|e-(v^p)^2\right\|\|\omega(\mu_p)\|_\infty+\frac{1}{2}\theta\mu\|\omega(\mu)\|+\|x_0s_0-\omega\|\mu.
\end{aligned}\tag{41}
$$

Considering the first two terms in the second inequality of (41), we compute that

$$
\begin{aligned}
&\left\|e-(v^p)^2\right\| \|\omega(\mu_p)\|_\infty + \frac{1}{2}\theta\mu\|\omega(\mu)\| \\
&\leq \left\{\left(6+\sqrt{1+(1+4\kappa')^2}\right)\delta^2+\frac{1}{2}\sqrt{n}\theta\mu\right\}\|\omega(\mu)\|_\infty \\
&\quad+\frac{(2+\kappa')\theta^2\mu^2\left[n+(2(1+2\kappa')+8)\delta^2\right]}{2(2-\theta\mu)}\|\omega(\mu)\|_\infty \qquad (42)\\
&\leq \left\{\frac{\mu}{64(2+\kappa')}+\frac{(2+\kappa')\theta^2\mu}{2(2-\theta\mu)}\left[n+\frac{(5+2\kappa')\mu^2}{128(2+\kappa')^2}\right]+\frac{1}{2}\sqrt{n}\theta\right\}\beta\mu \\
&\leq \left\{\frac{1}{64(2+\kappa')}+\frac{1}{2(2-\theta)}\times\frac{(2+\kappa')(1+\mu)^2}{144(2+\kappa')^2 n}\left[n+\frac{5+2\kappa'}{128(2+\kappa')^2}\right]\right\}\beta\mu \\
&\quad+\frac{1+\mu}{24(2+\kappa')}\beta\mu \\
&\leq \left\{\frac{1}{64(2+\kappa')}+\frac{12}{23}\times\frac{1}{72(2+\kappa')n}\left[n+\frac{5+2\kappa'}{128(2+\kappa')^2}\right]+\frac{1}{12(2+\kappa')}\right\}\beta\mu \\
&\leq \left[\frac{1}{64(2+\kappa')}+\frac{1}{138(2+\kappa')n}\left(n+\frac{5}{512}\right)+\frac{1}{12(2+\kappa')}\right]\beta\mu \\
&< \frac{1}{9(2+\kappa')}\beta\mu,
\end{aligned}
$$

where the first inequality is due to (33) and (36). The first term in the second inequality of (42) is derived from (40), $\delta \leq \frac{\mu}{16(2+\kappa')}$ and $\sqrt{1+(1+4\kappa')^2} < 2(1+2\kappa')$. The fourth inequality of (42) follows from (38). Then, from (41) and (42), we obtain

$$
\|x_p s_p - \omega\| < \left[\frac{1}{9(2+\kappa')}\beta + \|x_0 s_0 - \omega\|\right]\mu,
$$

which is the desired result. □

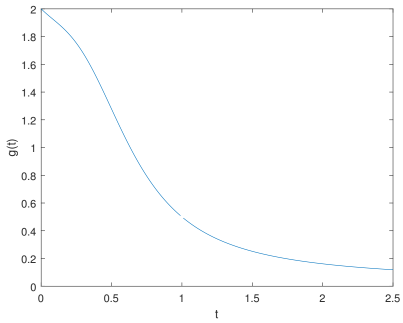

Fig. 2: The figure of $g(t)$

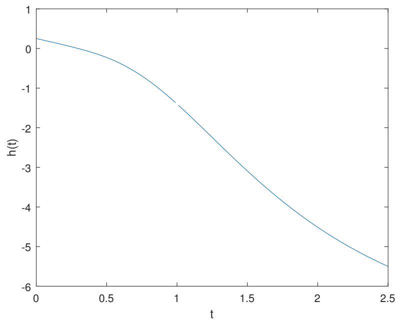

Fig. 3: The figure of $h(t)$

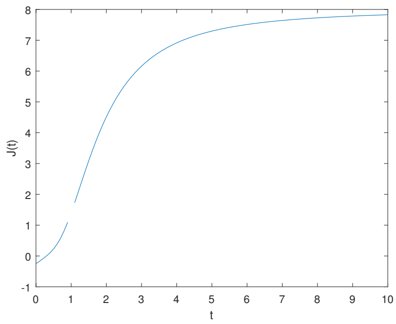

Fig. 4: Graph of the function $J(t)$