Mean Inequalities Associated with Circular Cones

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Abstract: Mean inequalities on the second order cone have been studied as an extension of the SPD cone. In this article, we turn our eyes on non-symmetric cones. In fact, we investigate two types of decompositions associated with circular cones, and establish their own mean inequalities. These inequalities are ground bricks for further study regarding circular cone optimization. We also find under the condition $0 < \theta < \frac{\pi}{4}$ some inequalities cannot hold if we apply different decomposition, and correspondingly we raise a conjecture.

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Mathematics Subject Classification: 26A27, 26B05, 26B35, 49J52, 90C33.

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1 Introduction

The second-order cone (SOC) in $\mathbb{R}^n$, also called Lorentz cone, is defined by

$$\mathcal{K}^n = \{(x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, x_1 \geq \|\mathbf{x}_2\|\},$$

where $\|\cdot\|$ denotes the Euclidean norm. If $n = 1$, then $\mathcal{K}^n$ reduces to the set of nonnegative real numbers $\mathbb{R}_+$. As a natural extension of the second-order cone, the circular cone $\mathcal{L}_\theta$ was first considered and investigated in [9]. In particular, let the half-aperture angle be θ with $\theta \in (0, \frac{\pi}{2})$, the circular cone $\mathcal{L}_\theta$ is defined as

$$\mathcal{L}_\theta = \{(x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|\mathbf{x}_2\| \cot\theta\},$$

where $\|\cdot\|$ is the Euclidean norm, see Figures 1-2. It is clear that the SOC is a special case of circular cone, corresponding to $\theta = \frac{\pi}{4}$.

There holds a relationship between these two cones, see [9]. More specifically, it has been shown that

$$T\mathcal{L}_\theta = \mathcal{K}^n, \quad \text{where} \quad T = \begin{bmatrix} \tan\theta & \mathbf{0}^T \\ \mathbf{0} & I \end{bmatrix},$$

which is equivalent to saying

$$\mathbf{x} \in \mathcal{L}_\theta \quad \Longleftrightarrow \quad T\mathbf{x} \in \mathcal{K}^n.$$

Since $\mathcal{L}_\theta$ is a pointed and salient closed convex cone in $\mathbb{R}^n$, we introduce a partial order on $\mathbb{R}^n$. For any $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, we write $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{y}$ if and only if $\mathbf{x} - \mathbf{y} \in \mathcal{L}_\theta$; and write $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{y}$ if and only if $\mathbf{x} - \mathbf{y} \in \text{int}(\mathcal{L}_\theta)$. With this partial ordering, it is easy to verify the following facts.

Lemma 1.1. *Suppose that* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*. Then, the following holds.*

(a) *If* $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}$ *and* $\mathbf{y} \succeq_{\mathcal{L}_\theta} \mathbf{0}$*, then* $\mathbf{x} + \mathbf{y} \succeq_{\mathcal{L}_\theta} \mathbf{0}$.

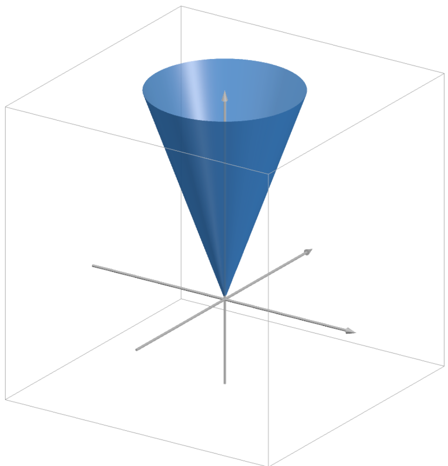

Figure 1: Circular cone with $\theta \in (0, \frac{\pi}{4})$.

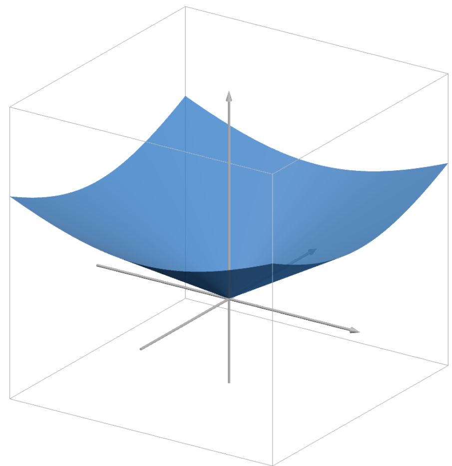

Figure 2: Circular cone with $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

(b) *If* $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{y}$, $\mathbf{y} \succeq_{\mathcal{L}_\theta} \mathbf{z}$, *then* $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{z}$.

(c) *If* $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}$, *then* $-\mathbf{x} \preceq_{\mathcal{L}_\theta} \mathbf{0}$.

For any real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$, the SOC function f^{soc} (a vector-valued function) is defined as

$$f^{\mathrm{soc}}(\mathbf{x}) = f(\lambda_1(\mathbf{x}))\mathbf{u}_{\mathbf{x}}^{(1)} + f(\lambda_2(\mathbf{x}))\mathbf{u}_{\mathbf{x}}^{(2)}, \quad \forall \mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{1.1}$$

Here $\mathbf{x}$ is decomposed as

$$\mathbf{x} = \lambda_1(\mathbf{x})\mathbf{u}_{\mathbf{x}}^{(1)} + \lambda_2(\mathbf{x})\mathbf{u}_{\mathbf{x}}^{(2)}, \tag{1.2}$$

where $\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x})$ and $\mathbf{u}_{\mathbf{x}}^{(1)}, \mathbf{u}_{\mathbf{x}}^{(2)}$ are the spectral values and the associated spectral vectors of $\mathbf{x}$ given by

$$\lambda_i(\mathbf{x}) = x_1 + (-1)^i \|\mathbf{x}_2\|, \tag{1.3}$$

$$\mathbf{u}_{\mathbf{x}}^{(i)} = \begin{cases} \frac{1}{2}\left(1\ ,\ (-1)^i \dfrac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\right), & \text{if} \quad \mathbf{x}_2 \neq \mathbf{0}, \\ \frac{1}{2}\left(1\ ,\ (-1)^i \mathbf{w}\right), & \text{if} \quad \mathbf{x}_2 = \mathbf{0}, \end{cases} \tag{1.4}$$

for $i = 1, 2$ with $\mathbf{w}$ being any vector in $\mathbb{R}^{n-1}$ satisfying $\|\mathbf{w}\| = 1$. If $\mathbf{x}_2 \neq \mathbf{0}$, the decomposition is unique. The SOC function was first introduced in [1] and also contributes a lot of applications to second-order cone program (SOCP) and second-order cone complementarity problem (SOCCP), see [4, 5, 6, 8]. In addition, SOC-convex and SOC-monotone functions, and mean inequalities associated with second-order cone play important roles. In this article we turn our eyes on circular cones. We will consider two types of decomposition on circular cones, and develop their own mean inequalities correspondingly. First, once the circular cone has its own decomposition like (1.2), we introduce the circular cone function analogous to (1.1) as follows.

Definition 1.2. (Circular Cone Function) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function, we define a vector-valued function, $f^{\mathcal{L}_\theta} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$f^{\mathcal{L}_\theta}(\mathbf{x}) = f(\lambda_1(\mathbf{x}))\mathbf{u}_{\mathbf{x}}^{(1)} + f(\lambda_2(\mathbf{x}))\mathbf{u}_{\mathbf{x}}^{(2)}, \quad \forall \mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1},$$

which is called an $\mathcal{L}_\theta$ function.

If f is defined only on a subset of $\mathbb{R}$, then $f^{\mathcal{L}_\theta}$ is defined on the corresponding subset of $\mathbb{R}^n$. With Definition 1.2 the arithmetic mean $A(\mathbf{x},\mathbf{y})$ and the harmonic mean $H(\mathbf{x},\mathbf{y})$ are defined. In particular, for any $\mathbf{x},\mathbf{y} \in \mathcal{L}_\theta$, we define

$$\begin{aligned}
|\mathbf{x}| &= |\lambda_1(\mathbf{x})|\mathbf{u}_{\mathbf{x}}^{(1)} + |\lambda_2(\mathbf{x})|\mathbf{u}_{\mathbf{x}}^{(2)}, \\
\mathbf{x}^{-1} &= \lambda_1(\mathbf{x})^{-1}\mathbf{u}_{\mathbf{x}}^{(1)} + \lambda_2(\mathbf{x})^{-1}\mathbf{u}_{\mathbf{x}}^{(2)}, \quad \text{if } \lambda_1(\mathbf{x})\lambda_2(\mathbf{x}) \neq 0, \\
A(\mathbf{x},\mathbf{y}) &= \frac{\mathbf{x}+\mathbf{y}}{2}, \\
H(\mathbf{x},\mathbf{y}) &= \left(\frac{\mathbf{x}^{-1}+\mathbf{y}^{-1}}{2}\right)^{-1}, \quad \text{if } \lambda_1(\mathbf{x})\lambda_1(\mathbf{y}) \neq 0.
\end{aligned}$$

Furthermore, the maximum value $\mathbf{x} \vee \mathbf{y}$ and minimum value $\mathbf{x} \wedge \mathbf{y}$ are defined:

$$\begin{aligned}
\mathbf{x} \vee \mathbf{y} &= \frac{1}{2}(\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|), \\
\mathbf{x} \wedge \mathbf{y} &= \begin{cases} \frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|), & \text{if } \mathbf{x}+\mathbf{y} \succeq_{\mathcal{K}^n} |\mathbf{x}-\mathbf{y}|, \\ \mathbf{0}, & \text{otherwise.} \end{cases}
\end{aligned}$$

In the SOC setting [3], there holds

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{K}^n} A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{K}^n} H(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{K}^n} \mathbf{x} \wedge \mathbf{y}. \tag{1.5}$$

To prove the inequalities in (1.5), SOC-monotone functions and SOC-convex functions play important roles [2, 3]. Likewise, in order to show these inequalities in the circular cone setting, we introduce $\mathcal{L}_\theta$-monotone and $\mathcal{L}_\theta$-convex functions as below.

Definition 1.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function.

(a) f is said to be $\mathcal{L}_\theta$-monotone if it satisfies the following implication:

$$\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{y} \quad \Longrightarrow \quad f^{\mathcal{L}_\theta}(\mathbf{x}) \succeq_{\mathcal{L}_\theta} f^{\mathcal{L}_\theta}(\mathbf{y}).$$

(b) f is said to be $\mathcal{L}_\theta$-convex if it satisfies the following condition:

$$\lambda f^{\mathcal{L}_\theta}(\mathbf{x}) + (1-\lambda) f^{\mathcal{L}_\theta}(\mathbf{y}) \succeq_{\mathcal{L}_\theta} f^{\mathcal{L}_\theta}\left(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\right), \quad \forall \lambda \in [0,1].$$

In this paper, we will raise two types of decomposition for the circular cone $\mathcal{L}_\theta$, and study their own mean inequalities.

2 First type of decomposition

In this section, we introduce the first type of decomposition for the circular cone $\mathcal{L}_\theta$, which is new to the literature. For any $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{x}$ is decomposed as

$$\mathbf{x} = \lambda_1(\mathbf{x})\mathbf{u}_{\mathbf{x}}^{(1)} + \lambda_2(\mathbf{x})\mathbf{u}_{\mathbf{x}}^{(2)}, \tag{2.1}$$

where the spectral values of $\mathbf{x}$ are given by

$$\begin{aligned}
\lambda_1(\mathbf{x}) &= x_1 - \|\mathbf{x}_2\|\cot\theta, \\
\lambda_2(\mathbf{x}) &= x_1 + \|\mathbf{x}_2\|\cot\theta.
\end{aligned}$$

In addition, the spectral vectors of $\mathbf{x}$ are respectively expressed as below:

$$\mathbf{u}_{\mathbf{x}}^{(i)} = \begin{cases} \frac{1}{2}\left(1\ ,\ (-1)^i \dfrac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta\right), & \text{if} \quad \mathbf{x}_2 \neq \mathbf{0}, \\ \frac{1}{2}\left(1\ ,\ (-1)^i \mathbf{w}\tan\theta\right), & \text{if} \quad \mathbf{x}_2 = \mathbf{0}, \end{cases}$$

for $i = 1, 2$ with $\mathbf{w}$ being any vector in $\mathbb{R}^{n-1}$ satisfying $\|\mathbf{w}\| = 1$. If $\mathbf{x}_2 \neq \mathbf{0}$, the decomposition is unique. When $\theta = \frac{\pi}{4}$, this decomposition coincides with those (1.3) and (1.4) in SOC setting. Note that it is true that $\lambda_2(\mathbf{x}) \geq \lambda_1(\mathbf{x})$, and

$$\lambda_1(\mathbf{x}) \geq 0 \iff \mathbf{x} \in \mathcal{L}_\theta \iff \mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}; \quad \lambda_1(\mathbf{x}) > 0 \iff \mathbf{x} \in \operatorname{int}(\mathcal{L}_\theta) \iff \mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}.$$

For subsequent needs, let us examine some basic properties of the absolute value $|\mathbf{x}|$ and the inverse $\mathbf{x}^{-1}$ based on the above decomposition.

Lemma 2.1. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

(a) *If* $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}$, *then* $\mathbf{x} = |\mathbf{x}|$.

(b) *If* $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$, *then* $\mathbf{x}^{-1} = \frac{1}{x_1^2 - \|\mathbf{x}_2\|^2 \cot^2\theta}(x_1, -\mathbf{x}_2) \succ_{\mathcal{L}_\theta} \mathbf{0}$.

(c) *If* $r \in \mathbb{R}$ *and* $r \neq 0$, *then* $(r\mathbf{x})^{-1} = \frac{1}{r}\mathbf{x}^{-1}$.

(d) *If* $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$, *then* $(\mathbf{x}^{-1})^{-1} = \mathbf{x}$.

Proof. (a) By decomposition (2.1), it is clear to see that $|\mathbf{x}| = |\lambda_1(\mathbf{x})|\mathbf{u}_{\mathbf{x}}^{(1)} + |\lambda_2(\mathbf{x})|\mathbf{u}_{\mathbf{x}}^{(2)}$. Since $\mathbf{x} \in \mathcal{L}_\theta$, $\lambda_1(\mathbf{x}) = x_1 - \|\mathbf{x}_2\|\cot\theta \geq 0$, and $\lambda_2(\mathbf{x}) = x_1 + \|\mathbf{x}_2\|\cot\theta \geq \lambda_1(\mathbf{x}) \geq 0$. Then, we have $|\lambda_1(\mathbf{x})| = \lambda_1(\mathbf{x})$, $|\lambda_2(\mathbf{x})| = \lambda_2(\mathbf{x})$, which concludes $\mathbf{x} = |\mathbf{x}|$.

(b) It is easy to verify the case for $\mathbf{x}_2 = \mathbf{0}$. Assume $\mathbf{x}_2 \neq \mathbf{0}$ now. Then, the desired result follows directly from below verifications:

$$\begin{aligned} \mathbf{x}^{-1} &= \lambda_1(\mathbf{x})^{-1}\mathbf{u}_{\mathbf{x}}^{(1)} + \lambda_2(\mathbf{x})^{-1}\mathbf{u}_{\mathbf{x}}^{(2)} \\ &= \frac{1}{x_1 - \|\mathbf{x}_2\|\cot\theta}\frac{1}{2}\begin{bmatrix} 1 \\ -\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta \end{bmatrix} + \frac{1}{x_1 + \|\mathbf{x}_2\|\cot\theta}\frac{1}{2}\begin{bmatrix} 1 \\ \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta \end{bmatrix} \\ &= \frac{1}{x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta}(x_1, -\mathbf{x}_2) \succ_{\mathcal{L}_\theta} \mathbf{0}. \end{aligned}$$

(c) Applying part(b) yields

$$\begin{aligned} (r\mathbf{x})^{-1} &= \left(\begin{bmatrix} rx_1 \\ r\mathbf{x}_2 \end{bmatrix}\right)^{-1} \\ &= \frac{1}{r^2x_1^2 - r^2\|\mathbf{x}_2\|^2\cot^2\theta}(rx_1, -r\mathbf{x}_2) \\ &= \frac{1}{r}\mathbf{x}^{-1}, \end{aligned}$$

which shows the desired result.

(d) It is an immediate consequence of part(b) and part(c). □

2.1 First relation: $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$

Now, we aim to show the first relation,

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}.$$

To this end, we need the following basic property regrading absolute value $|\mathbf{x}|$.

Proposition 2.2. *For any* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $|\mathbf{x}| \in \mathcal{L}_\theta$.

Proof. For $\mathbf{x}_2 = \mathbf{0}$, it is trivial. Assume $\mathbf{x}_2 \neq \mathbf{0}$ now. For convenience, we denote $\rho_1 := |\lambda_1(\mathbf{x})| \geq 0$ and $\rho_2 := |\lambda_2(\mathbf{x})| \geq 0$. Then, we have

$$\begin{aligned} |\mathbf{x}| &= \frac{\rho_1}{2} \begin{bmatrix} 1 \\ -\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \tan\theta \end{bmatrix} + \frac{\rho_2}{2} \begin{bmatrix} 1 \\ \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \tan\theta \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \rho_1 + \rho_2 \\ (\rho_2 - \rho_1) \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \tan\theta \end{bmatrix} \succeq_{\mathcal{L}_\theta} \mathbf{0}. \end{aligned}$$

Thus, the proof is complete. □

Proposition 2.3. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ *and* $\mathbf{x}, \mathbf{y} \in \mathcal{L}_\theta$. *Then, the following hold.*

(a) $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y})$,

(b) $A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$.

Proof. (a) It is easy to see that

$$\mathbf{x} \vee \mathbf{y} - A(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|) - \frac{\mathbf{x} + \mathbf{y}}{2} = \frac{|\mathbf{x} - \mathbf{y}|}{2}.$$

From Proposition 2.2, we have $|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} \mathbf{0}$, which proves $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y})$.

(b) Similarly, there has

$$A(\mathbf{x}, \mathbf{y}) - \mathbf{x} \wedge \mathbf{y} = \frac{\mathbf{x} + \mathbf{y}}{2} - \frac{1}{2}(\mathbf{x} + \mathbf{y} - |\mathbf{x} - \mathbf{y}|) = \frac{|\mathbf{x} - \mathbf{y}|}{2} \succeq_{\mathcal{L}_\theta} \mathbf{0}.$$

Then, the proof is complete. □

2.2 Second relation: $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$

The second relation we want to claim is

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}.$$

Likewise, we need the following properties regarding the absolute value $|\mathbf{x}|$.

Proposition 2.4. *For any* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, *we have*

(a) $|\mathbf{x}| \succeq_{\mathcal{L}_\theta} \mathbf{x}$,

(b) $|\mathbf{x}| \succeq_{\mathcal{L}_\theta} -\mathbf{x}$.

Proof. (a) Again, it is easy to verify the case for $\mathbf{x}_2 = \mathbf{0}$. Assume $\mathbf{x}_2 \neq \mathbf{0}$ now. Let $\rho_1 := |\lambda_1(\mathbf{x})| - \lambda_1(\mathbf{x}) \geq 0$ and $\rho_2 := |\lambda_2(\mathbf{x})| - \lambda_2(\mathbf{x}) \geq 0$. Then, we have

$$\begin{aligned} |\mathbf{x}| - \mathbf{x} &= \frac{\rho_1}{2}\begin{bmatrix} 1 \\ -\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta \end{bmatrix} + \frac{\rho_2}{2}\begin{bmatrix} 1 \\ \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \rho_1 + \rho_2 \\ (\rho_2 - \rho_1)\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta \end{bmatrix} \succeq_{\mathcal{L}_\theta} \mathbf{0}, \end{aligned}$$

which shows the proof.

(b) It is trivial for $\mathbf{x}_2 = \mathbf{0}$. Assume $\mathbf{x}_2 \neq \mathbf{0}$ now. Similarly, we denote $\rho_1 := |\lambda_1(\mathbf{x})| + \lambda_1(\mathbf{x}) \geq 0$ and $\rho_2 := |\lambda_2(\mathbf{x})| + \lambda_2(\mathbf{x}) \geq 0$. Then, we have

$$\begin{aligned} |\mathbf{x}| + \mathbf{x} &= \frac{\rho_1}{2}\begin{bmatrix} 1 \\ -\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta \end{bmatrix} + \rho_2\frac{1}{2}\begin{bmatrix} 1 \\ \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \rho_1 + \rho_2 \\ (\rho_2 - \rho_1)\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\tan\theta \end{bmatrix} \succeq_{\mathcal{L}_\theta} \mathbf{0}. \end{aligned}$$

Thus, the proof is complete. $\square$

In order to link the inequalities, we still need the concept of circular cone monotonicity.

Proposition 2.5. *Suppose that $f : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ is given by $f(t) = -t^{-1}$. Then, f is $\mathcal{L}_\theta$-monotone.*

Proof. It suffices to show that $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$ implies $\mathbf{y}^{-1} \succ_{\mathcal{L}_\theta} \mathbf{x}^{-1}$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{L}_\theta$, by Lemma 2.1, we know that $\mathbf{y}^{-1} = \frac{1}{\det(\mathbf{y})}(y_1, -\mathbf{y}_2)$, $\mathbf{x}^{-1} = \frac{1}{\det(\mathbf{x})}(x_1, -\mathbf{x}_2)$, where $\det(\mathbf{x}) = x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta$ and $\det(\mathbf{y}) = y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta$. Thus, we obtain

$$\begin{aligned} \mathbf{y}^{-1} - \mathbf{x}^{-1} &= \left(\frac{y_1}{\det(\mathbf{y})} - \frac{x_1}{\det(\mathbf{x})}, \frac{\mathbf{x}_2}{\det(\mathbf{x})} - \frac{\mathbf{y}_2}{\det(\mathbf{y})}\right) \\ &= \frac{1}{\det(\mathbf{x})\det(\mathbf{y})}\Big(\det(\mathbf{x})y_1 - \det(\mathbf{y})x_1 \,,\, \det(\mathbf{y})\mathbf{x}_2 - \det(\mathbf{x})\mathbf{y}_2\Big). \end{aligned}$$

Note that $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{y}$ implies

$$x_1 - y_1 \geq \|\mathbf{x}_2 - \mathbf{y}_2\|\cot\theta \geq |\|\mathbf{x}_2\| - \|\mathbf{y}_2\||\cot\theta.$$

In view of these, to complete the proof, we need to verify two things.
First, we have to show that $\det(\mathbf{x})y_1 - \det(\mathbf{y})x_1 \geq 0$. Indeed, we compute

$$\frac{\det(\mathbf{x})}{\det(\mathbf{y})} = \frac{x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta}{y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta} = \left(\frac{x_1 + \|\mathbf{x}_2\|\cot\theta}{y_1 + \|\mathbf{y}_2\|\cot\theta}\right)\left(\frac{x_1 - \|\mathbf{x}_2\|\cot\theta}{y_1 - \|\mathbf{y}_2\|\cot\theta}\right) \geq \frac{2x_1}{2y_1} = \frac{x_1}{y_1}.$$

Here we use the inequality

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) \geq \frac{a+c}{b+d},$$

provided that $a \geq b > 0$ and $c \geq d > 0$. Then, cross multiplying yields $\det(\mathbf{x})y_1 \geq \det(\mathbf{y})x_1$, i.e., $\det(\mathbf{x})y_1 - \det(\mathbf{y})x_1 \geq 0$.

Secondly, we shall show that $\|\det(\mathbf{y})\mathbf{x}_2 - \det(\mathbf{x})\mathbf{y}_2\|\cot\theta \le \det(\mathbf{x})y_1 - \det(\mathbf{y})x_1$. To see this, we compute

$$\begin{aligned}
& \left[\det(\mathbf{x})y_1 - \det(\mathbf{y})x_1\right]^2 - \|\det(\mathbf{y})\mathbf{x}_2 - \det(\mathbf{x})\mathbf{y}_2\|^2\cot^2\theta \\
= \; & (\det(\mathbf{x}))^2 y_1^2 - 2\det(\mathbf{x})\det(\mathbf{y})x_1y_1 + (\det(\mathbf{y}))^2 x_1^2 \\
& - \left[(\det(\mathbf{y}))^2\|\mathbf{x}_2\|^2 - 2\det(\mathbf{x})\det(\mathbf{y})\langle\mathbf{x}_2,\mathbf{y}_2\rangle + (\det(\mathbf{x}))^2\|\mathbf{y}_2\|^2\right]\cot^2\theta \\
= \; & (\det(\mathbf{x}))^2(y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta) + (\det(\mathbf{y}))^2(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta) \\
& -2\det(\mathbf{x})\det(\mathbf{y})(x_1y_1 - \langle\mathbf{x}_2,\mathbf{y}_2\rangle\cot^2\theta) \\
= \; & (\det(\mathbf{x}))^2\det(\mathbf{y}) + (\det(\mathbf{y}))^2\det(\mathbf{x}) - 2\det(\mathbf{x})\det(\mathbf{y})(x_1y_1 - \langle\mathbf{x}_2,\mathbf{y}_2\rangle\cot^2\theta) \\
= \; & \det(\mathbf{x})\det(\mathbf{y})\left[\det(\mathbf{x}) + \det(\mathbf{y}) - 2x_1y_1 + 2\langle\mathbf{x}_2,\mathbf{y}_2\rangle\cot^2\theta\right] \\
= \; & \det(\mathbf{x})\det(\mathbf{y})\left[(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta) + (y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta) - 2x_1y_1 + 2\langle\mathbf{x}_2,\mathbf{y}_2\rangle\cot^2\theta\right] \\
= \; & \det(\mathbf{x})\det(\mathbf{y})\left[(x_1 - y_1)^2 - (\|\mathbf{x}_2\|^2 + \|\mathbf{y}_2\|^2 - 2\langle\mathbf{x}_2,\mathbf{y}_2\rangle)\cot^2\theta\right] \\
= \; & \det(\mathbf{x})\det(\mathbf{y})\left[(x_1 - y_1)^2 - (\|\mathbf{x}_2 - \mathbf{y}_2\|^2)\cot^2\theta\right] \\
\ge \; & 0,
\end{aligned}$$

where the last step holds by the inequality $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$, which is equivalent to $\mathbf{x}-\mathbf{y} \succeq_{\mathcal{L}_\theta} \mathbf{0}$. To sum up, from all the above, we prove $\mathbf{y}^{-1} - \mathbf{x}^{-1} \in \mathrm{int}(\mathcal{L}_\theta)$, i.e., $\mathbf{y}^{-1} \succ_{\mathcal{L}_\theta} \mathbf{x}^{-1}$. Then, the proof is complete. □

Proposition 2.6. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$, *and* $\mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$. *Then, we have*

(a) $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y})$,

(b) $H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$.

Proof. (a) Applying Proposition 2.4 gives

$$\begin{aligned}
& |\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} (\mathbf{x} - \mathbf{y}) \\
\Longrightarrow \; & \frac{1}{2}|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} \frac{1}{2}(\mathbf{x} - \mathbf{y}) \\
\Longrightarrow \; & \frac{1}{2}|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} -\frac{1}{2}(\mathbf{x} + \mathbf{y}) + \mathbf{x} \\
\Longrightarrow \; & \frac{1}{2}(\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|) \succeq_{\mathcal{L}_\theta} \mathbf{x},
\end{aligned}$$

and

$$\begin{aligned}
& |\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} -(\mathbf{x} - \mathbf{y}) \\
\Longrightarrow \; & \frac{1}{2}|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} -\frac{1}{2}(\mathbf{x} - \mathbf{y}) \\
\Longrightarrow \; & \frac{1}{2}|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} -\frac{1}{2}(\mathbf{x} + \mathbf{y}) + \mathbf{y} \\
\Longrightarrow \; & \frac{1}{2}(\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|) \succeq_{\mathcal{L}_\theta} \mathbf{y}.
\end{aligned}$$

With these, we conclude that $\frac{1}{2}(\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|) \succeq_{\mathcal{L}_\theta} \mathbf{x}$ and $\frac{1}{2}(\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|) \succeq_{\mathcal{L}_\theta} \mathbf{y}$. Then, using $\mathcal{L}_\theta$-monotonicity of $f(t) = -t^{-1}$ shown in Proposition 2.5, we obtain

$$\mathbf{x}^{-1} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|}{2}\right)^{-1} \quad \text{and} \quad \mathbf{y}^{-1} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|}{2}\right)^{-1},$$

which further imply

$$\frac{\mathbf{x}^{-1}+\mathbf{y}^{-1}}{2} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|}{2}\right)^{-1}.$$

Using $\mathcal{L}_\theta$-monotonicity of f again, it yields

$$\frac{\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|}{2} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x}^{-1}+\mathbf{y}^{-1}}{2}\right)^{-1}.$$

Thus, $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y})$ is proved.

(b) For $\frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \notin \mathcal{L}_\theta$, the inequality holds clearly. For $\frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \in \mathcal{L}_\theta$, applying Proposition 2.5 gives

$$\begin{aligned} & |\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} -(\mathbf{x}-\mathbf{y}) \\ \Longrightarrow\quad & \frac{1}{2}|\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} \frac{1}{2}(\mathbf{x}+\mathbf{y}) - \mathbf{x} \\ \Longrightarrow\quad & \frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \preceq_{\mathcal{L}_\theta} \mathbf{x}, \end{aligned}$$

and

$$\begin{aligned} & |\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} (\mathbf{x}-\mathbf{y}) \\ \Longrightarrow\quad & \frac{1}{2}|\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} \frac{1}{2}(\mathbf{x}+\mathbf{y}) - \mathbf{y} \\ \Longrightarrow\quad & \frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \preceq_{\mathcal{L}_\theta} \mathbf{y}. \end{aligned}$$

Hence, we can conclude that $\frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \preceq_{\mathcal{L}_\theta} \mathbf{x}$ and $\frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \preceq_{\mathcal{L}_\theta} \mathbf{y}$. Again, using the $\mathcal{L}_\theta$-monotonicity of $f(t) = -t^{-1}$ shown in Proposition 2.5, we obtain

$$\left(\frac{\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|}{2}\right)^{-1} \succeq_{\mathcal{L}_\theta} \mathbf{x}^{-1} \quad \text{and} \quad \left(\frac{\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|}{2}\right)^{-1} \succeq_{\mathcal{L}_\theta} \mathbf{y}^{-1},$$

which imply

$$\left(\frac{\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|}{2}\right)^{-1} \succeq_{\mathcal{L}_\theta} \frac{\mathbf{x}^{-1}+\mathbf{y}^{-1}}{2}.$$

Besides, employing $\mathcal{L}_\theta$-monotonicity of f one more time, we achieve

$$\left(\frac{\mathbf{x}^{-1}+\mathbf{y}^{-1}}{2}\right)^{-1} \succeq_{\mathcal{L}_\theta} \frac{\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|}{2}.$$

Thus, $H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$. $\square$

2.3 Third relation: $A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x},\mathbf{y})$

Next, under the first type of decomposition, we clarify the relation between $A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x},\mathbf{y})$. In fact, we will show

$$A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x},\mathbf{y}),$$

for which we need the concept of the circular cone convexity. However, showing $f(t) = -t^{-1}$ is $\mathcal{L}_\theta$-convex is a bit complicated. Instead, we only need to prove the inequality as shown in Proposition 2.7, since it is what will be employed.

Proposition 2.7. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathrm{IR} \times \mathrm{IR}^{n-1}$, $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathrm{IR} \times \mathrm{IR}^{n-1}$, $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$, *and* $\mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$. *Then, we have*

$$\frac{\mathbf{x}^{-1} + \mathbf{y}^{-1}}{2} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)^{-1}.$$

Proof. For any $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$ and $\mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$, we know that

$$\left\{\begin{array}{l} x_1 - \|\mathbf{x}_2\| \cot\theta > 0, \\ y_1 - \|\mathbf{y}_2\| \cot\theta > 0, \\ |\langle \mathbf{x}_2, \mathbf{y}_2 \rangle| \cot^2\theta \le \|\mathbf{x}_2\| \cdot \|\mathbf{y}_2\| \cot^2\theta \le x_1 y_1. \end{array}\right. \tag{2.2}$$

From $\mathbf{x}^{-1} = \frac{1}{\det(\mathbf{x})}(x_1, -\mathbf{x}_2)$ and $\mathbf{y}^{-1} = \frac{1}{\det(\mathbf{y})}(y_1, -\mathbf{y}_2)$, there have

$$\frac{1}{2}\left(\mathbf{x}^{-1} + \mathbf{y}^{-1}\right) = \frac{1}{2}\left(\frac{x_1}{\det(\mathbf{x})} + \frac{y_1}{\det(\mathbf{y})}\ ,\ -\frac{\mathbf{x}_2}{\det(\mathbf{x})} - \frac{\mathbf{y}_2}{\det(\mathbf{y})}\right)$$

and

$$\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)^{-1} = \frac{2}{\det(\mathbf{x}+\mathbf{y})}\Big(x_1 + y_1\ ,\ -(\mathbf{x}_2 + \mathbf{y}_2)\Big).$$

For notational convenience, we denote $\frac{1}{2}\left(\mathbf{x}^{-1} + \mathbf{y}^{-1}\right) - \left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)^{-1} := \frac{1}{2}(\Xi_1, \Xi_2)$, where $\Xi_1 \in \mathrm{IR}$ and $\Xi_2 \in \mathrm{IR}^{n-1}$ are given by

$$\left\{\begin{array}{lcl} \Xi_1 & = & \left(\frac{x_1}{\det(\mathbf{x})} + \frac{y_1}{\det(\mathbf{y})}\right) - \frac{4(x_1+y_1)}{\det(\mathbf{x}+\mathbf{y})}, \\ \Xi_2 & = & \frac{4(\mathbf{x}_2+\mathbf{y}_2)}{\det(\mathbf{x}+\mathbf{y})} - \left(\frac{\mathbf{x}_2}{\det(\mathbf{x})} + \frac{\mathbf{y}_2}{\det(\mathbf{y})}\right). \end{array}\right.$$

Now it suffices to verify two things : $\Xi_1 \ge 0$ and $\|\Xi_2\| \cot\theta \le \Xi_1$.

First, we verify that $\Xi_1 \ge 0$. In fact, by defining the function

$$g(\mathbf{x}) := \frac{x_1}{x_1^2 - \|\mathbf{x}_2\|^2 \cot^2\theta} = \frac{x_1}{\det(\mathbf{x})},$$

then we observe that

$$g\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \le \frac{1}{2}\Big(g(\mathbf{x}) + g(\mathbf{y})\Big) \quad \Longleftrightarrow \quad \Xi_1 \ge 0.$$

Hence, to prove $\Xi_1 \geq 0$, it is sufficient to show g is convex on $\text{int}(\mathcal{L}_\theta)$. Since $\text{int}(\mathcal{L}_\theta)$ is a convex set, it is equivalent to verifying that $\nabla^2 g(\mathbf{x})$ is a positive semidefinite matrix. From direct computations, we have

$$\nabla^2 g(\mathbf{x}) = \frac{1}{(x_1^2 - \|\mathbf{x}_2\|^2 \cot^2\theta)^3} \left[\begin{array}{cc} A & B \\ B^T & C \end{array} \right],$$

where

$$\left[\begin{array}{cc} A & B \\ B^T & C \end{array} \right] = \left[\begin{array}{cc} 2x_1^3 + 6x_1\|\mathbf{x}_2\|^2 \cot^2\theta & -(6x_1^2 + 2\|\mathbf{x}_2\|^2\cot^2\theta)\cot\theta \mathbf{x}_2^T \\ -(6x_1^2 + 2\|\mathbf{x}_2\|^2\cot^2\theta)\cot\theta \mathbf{x}_2 & 2x_1\Big((x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta)I + 4\cot^2\theta \mathbf{x}_2\mathbf{x}_2^T\Big) \end{array} \right].$$

Obviously, A is a positive scalar. By (2.2) and the Schur Complement Theorem (see [7, Theorem 7.7.6]), it suffices to claim that $AC - B^T B$ is positive semidefinite. To this end, we compute

$$\begin{aligned}
& AC - B^T B \\
= \;& 2x_1\Big(2x_1^3 + 6x_1\|\mathbf{x}_2\|^2\cot^2\theta\Big)\Big((x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta)I + 4\cot^2\theta\mathbf{x}_2\mathbf{x}_2^T\Big) \\
& -\cot^2\theta\Big(6x_1^2 + 2\|\mathbf{x}_2\|^2\cot^2\theta\Big)^2 \mathbf{x}_2\mathbf{x}_2^T \\
= \;& \Big(4x_1^4 + 12x_1^2\|\mathbf{x}_2\|^2\cot^2\theta\Big)\Big(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta\Big)I \\
& -\cot^2\theta\Big(20x_1^4 - 24x_1^2\|\mathbf{x}_2\|^2\cot^2\theta + 4\|\mathbf{x}_2\|^4\cot^4\theta\Big)\mathbf{x}_2\mathbf{x}_2^T \\
= \;& \Big(4x_1^4 + 12x_1^2\|\mathbf{x}_2\|^2\cot^2\theta\Big)\Big(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta\Big)I \\
& -4\cot^2\theta\Big(5x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta\Big)\Big(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta\Big)\mathbf{x}_2\mathbf{x}_2^T \\
= \;& \Big(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta\Big)\Big[\Big(4x_1^4 + 12x_1^2\|\mathbf{x}_2\|^2\cot^2\theta\Big)I - 4\cot^2\theta\Big(5x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta\Big)\mathbf{x}_2\mathbf{x}_2^T\Big] \\
= \;& \Big(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta\Big)\cdot M.
\end{aligned}$$

We know that $\mathbf{x}_2\mathbf{x}_2^T$ is positive semidefinite with only nonzero eigenvalue $\|\mathbf{x}_2\|^2$. Hence, all the eigenvalues of the matrix M are $4x_1^4 + 12x_1^2\|\mathbf{x}_2\|^2\cot^2\theta - 20x_1^2\|\mathbf{x}_2\|^2\cot^2\theta + 4\|\mathbf{x}_2\|^4\cot^4\theta$ with multiplicity of 1 and $4x_1^4 + 12x_1^2\|\mathbf{x}_2\|^2\cot^2\theta$ with multiplicity of $n-2$, which are all positive because

$$\begin{aligned}
& 4x_1^4 + 12x_1^2\|\mathbf{x}_2\|^2\cot^2\theta - 20x_1^2\|\mathbf{x}_2\|^2\cot^2\theta + 4\|\mathbf{x}_2\|^4\cot^4\theta \\
= \;& 4x_1^4 - 8x_1^2\|\mathbf{x}_2\|^2\cot^2\theta + 4\|\mathbf{x}_2\|^4\cot^4\theta \\
= \;& 4\Big(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta\Big)^2 \\
> \;& 0.
\end{aligned}$$

Thus, we conclude that $\nabla^2 g(\mathbf{x})$ is positive semidefinite. Then, it follows that g is convex on $\text{int}(\mathcal{L}_\theta)$, which says $\Xi_1 \geq 0$.

It remains to verify that $\Xi_1^2 - \|\Xi_2\|^2 \cot^2\theta \geq 0$.

$$
\begin{aligned}
& \Xi_1^2 - \|\Xi_2\|^2 \cot^2\theta \\
= & \left[\left(\frac{x_1^2}{\det(\mathbf{x})^2} + \frac{2x_1y_1}{\det(\mathbf{x})\det(\mathbf{y})} + \frac{y_1^2}{\det(\mathbf{y})^2}\right) - \frac{8(x_1+y_1)}{\det(\mathbf{x}+\mathbf{y})}\left(\frac{x_1}{\det(\mathbf{x})} + \frac{y_1}{\det(\mathbf{y})}\right)\right. \\
& \left. + \frac{16}{\det(\mathbf{x}+\mathbf{y})^2}\left(x_1^2 + 2x_1y_1 + y_1^2\right)\right] - \left\|\frac{4(\mathbf{x}_2+\mathbf{y}_2)}{\det(\mathbf{x}+\mathbf{y})} - \left(\frac{\mathbf{x}_2}{\det(\mathbf{x})} + \frac{\mathbf{y}_2}{\det(\mathbf{y})}\right)\right\|^2 \cot^2\theta \\
= & \left[\left(\frac{x_1^2}{\det(\mathbf{x})^2} + \frac{2x_1y_1}{\det(\mathbf{x})\det(\mathbf{y})} + \frac{y_1^2}{\det(\mathbf{y})^2}\right) - \frac{8(x_1+y_1)}{\det(\mathbf{x}+\mathbf{y})}\left(\frac{x_1}{\det(\mathbf{x})} + \frac{y_1}{\det(\mathbf{y})}\right)\right. \\
& \left. + \frac{16}{\det(\mathbf{x}+\mathbf{y})^2}\left(x_1^2 + 2x_1y_1 + y_1^2\right)\right] - \left[\frac{16\cot^2\theta}{\det(\mathbf{x}+\mathbf{y})^2}\left(\|\mathbf{x}_2\|^2 + 2\langle \mathbf{x}_2, \mathbf{y}_2\rangle + \|\mathbf{y}_2\|^2\right)\right. \\
& \left. - 8\cot^2\theta \left\langle \frac{\mathbf{x}_2+\mathbf{y}_2}{\det(\mathbf{x}+\mathbf{y})}\, ,\, \frac{\mathbf{x}_2}{\det(\mathbf{x})} + \frac{\mathbf{y}_2}{\det(\mathbf{y})}\right\rangle + \cot^2\theta\left(\frac{\|\mathbf{x}_2\|^2}{\det(\mathbf{x})^2} + \frac{2\langle \mathbf{x}_2, \mathbf{y}_2\rangle}{\det(\mathbf{x})\det(\mathbf{y})} + \frac{\|\mathbf{y}_2\|^2}{\det(\mathbf{y})^2}\right)\right] \\
= & \left[\frac{x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta}{\det(\mathbf{x})^2} + \frac{2(x_1y_1 - \cot^2\theta\langle \mathbf{x}_2, \mathbf{y}_2\rangle)}{\det(\mathbf{x})\det(\mathbf{y})} + \frac{y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta}{\det(\mathbf{y})^2}\right] \\
& + \frac{16}{\det(\mathbf{x}+\mathbf{y})^2}\left[(x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta) + 2(x_1y_1 - \cot^2\theta\langle \mathbf{x}_2, \mathbf{y}_2\rangle) + (y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta)\right] \\
& - 8\left[\frac{x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta}{\det(\mathbf{x}+\mathbf{y})\det(\mathbf{x})} + \frac{x_1y_1 - \cot^2\theta\langle \mathbf{x}_2, \mathbf{y}_2\rangle}{\det(\mathbf{x}+\mathbf{y})\det(\mathbf{x})} + \frac{x_1y_1 - \cot^2\theta\langle \mathbf{x}_2, \mathbf{y}_2\rangle}{\det(\mathbf{x}+\mathbf{y})\det(\mathbf{y})} + \frac{y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta}{\det(\mathbf{x}+\mathbf{y})\det(\mathbf{y})}\right] \\
= & (x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta)\left(\frac{1}{\det(\mathbf{x})^2} + \frac{16}{\det(\mathbf{x}+\mathbf{y})^2} - \frac{8}{\det(\mathbf{x}+\mathbf{y})\det(\mathbf{x})}\right) \\
& + (y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta)\left(\frac{1}{\det(\mathbf{y})^2} + \frac{16}{\det(\mathbf{x}+\mathbf{y})^2} - \frac{8}{\det(\mathbf{x}+\mathbf{y})\det(\mathbf{y})}\right) \\
& + 2(x_1y_1 - \cot^2\theta\langle \mathbf{x}_2, \mathbf{y}_2\rangle)\left(\frac{1}{\det(\mathbf{x})\det(\mathbf{y})} + \frac{16}{\det(\mathbf{x}+\mathbf{y})^2} - \frac{4(\det(\mathbf{x}) + \det(\mathbf{y}))}{\det(\mathbf{x}+\mathbf{y})}\right) \\
= & (x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta)\left(\frac{\det(\mathbf{x}+\mathbf{y}) - 4\det(\mathbf{x})}{\det(\mathbf{x})\det(\mathbf{x}+\mathbf{y})}\right)^2 + (y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta)\left(\frac{\det(\mathbf{x}+\mathbf{y}) - 4\det(\mathbf{y})}{\det(\mathbf{y})\det(\mathbf{x}+\mathbf{y})}\right)^2 \\
& + 2(x_1y_1 - \cot^2\theta\langle \mathbf{x}_2, \mathbf{y}_2\rangle)\left(\frac{(\det(\mathbf{x}+\mathbf{y}) - 4\det(\mathbf{x}))(\det(\mathbf{x}+\mathbf{y}) - 4\det(\mathbf{y}))}{\det(\mathbf{x})\det(\mathbf{y})\det(\mathbf{x}+\mathbf{y})^2}\right).
\end{aligned}
$$

Now applying the fact that $\det(\mathbf{x}) = x_1^2 - \|\mathbf{x}_2\|^2\cot^2\theta$, $\det(\mathbf{y}) = y_1^2 - \|\mathbf{y}_2\|^2\cot^2\theta$ and $\det(\mathbf{x}+\mathbf{y}) - \det(\mathbf{x}) - \det(\mathbf{y}) = 2(x_1y_1 - \cot^2\theta\langle \mathbf{x}_2, \mathbf{y}_2\rangle)$ are all nonnegative by (2.2), we can simplify the last term (after a lot of algebra simplifications) and achieve

$$
\Xi_1^2 - \|\Xi_2\|^2\cot^2\theta = \frac{\Big[\det(\mathbf{x}+\mathbf{y}) - 2\det(\mathbf{x}) - 2\det(\mathbf{y})\Big]^2}{\det(\mathbf{x})\det(\mathbf{y})\det(\mathbf{x}+\mathbf{y})} \geq 0.
$$

Hence,

$$
\frac{\mathbf{x}^{-1} + \mathbf{y}^{-1}}{2} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)^{-1}.
$$

Then, the proof is complete. □

Proposition 2.8. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$*,* $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$*,* $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$*, and* $\mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$*. Then, there holds*

$$A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}).$$

Proof. From Proposition 2.7, we have

$$\frac{\mathbf{x}^{-1} + \mathbf{y}^{-1}}{2} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x} + \mathbf{y}}{2} \right)^{-1}.$$

Then, applying Proposition 2.5 and Lemma 2.1(d), it leads to $A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y})$. □

To sum up, from all the aforementioned three relations, we conclude that $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$ which is stated in Theorem 2.9.

Theorem 2.9. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$*,* $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$*,* $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$*, and* $\mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$*. Then, we have*

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}.$$

3 Second type of decomposition

This section is devoted to presenting another type of decomposition for the circular cone $\mathcal{L}_\theta$. Indeed, it is a traditional decomposition already studied in [9]. For any $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{x}$ is decomposed as

$$\mathbf{x} = \lambda_1(\mathbf{x}) \mathbf{u}_{\mathbf{x}}^{(1)} + \lambda_2(\mathbf{x}) \mathbf{u}_{\mathbf{x}}^{(2)}, \tag{3.1}$$

where the spectral values of $\mathbf{x}$ are defined as

$$\begin{aligned} \lambda_1(\mathbf{x}) &= x_1 - \|\mathbf{x}_2\| \cot\theta, \\ \lambda_2(\mathbf{x}) &= x_1 + \|\mathbf{x}_2\| \tan\theta, \end{aligned}$$

with the spectral vectors of $\mathbf{x}$ respectively given by

$$\mathbf{u}_{\mathbf{x}}^{(1)} = \begin{cases} \left(\sin^2\theta \ , \ -(\sin\theta\cos\theta) \dfrac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \right), & \text{if} \quad \mathbf{x}_2 \neq \mathbf{0}, \\ \left(\sin^2\theta \ , \ -(\sin\theta\cos\theta)\mathbf{w} \right), & \text{if} \quad \mathbf{x}_2 = \mathbf{0}. \end{cases}$$

$$\mathbf{u}_{\mathbf{x}}^{(2)} = \begin{cases} \left(\cos^2\theta \ , \ (\sin\theta\cos\theta) \dfrac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \right), & \text{if} \quad \mathbf{x}_2 \neq \mathbf{0}, \\ \left(\cos^2\theta \ , \ (\sin\theta\cos\theta)\mathbf{w} \right), & \text{if} \quad \mathbf{x}_2 = \mathbf{0}. \end{cases}$$

The above $\mathbf{w}$ could be any vector in $\mathbb{R}^{n-1}$ satisfying $\|\mathbf{w}\| = 1$. If $\mathbf{x}_2 \neq \mathbf{0}$, the decomposition is unique. When $\theta = \frac{\pi}{4}$, this decomposition coincides with those (1.3) and (1.4) in SOC setting. Note that it is true that $\lambda_2(\mathbf{x}) \geq \lambda_1(\mathbf{x})$, and

$$\lambda_1(\mathbf{x}) \geq 0 \iff \mathbf{x} \in \mathcal{L}_\theta \iff \mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}; \quad \lambda_1(\mathbf{x}) > 0 \iff \mathbf{x} \in \operatorname{int}(\mathcal{L}_\theta) \iff \mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}.$$

To proceed, we examine some basic properties of the absolute value $|\mathbf{x}|$ and the inverse $\mathbf{x}^{-1}$ based on the second type of decomposition.

Proposition 3.1. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

(a) *If* $\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}$, *then* $\mathbf{x} = |\mathbf{x}|$.

(b) *If* $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$ *and* $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, *then* $\mathbf{x}^{-1} \succ_{\mathcal{L}_\theta} \mathbf{0}$.

(c) *Let* $r \in \mathbb{R}$ *and* $r \neq 0$, *then* $(r\mathbf{x})^{-1} = \frac{1}{r}\mathbf{x}^{-1}$.

Proof. (a) From decomposition (3.1), we have $|\mathbf{x}| = |\lambda_1(\mathbf{x})|\mathbf{u}_{\mathbf{x}}^{(1)} + |\lambda_2(\mathbf{x})|\mathbf{u}_{\mathbf{x}}^{(2)}$. Since $\mathbf{x} \in \mathcal{L}_\theta$, $\lambda_1(\mathbf{x}) = x_1 - \|\mathbf{x}_2\| \cot\theta \geq 0$, and $\lambda_2(\mathbf{x}) = x_1 + \|\mathbf{x}_2\| \tan\theta \geq \lambda_1(\mathbf{x}) \geq 0$, we have $|\lambda_1(\mathbf{x})| = \lambda_1(\mathbf{x})$, $|\lambda_2(\mathbf{x})| = \lambda_2(\mathbf{x})$, which proves $\mathbf{x} = |\mathbf{x}|$.

(b) It is easy to check the case of $\mathbf{x}_2 = \mathbf{0}$. Assume $\mathbf{x}_2 \neq \mathbf{0}$ now. Then, we have

$$\begin{aligned}
\mathbf{x}^{-1} &= \lambda_1(\mathbf{x})^{-1}\mathbf{u}_{\mathbf{x}}^{(1)} + \lambda_2(\mathbf{x})^{-1}\mathbf{u}_{\mathbf{x}}^{(2)} \\
&= \frac{1}{x_1 - \|\mathbf{x}_2\|\cot\theta}\begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \end{bmatrix} + \frac{1}{x_1 + \|\mathbf{x}_2\|\tan\theta}\begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sin^2\theta}{x_1 - \|\mathbf{x}_2\|\cot\theta} + \frac{\cos^2\theta}{x_1 + \|\mathbf{x}_2\|\tan\theta} \\ \frac{-(\sin\theta\cos\theta)\mathbf{x}_2}{(x_1 - \|\mathbf{x}_2\|\cot\theta)\|\mathbf{x}_2\|} + \frac{(\sin\theta\cos\theta)\mathbf{x}_2}{(x_1 + \|\mathbf{x}_2\|\tan\theta)\|\mathbf{x}_2\|} \end{bmatrix} \\
&= \frac{1}{(x_1 - \|\mathbf{x}_2\|\cot\theta)(x_1 + \|\mathbf{x}_2\|\tan\theta)}\begin{bmatrix} x_1 + (\frac{\sin^3\theta}{\cos\theta} - \frac{\cos^3\theta}{\sin\theta})\|\mathbf{x}_2\| \\ \sin\theta\cos\theta(-\mathbf{x}_2(\tan\theta + \cot\theta)) \end{bmatrix} \\
&= \frac{1}{(x_1 - \|\mathbf{x}_2\|\cot\theta)(x_1 + \|\mathbf{x}_2\|\tan\theta)}\begin{bmatrix} x_1 + (\frac{\sin^4\theta - \cos^4\theta}{\cos\theta\sin\theta})\|\mathbf{x}_2\| \\ -\mathbf{x}_2 \end{bmatrix}.
\end{aligned}$$

Since $(x_1 - \|\mathbf{x}_2\|\cot\theta)(x_1 + \|\mathbf{x}_2\|\tan\theta) > 0$, it suffices to show $x_1 + (\frac{\sin^4\theta - \cos^4\theta}{\cos\theta\sin\theta})\|\mathbf{x}_2\| > \cot\theta\|\mathbf{x}_2\|$. To see this, using $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0} \implies x_1 > \|\mathbf{x}_2\|\cot\theta$, we have

$$\begin{aligned}
& x_1 + (\frac{\sin^4\theta - \cos^4\theta}{\cos\theta\sin\theta})\|\mathbf{x}_2\| \\
> \;& \cot\theta\|\mathbf{x}_2\| + \left(\frac{\sin^4\theta - \cos^4\theta}{\cos\theta\sin\theta}\right)\|\mathbf{x}_2\| \\
= \;& \left[\cot\theta + \left(\frac{\sin^4\theta - \cos^4\theta}{\cos\theta\sin\theta}\right)\right]\|\mathbf{x}_2\|.
\end{aligned}$$

Note that $\theta \in (\frac{\pi}{4}, \frac{\pi}{2}) \implies \sin\theta > \cos\theta$, it yields

$$\left[\cot\theta + \left(\frac{\sin^4\theta - \cos^4\theta}{\cos\theta\sin\theta}\right)\right]\|\mathbf{x}_2\| \geq \cot\theta\|\mathbf{x}_2\|,$$

which is equivalent to saying $\mathbf{x}^{-1} \succ_{\mathcal{L}_\theta} \mathbf{0}$.

(c) Again, the case of $\mathbf{x}_2 = \mathbf{0}$ is trivial. Assume $\mathbf{x}_2 \neq \mathbf{0}$ now. Then, we have

$$\begin{aligned}
(r\mathbf{x})^{-1} &= \left(\begin{bmatrix} rx_1 \\ r\mathbf{x}_2 \end{bmatrix}\right)^{-1} \\
&= \frac{1}{rx_1 - r\|\mathbf{x}_2\|\cot\theta}\begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)\frac{r\mathbf{x}_2}{r\|\mathbf{x}_2\|} \end{bmatrix} + \frac{1}{rx_1 + r\|\mathbf{x}_2\|\tan\theta}\begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)\frac{r\mathbf{x}_2}{r\|\mathbf{x}_2\|} \end{bmatrix} \\
&= \left(\frac{1}{r}\right)\left(\frac{1}{x_1 - \|\mathbf{x}_2\|\cot\theta}\begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \end{bmatrix} + \frac{1}{x_1 + \|\mathbf{x}_2\|\tan\theta}\begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \end{bmatrix}\right) \\
&= \frac{1}{r}\mathbf{x}^{-1}.
\end{aligned}$$

which is the desired result. $\square$

In view of the above arguments, it can be concluded that when an element $\mathbf{x}$ falls in the circular cone $\mathcal{L}_\theta$, its absolute value $|\mathbf{x}|$ and inverse $\mathbf{x}^{-1}$ will also fall in the circular cone under the condition $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$. Moreover, when $\theta < \frac{\pi}{4}$, it is **no** longer true. This indicates that the second type of decomposition is very different from the first type one. The differences between decomposition (2.1) and decomposition (3.1) will be elaborated in the final section.

3.1 The relation $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$ does not hold under $\theta \in (0, \frac{\pi}{4})$

Under the second type of decomposition, when $\theta \in (0, \frac{\pi}{4})$, the inequalities

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$$

do not hold in general. Here are counterexamples.

Example 3.2. Consider $\theta = \frac{\pi}{6}$, $\mathbf{x} = (1.6, 0.3, -0.1)$, $\mathbf{y} = (1.7, 0, 0.5)$. Then, $A(\mathbf{x}, \mathbf{y}) = (1.65, 0.15, 0.2)$ and $\mathbf{x} \vee \mathbf{y} = (1.91, 0.06, 0.39)$, which says

$$\mathbf{x} \vee \mathbf{y} - A(\mathbf{x}, \mathbf{y}) = (0.27, -0.09, 0.19) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}.$$

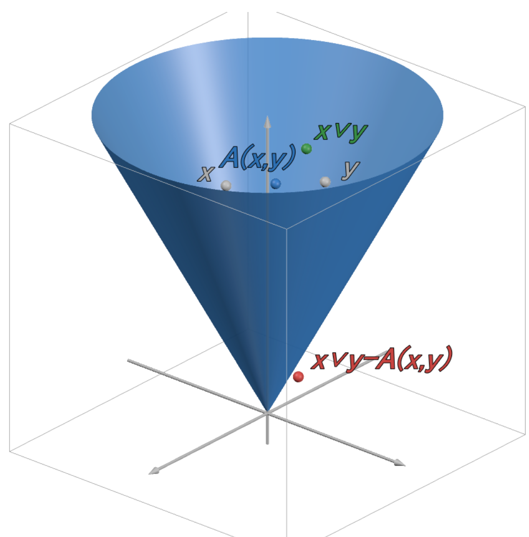

Figure 3: $\mathbf{x} \vee \mathbf{y} - A(\mathbf{x}, \mathbf{y}) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}$ with $\theta = \frac{\pi}{6}$.

Example 3.3. Consider $\theta = \frac{\pi}{6}$, $\mathbf{x} = (1.6, 0.3, -0.1)$, $\mathbf{y} = (1.7, 0, 0.5)$. Then, $A(\mathbf{x}, \mathbf{y}) = (1.65, 0.15, 0.2)$ and $\mathbf{x} \wedge \mathbf{y} = (1.38, 0.24, 0.01)$, which says

$$A(\mathbf{x}, \mathbf{y}) - \mathbf{x} \wedge \mathbf{y} = (0.27, -0.09, 0.19) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}.$$

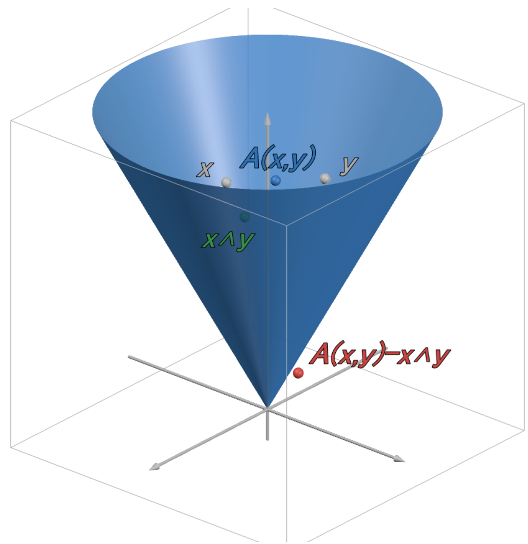

Figure 4: $A(\mathbf{x}, \mathbf{y}) - \mathbf{x} \wedge \mathbf{y} \not\succeq_{\mathcal{L}_\theta} \mathbf{0}$ with $\theta = \frac{\pi}{6}$.

In order to conquer the hurdle and build up new inequalities, we need the following inequality. In fact, it does not hold under $\theta \in (0, \frac{\pi}{4})$ in general.

Proposition 3.4. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. *Then, we have* $|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} \mathbf{0}$ *for* $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

Proof. It is easy to verify the case of $\mathbf{x}_2 = \mathbf{y}_2$. Assume $\mathbf{x}_2 \neq \mathbf{y}_2$ now. With this, we know $\lambda_1(\mathbf{x} - \mathbf{y}) = (x_1 - y_1) - \|\mathbf{x}_2 - \mathbf{y}_2\| \cot\theta$ and $\lambda_2(\mathbf{x} - \mathbf{y}) = (x_1 - y_1) + \|\mathbf{x}_2 - \mathbf{y}_2\| \tan\theta$. To proceed, we divide the arguments into three cases.

Case 1. $\lambda_1(\mathbf{x} - \mathbf{y}) \geq 0$, and $\lambda_2(\mathbf{x} - \mathbf{y}) \geq 0$.
If $\lambda_1(\mathbf{x} - \mathbf{y}) \geq 0$, $\lambda_2(\mathbf{x} - \mathbf{y}) \geq 0$, then $|\mathbf{x} - \mathbf{y}| = \mathbf{x} - \mathbf{y}$ which clearly says $|\mathbf{x} - \mathbf{y}| = \mathbf{x} - \mathbf{y} \in \mathcal{L}_\theta$.

Case 2. $\lambda_1(\mathbf{x} - \mathbf{y}) \leq 0$, and $\lambda_2(\mathbf{x} - \mathbf{y}) \leq 0$.
If $\lambda_1(\mathbf{x} - \mathbf{y}) \leq 0$ and $\lambda_2(\mathbf{x} - \mathbf{y}) \leq 0$, we have

$$
\begin{aligned}
|\mathbf{x} - \mathbf{y}| &= (y_1 - x_1 + \|\mathbf{x}_2 - \mathbf{y}_2\| \cot\theta) \begin{bmatrix} \sin^2\theta \\ -\sin\theta\cos\theta \frac{\mathbf{x}_2 - \mathbf{y}_2}{\|\mathbf{x}_2 - \mathbf{y}_2\|} \end{bmatrix} \\
&\quad + (y_1 - x_1 - \|\mathbf{x}_2 - \mathbf{y}_2\| \tan\theta) \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \frac{\mathbf{x}_2 - \mathbf{y}_2}{\|\mathbf{x}_2 - \mathbf{y}_2\|} \end{bmatrix} \\
&= \begin{bmatrix} y_1 - x_1 \\ \mathbf{y}_2 - \mathbf{x}_2 \end{bmatrix}.
\end{aligned}
$$

For $\theta > \frac{\pi}{4}$, the below implication is true:

$$
\lambda_2(\mathbf{x} - \mathbf{y}) \leq 0 \quad \Longrightarrow \quad y_1 - x_1 \geq \|\mathbf{y}_2 - \mathbf{x}_2\| \tan\theta \geq \|\mathbf{y}_2 - \mathbf{x}_2\| \cot\theta.
$$

Hence, it says that $|\mathbf{x} - \mathbf{y}| = \mathbf{y} - \mathbf{x} \in \mathcal{L}_\theta$.

Case 3. $\lambda_1(\mathbf{x}-\mathbf{y}) \le 0$, and $\lambda_2(\mathbf{x}-\mathbf{y}) \ge 0$.
For $\lambda_1(\mathbf{x}-\mathbf{y}) \le 0$, $\lambda_2(\mathbf{x}-\mathbf{y}) \ge 0$, letting $A = -\lambda_1(\mathbf{x}-\mathbf{y})$ and $B = \lambda_2(\mathbf{x}-\mathbf{y})$ yields

$$\begin{aligned} |\mathbf{x}-\mathbf{y}| &= A\begin{bmatrix} \sin^2\theta \\ -\sin\theta\cos\theta\frac{\mathbf{x}_2-\mathbf{y}_2}{\|\mathbf{x}_2-\mathbf{y}_2\|}\end{bmatrix} + B\begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta\frac{\mathbf{x}_2-\mathbf{y}_2}{\|\mathbf{x}_2-\mathbf{y}_2\|}\end{bmatrix} \\ &= \begin{bmatrix} A\sin^2\theta + B\cos^2\theta \\ (-A+B)\sin\theta\cos\theta\frac{\mathbf{x}_2-\mathbf{y}_2}{\|\mathbf{x}_2-\mathbf{y}_2\|}\end{bmatrix}. \end{aligned}$$

It is obvious that

$$A\sin^2\theta + B\cos^2\theta \ge \|(-A+B)\sin\theta\cos\theta\frac{\mathbf{x}_2-\mathbf{y}_2}{\|\mathbf{x}_2-\mathbf{y}_2\|}\|\cot\theta,$$

since $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$. Hence, it is clear that $|\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} \mathbf{0}$ holds.

Notice that $\lambda_2(\mathbf{x}-\mathbf{y}) \ge \lambda_1(\mathbf{x}-\mathbf{y})$ and based on the above three cases, we conclude that $|\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} \mathbf{0}$ for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$. □

Next, we establish the inequalities:

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}, \quad \text{for} \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

Proposition 3.5. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2), \mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, *and* $\mathbf{x}, \mathbf{y} \in \mathcal{L}_\theta$. *Then, we have*

(a) $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x},\mathbf{y})$ *for* $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$,

(b) $A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$ *for* $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

Proof. (a) It is clear to see

$$\mathbf{x} \vee \mathbf{y} - A(\mathbf{x},\mathbf{y}) = \frac{1}{2}(\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|) - \frac{\mathbf{x}+\mathbf{y}}{2} = \frac{|\mathbf{x}-\mathbf{y}|}{2}.$$

By Proposition 3.4, we know $|\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} \mathbf{0}$ for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$. Hence, $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x},\mathbf{y})$ for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.
(b) Similarly, there holds

$$A(\mathbf{x},\mathbf{y}) - \mathbf{x} \wedge \mathbf{y} = \frac{\mathbf{x}+\mathbf{y}}{2} - \frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) = \frac{|\mathbf{x}-\mathbf{y}|}{2}.$$

Then, the proof is trivial with the same arguments. □

3.2 The relation $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$ does not hold under $\theta \in (0, \frac{\pi}{4})$

Likewise, under the second type of decomposition, the inequality $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x},\mathbf{y})$ does not hold in the circular cone setting when $\theta \in (0, \frac{\pi}{4})$. Here is a counterexample.

Example 3.6. Consider $\theta = \frac{\pi}{6}$, $\mathbf{x} = (1.6, 0.3, -0.1)$, $\mathbf{y} = (1.7, 0, 0.5)$. Then, $H(\mathbf{x}, \mathbf{y}) = (1.6, 0.26, 0.4)$ and $\mathbf{x} \vee \mathbf{y} = (1.91, 0.06, 0.39)$, which says

$$\mathbf{x} \vee \mathbf{y} - H(\mathbf{x}, \mathbf{y}) = (0.31, -0.2, -0.01) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}.$$

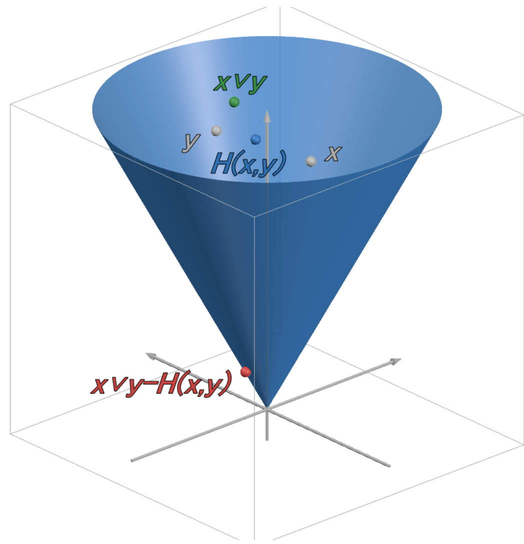

Figure 5: $\mathbf{x} \vee \mathbf{y} - H(\mathbf{x}, \mathbf{y}) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}$ with $\theta = \frac{\pi}{6}$.

In order to fix the problem, we need to propose the following more delicate property regarding the absolute value $|\mathbf{x}|$.

Proposition 3.7. *For any* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, *we have*

(a) $|\mathbf{x}| \succeq_{\mathcal{L}_\theta} \mathbf{x}$ *for* $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$;

(b) $|\mathbf{x}| \succeq_{\mathcal{L}_\theta} -\mathbf{x}$ *for* $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

Proof. (a) From definition, we have

$$|\mathbf{x}| - \mathbf{x} = [|\lambda_1(\mathbf{x})| - \lambda_1(\mathbf{x})]\mathbf{u}_{\mathbf{x}}^{(1)} + [|\lambda_2(\mathbf{x})| - \lambda_2(\mathbf{x})]\mathbf{u}_{\mathbf{x}}^{(2)}.$$

To proceed, we discuss three cases.

Case 1. For $\lambda_1(\mathbf{x}) \geq 0$, $\lambda_2(\mathbf{x}) \geq 0$, it is clear that $|\mathbf{x}| - \mathbf{x} = \mathbf{0} \succeq_{\mathcal{L}_\theta} \mathbf{0}$.

Case 2. For $\lambda_1(\mathbf{x}) \leq 0$, $\lambda_2(\mathbf{x}) \leq 0$, we know $|\mathbf{x}| - \mathbf{x} = -2(\lambda_1(\mathbf{x})\mathbf{u}_{\mathbf{x}}^{(1)} + \lambda_2(\mathbf{x})\mathbf{u}_{\mathbf{x}}^{(2)}) = -2\mathbf{x}$. Since $\lambda_2(\mathbf{x}) = x_1 + \|\mathbf{x}_2\| \tan\theta \leq 0$, we have $-x_1 \geq \| - \mathbf{x}_2\| \tan\theta \geq \| - \mathbf{x}_2\| \cot\theta \geq 0$, which says $|\mathbf{x}| - \mathbf{x} = -2\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}$.

Case 3. For $\lambda_1(\mathbf{x}) \leq 0$, $\lambda_2(\mathbf{x}) \geq 0$, there have

$$|\mathbf{x}| - \mathbf{x} = -2\lambda_1(\mathbf{x})\mathbf{u}_{\mathbf{x}}^{(1)} = -2\lambda_1(\mathbf{x}) \begin{bmatrix} \sin^2\theta \\ -\sin\theta\cos\theta\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \end{bmatrix}.$$

Since $-2\lambda_1(\mathbf{x}) \geq 0$, we only need to verify $\sin^2\theta \geq \| - \sin\theta\cos\theta\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\| \cot\theta = \cos^2\theta$. In fact, from the hypothesis $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, we have $\sin^2\theta \geq \cos^2\theta$, which says $|\mathbf{x}| - \mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}$.

(b) Similarly, from definition, we have

$$|\mathbf{x}| + \mathbf{x} = [|\lambda_1(\mathbf{x})| + \lambda_1(\mathbf{x})]\, \mathbf{u}_{\mathbf{x}}^{(1)} + [|\lambda_2(\mathbf{x})| + \lambda_2(\mathbf{x})]\, \mathbf{u}_{\mathbf{x}}^{(2)}.$$

Again, three cases are discussed.

Case 1. For $\lambda_1(\mathbf{x}) \geq 0$, $\lambda_2(\mathbf{x}) \geq 0$, it gives $|\mathbf{x}| + \mathbf{x} = 2\mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}$.

Case 2. For $\lambda_1(\mathbf{x}) \leq 0$, $\lambda_2(\mathbf{x}) \leq 0$, it gives $|\mathbf{x}| + \mathbf{x} = \mathbf{0} \succeq_{\mathcal{L}_\theta} \mathbf{0}$.

Case 3. For $\lambda_1(\mathbf{x}) \leq 0$, $\lambda_2(\mathbf{x}) \geq 0$, there has

$$|\mathbf{x}| + \mathbf{x} = 2\lambda_2(\mathbf{x})\mathbf{u}_{\mathbf{x}}^{(2)} = 2\lambda_2(\mathbf{x}) \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \end{bmatrix}.$$

Since $2\lambda_2(\mathbf{x}) \geq 0$, we only need to check $\cos^2\theta \geq \| \sin\theta\cos\theta \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \| \cot\theta = \cos^2\theta$, which is always true for any angle. Thus, $|\mathbf{x}| + \mathbf{x} \succeq_{\mathcal{L}_\theta} \mathbf{0}$. $\square$

In order to establish more inequalities under this type of decomposition, we still need the concept of circular cone monotonicity. Zhou and Chen [9] are the pioneers who studied the second decomposition on $\mathcal{L}_\theta$, and they provided a sufficient condition for f to be $\mathcal{L}_\theta$-monotone within the second decomposition.

Theorem 3.8. *[11, Theorem 2.3] Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable. If for all real numbers $t_1 \leq t_2$, we have*

$$(\tan\theta - \cot\theta)(f'(t_1) - f'(t_2)) \geq 0,$$

and

$$\begin{bmatrix} f'(t_1) & (f(t_2) - f(t_1))(t_2 - t_1)^{-1} \\ (f(t_2) - f(t_1))(t_2 - t_1)^{-1} & f'(t_2) \end{bmatrix}$$

is a positive semidefinite matrix, then f is $\mathcal{L}_\theta$-monotone.

Recall that the fact a matrix is positive semidefinite if and only if all its eigenvalues are nonnegative. Consequently, we obtain Corollary 3.9.

Corollary 3.9. *Suppose that $f : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ is given by $f(t) = -t^{-1}$, and $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$. Then f is $\mathcal{L}_\theta$-monotone.*

Proof. Only the case $t_2 > t_1 > 0$ needs to be considered. Suppose $t_2 > t_1 > 0$, then we have $f'(t_1) - f'(t_2) = t_1^{-2} - t_2^{-2} \geq 0$. When $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, obviously we have $\tan\theta - \cot\theta \geq 0$. Hence, $(\tan\theta - \cot\theta)(f'(t_1) - f'(t_2)) \geq 0$ is satisfied.

Next, we will show $\begin{bmatrix} f'(t_1) & (f(t_2) - f(t_1))(t_2 - t_1)^{-1} \\ (f(t_2) - f(t_1))(t_2 - t_1)^{-1} & f'(t_2) \end{bmatrix}$ is a positive semidefinite matrix.

It is easy to see that $(t_1^{-1} - t_2^{-1})(t_2 - t_1)^{-1} = (t_1 t_2)^{-1}$. The eigenvalues of the matrix will be shown all nonnegative below.

$$\begin{aligned} \det \begin{bmatrix} t_1^{-2} - \lambda & (t_1 t_2)^{-1} \\ (t_1 t_2)^{-1} & t_2^{-2} - \lambda \end{bmatrix} &= (t_1^{-2} - \lambda)(t_2^{-2} - \lambda) - ((t_1 t_2)^{-1})^2 \\ &= (t_1 t_2)^{-2} - (t_1^{-2} + t_2^{-2})\lambda + \lambda^2 - (t_1 t_2)^{-2} \\ &= \lambda^2 - (t_1^{-2} + t_2^{-2})\lambda. \end{aligned}$$

When $\lambda^2 - (t_1^{-2} + t_2^{-2})\lambda = 0$, we have $\lambda = 0$ or $\lambda = t_1^{-2} + t_2^{-2}$, which implies

$$\begin{bmatrix} t_1^{-2} & (t_1^{-1} - t_2^{-1})(t_2 - t_1)^{-1} \\ (t_1^{-1} - t_2^{-1})(t_2 - t_1)^{-1} & t_2^{-2} \end{bmatrix}$$

is a positive semidefinite matrix. Then, applying Theorem 3.8 yields that $f(t) = -t^{-1}$ is $\mathcal{L}_\theta$-monotone. $\square$

Use Proposition 3.7 and Corollary 3.9, we also establish the following inequalities under the second type of decomposition.

Proposition 3.10. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$, *and* $\mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$. *Then, we have*

(a) $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y})$ *for* $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$;

(b) $H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$ *for* $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

Proof. (a) For $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, from Proposition 3.7, we have

$$\begin{aligned} & |\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} (\mathbf{x} - \mathbf{y}) \\ \Longrightarrow\ & \frac{1}{2}|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} \frac{1}{2}(\mathbf{x} - \mathbf{y}) \\ \Longrightarrow\ & \frac{1}{2}|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} -\frac{1}{2}(\mathbf{x} + \mathbf{y}) + \mathbf{x} \\ \Longrightarrow\ & \frac{1}{2}(\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|) \succeq_{\mathcal{L}_\theta} \mathbf{x}, \end{aligned}$$

and

$$\begin{aligned} & |\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} -(\mathbf{x} - \mathbf{y}) \\ \Longrightarrow\ & \frac{1}{2}|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} -\frac{1}{2}(\mathbf{x} - \mathbf{y}) \\ \Longrightarrow\ & \frac{1}{2}|\mathbf{x} - \mathbf{y}| \succeq_{\mathcal{L}_\theta} -\frac{1}{2}(\mathbf{x} + \mathbf{y}) + \mathbf{y} \\ \Longrightarrow\ & \frac{1}{2}(\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|) \succeq_{\mathcal{L}_\theta} \mathbf{y}. \end{aligned}$$

Hence, we see that $\frac{1}{2}(\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|) \succeq_{\mathcal{L}_\theta} \mathbf{x}$ and $\frac{1}{2}(\mathbf{x}+\mathbf{y}+|\mathbf{x}-\mathbf{y}|) \succeq_{\mathcal{L}_\theta} \mathbf{y}$. Use $\mathcal{L}_\theta$-monotonicity of $f(t) = -t^{-1}$ shown in Corollary 3.9, we obtain

$$\mathbf{x}^{-1} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|}{2}\right)^{-1} \quad \text{and} \quad \mathbf{y}^{-1} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|}{2}\right)^{-1},$$

which imply

$$\frac{\mathbf{x}^{-1} + \mathbf{y}^{-1}}{2} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|}{2}\right)^{-1}.$$

Applying $\mathcal{L}_\theta$-monotonicity of f again, we obtain

$$\frac{\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|}{2} \succeq_{\mathcal{L}_\theta} \left(\frac{\mathbf{x}^{-1} + \mathbf{y}^{-1}}{2}\right)^{-1}.$$

Thus, $\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y})$ when $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

(b) If $\frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \notin \mathcal{L}_\theta$, the inequality holds clearly. Suppose $\frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \in \mathcal{L}_\theta$. For $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, by Proposition 3.7, we have

$$\begin{aligned} & |\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} -(\mathbf{x}-\mathbf{y}) \\ \Longrightarrow\ & \frac{1}{2}|\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} \frac{1}{2}(\mathbf{x}+\mathbf{y}) - \mathbf{x} \\ \Longrightarrow\ & \frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \preceq_{\mathcal{L}_\theta} \mathbf{x}, \end{aligned}$$

and

$$\begin{aligned} & |\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} (\mathbf{x}-\mathbf{y}) \\ \Longrightarrow\ & \frac{1}{2}|\mathbf{x}-\mathbf{y}| \succeq_{\mathcal{L}_\theta} \frac{1}{2}(\mathbf{x}+\mathbf{y}) - \mathbf{y} \\ \Longrightarrow\ & \frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \preceq_{\mathcal{L}_\theta} \mathbf{y}. \end{aligned}$$

Hence, we can conclude that $\frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \preceq_{\mathcal{L}_\theta} \mathbf{x}$ and $\frac{1}{2}(\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|) \preceq_{\mathcal{L}_\theta} \mathbf{y}$. Use $\mathcal{L}_\theta$-monotonicity of $f(t) = -t^{-1}$ shown in Corollary 3.9, we obtain

$$\Big(\frac{\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|}{2}\Big)^{-1} \succeq_{\mathcal{L}_\theta} \mathbf{x}^{-1} \quad \text{and} \quad \Big(\frac{\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|}{2}\Big)^{-1} \succeq_{\mathcal{L}_\theta} \mathbf{y}^{-1},$$

which imply

$$\Big(\frac{\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|}{2}\Big)^{-1} \succeq_{\mathcal{L}_\theta} \frac{\mathbf{x}^{-1}+\mathbf{y}^{-1}}{2}.$$

Applying $\mathcal{L}_\theta$-monotonicity of f again, we obtain

$$\Big(\frac{\mathbf{x}^{-1}+\mathbf{y}^{-1}}{2}\Big)^{-1} \succeq_{\mathcal{L}_\theta} \frac{\mathbf{x}+\mathbf{y}-|\mathbf{x}-\mathbf{y}|}{2}.$$

Thus, $H(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}$ when $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$. □

3.3 Relation between $A(\mathbf{x}, \mathbf{y})$ and $H(\mathbf{x}, \mathbf{y})$

With no doubt, under the second type of decomposition, the inequality

$$A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x},\mathbf{y})$$

does not hold when $0 < \theta < \frac{\pi}{4}$. To see this, we can find two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{L}_{\frac{\pi}{6}}$ such that the inequality $A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_{\frac{\pi}{6}}} H(\mathbf{x},\mathbf{y})$ does not hold.

Example 3.11. Consider $\theta = \frac{\pi}{6}$, $\mathbf{x} = (1.6, 0.3, -0.1)$, $\mathbf{y} = (1.7, 0, 0.5)$. Then, we have $A(\mathbf{x},\mathbf{y}) = (1.65, 0.15, 0.2)$ and $H(\mathbf{x},\mathbf{y}) = (1.6, 0.26, 0.4)$, which says

$$A(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) = (0.05, -0.11, -0.2) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}.$$

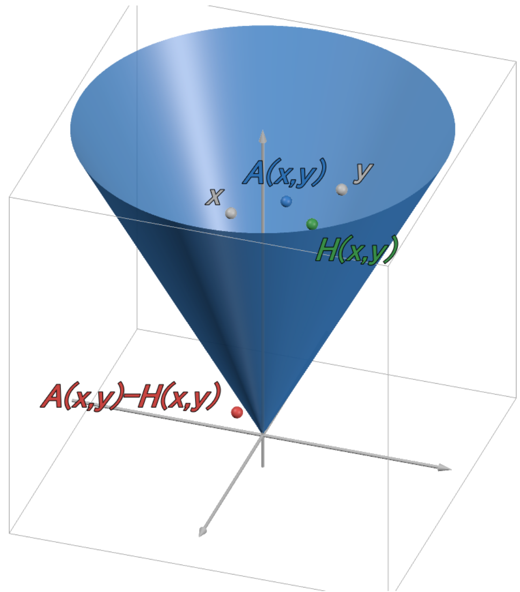

Figure 6: $A(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}$ with $\theta = \frac{\pi}{6}$.

Unfortunately, we can also find $\mathbf{x},\mathbf{y} \in \mathcal{L}_{\frac{\pi}{3}}$ such that the inequality $A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_{\frac{\pi}{3}}} H(\mathbf{x},\mathbf{y})$ does not hold.

Example 3.12. Let $\theta = \frac{\pi}{3}$, $\mathbf{x} = (0.6, 0, 0.4)$, $\mathbf{y} = (0.6, -0.2, 0.4)$, then $A(\mathbf{x},\mathbf{y}) = (0.6, -0.1, 0.4)$, and $H(\mathbf{x},\mathbf{y}) = (0.48, -0.03, 0.12)$, we have

$$A(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) = (0.12, -0.07, 0.28) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}.$$

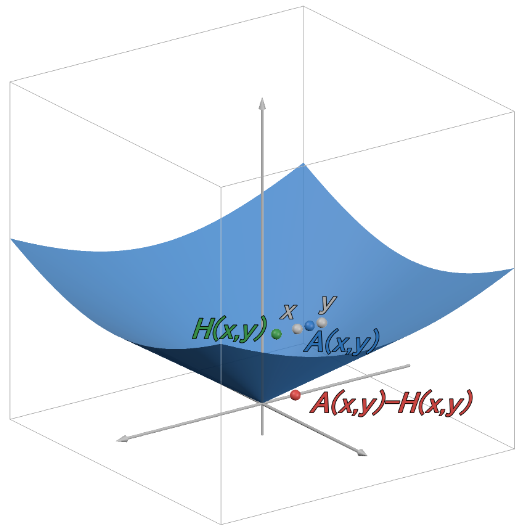

Figure 7: $A(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \not\succeq_{\mathcal{L}_\theta} \mathbf{0}$ with $\theta = \frac{\pi}{3}$.

According to Example 3.12, the inequality between $A(\mathbf{x},\mathbf{y})$ and $H(\mathbf{x},\mathbf{y})$ in the SOC setting cannot be directly generalized to the circular cone setting. Nonetheless, by letting

$$
\begin{aligned}
k(\mathbf{x},\mathbf{y}) &= \max\left\{1+\frac{8\tan^2\theta-2}{\phi\delta ab}, 1+\frac{8\tan^2\theta-2}{\phi\delta cd}\right\},\\
a &= x_1 - \cot\theta\|\mathbf{x}_2\|,\\
b &= x_1 + \tan\theta\|\mathbf{x}_2\|,\\
c &= y_1 - \cot\theta\|\mathbf{y}_2\|,\\
d &= y_1 + \tan\theta\|\mathbf{y}_2\|,\\
\phi &= \frac{1}{ab}\left(x_1 + \frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}\|\mathbf{x}_2\|\right) + \frac{1}{cd}\left(y_1 + \frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}\|\mathbf{y}_2\|\right) - \left\|\frac{1}{ab}\mathbf{x}_2 + \frac{1}{cd}\mathbf{y}_2\right\|\cot\theta,\\
\delta &= \frac{1}{ab}\left(x_1 + \frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}\|\mathbf{x}_2\|\right) + \frac{1}{cd}\left(y_1 + \frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}\|\mathbf{y}_2\|\right) + \left\|\frac{1}{ab}\mathbf{x}_2 + \frac{1}{cd}\mathbf{y}_2\right\|\tan\theta.
\end{aligned}
$$

we can achieve Proposition 3.13 though it is a tedious work (the arguments are omitted).

Proposition 3.13. *Suppose that* $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$, $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$, $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$, *and* $\mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$. *Let* $K = \begin{bmatrix} k(\mathbf{x},\mathbf{y}) & \mathbf{0}^T \\ \mathbf{0} & I \end{bmatrix}$, *then we have*

$$K \cdot A(\mathbf{x},\mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x},\mathbf{y}) \quad \textit{for } \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

In fact, the matrix K depends on $\mathbf{x}, \mathbf{y}$, which does not sound good. It is hoped that we can replace it by a constant matrix which does not depend on $\mathbf{x}$ and $\mathbf{y}$. After lots of experiments, it seem to be true that all examples support this idea. For instance, letting $T = \begin{bmatrix} \tan\theta & \mathbf{0}^T \\ \mathbf{0} & I \end{bmatrix}$, we obtain below example.

Example 3.14. Consider $\theta = \frac{\pi}{3}$, $\mathbf{x} = (0.6, 0, 0.4)$, $\mathbf{y} = (0.6, -0.2, 0.4)$. Then, we have $A(\mathbf{x}, \mathbf{y}) = (0.6, -0.1, 0.4)$, and $H(\mathbf{x}, \mathbf{y}) = (0.48, -0.03, 0.12)$, which says

$$T \cdot A(\mathbf{x}, y) - H(\mathbf{x}, \mathbf{y}) = (0.6\sqrt{3} - 0.48, -0.07, 0.28) \succeq_{\mathcal{L}_\theta} \mathbf{0}.$$

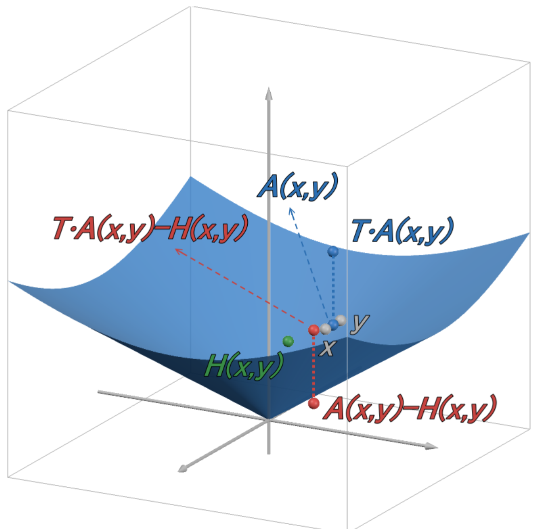

Figure 8: $T \cdot A(\mathbf{x}, \mathbf{y}) - H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{0}$ with $\theta = \frac{\pi}{3}$.

Based on the aforementioned examples and discussions, we make the following conjecture.

Conjecture 3.15. Suppose that $\mathbf{x} = (x_1, \mathbf{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\mathbf{x} \succ_{\mathcal{L}_\theta} \mathbf{0}$, and $\mathbf{y} \succ_{\mathcal{L}_\theta} \mathbf{0}$. Let $T = \begin{bmatrix} \tan\theta & \mathbf{0}^T \\ \mathbf{0} & I \end{bmatrix}$, then we have

$$T \cdot A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}) \quad \text{for } \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

4 Final remarks and conclusion

In this paper, we raise two types of decompositions regarding the circular cone $\mathcal{L}_\theta$. Within the first decomposition, we establish

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y},$$

as happened in the SOC setting. To the contrast, on the second decomposition with $0 < \theta < \pi/4$, things become complicated. Nonetheless, when $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, we achieve that

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y} \quad \text{and} \quad \mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y}.$$

However, in general the inequality $A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y})$ does not hold. Accordingly, we propose a conjecture for this. If the conjecture is true, then we have

$$T \cdot (\mathbf{x} \vee \mathbf{y}) \succeq_{\mathcal{L}_\theta} T \cdot A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{L}_\theta} \mathbf{x} \wedge \mathbf{y},$$

instead, where $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ and $T = \begin{bmatrix} \tan\theta & \mathbf{0}^T \\ \mathbf{0} & I \end{bmatrix}$.

At last, we point out the main differences between these two type of decompositions. They indeed arise from the choices of their eigenvectors, which are the main sources of achieving various outcomes.

	First Decomposition	Second Decomposition
Eigenvectors are in the cone?	Yes.	Yes, when $\frac{\pi}{4} \le \theta < \frac{\pi}{2}$.
Eigenvectors are orthogonal?	No, when $\theta \neq \frac{\pi}{4}$.	Yes.
$\forall \mathbf{x} \in \text{int}(\mathcal{L}_\theta)$, $(\mathbf{x}^{-1})^{-1}$ is still $\mathbf{x}$?	Yes.	No, when $\theta \neq \frac{\pi}{4}$.
$\forall \mathbf{x} \in \mathbb{R}^n$, do we have $\lvert\mathbf{x}\rvert \succeq_{\mathcal{L}_\theta} \mathbf{x}$?	Yes.	Yes, when $\frac{\pi}{4} \le \theta < \frac{\pi}{2}$.
$\forall \mathbf{x} \in \mathbb{R}^n$, do we have $\lvert\mathbf{x}\rvert \succeq_{\mathcal{L}_\theta} -\mathbf{x}$?	Yes.	Yes, when $\frac{\pi}{4} \le \theta < \frac{\pi}{2}$.

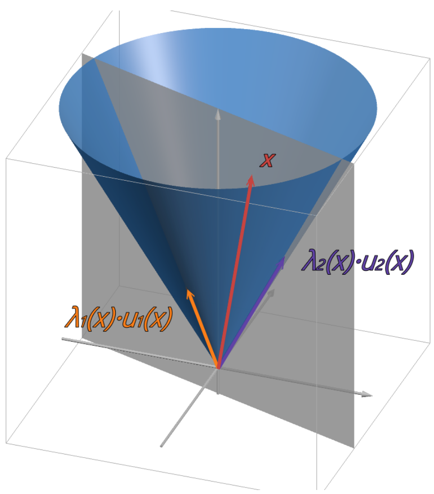

Figure 9: First Decomposition

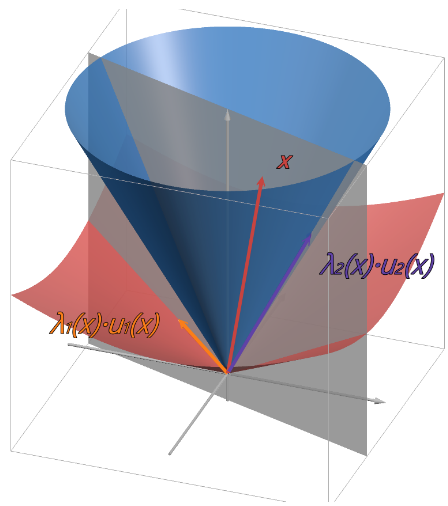

Figure 10: Second Decomposition

Once the eigenvectors fall outside the cone, the absolute value $|\mathbf{x}|$ and the inverse $\mathbf{x}^{-1}$ would fall outside the cone. That tells why we have lots of counterexamples when $0 < \theta < \frac{\pi}{4}$ within the second decomposition. On the other hand, the advantage of orthogonal eigenvectors helps us to clarify the characterization of monotonicity of circular function $f^{\mathcal{L}_\theta}$, see Theorem 3.8. If the eigenvectors are not orthogonal, the proof of monotonicity becomes complicated, see Proposition 2.5.

The geometric mean that we have not touched is also important. The SOC geometric mean $G(\mathbf{x}, \mathbf{y})$ is considered in [3]. Let V be a Euclidean Jordan algebra with a Jordan product $\circ$, let $\mathcal{K}$ be the set of all square elements of V (the associated symmetric cone), and $\Omega := \text{int}(\mathcal{K})$ (the interior of the symmetric cone). For $\mathbf{x} \in V$, let $\mathcal{L}(\mathbf{x})$ denote the linear operator given by $\mathcal{L}(\mathbf{x})\mathbf{y} := \mathbf{x} \circ \mathbf{y}$, and let $P(\mathbf{x}) := 2\mathcal{L}(\mathbf{x})^2 - \mathcal{L}(\mathbf{x}^2)$. Suppose that $\mathbf{x}, \mathbf{y} \in \Omega$. The geometric mean of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$G(\mathbf{x}, \mathbf{y}) := P(\mathbf{x}^{\frac{1}{2}})(P(\mathbf{x}^{-\frac{1}{2}})\mathbf{y})^{\frac{1}{2}}.$$

It can be shown that in the SOC setting [8]

$$\mathbf{x} \vee \mathbf{y} \succeq_{\mathcal{K}^n} A(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{K}^n} G(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{K}^n} H(\mathbf{x}, \mathbf{y}) \succeq_{\mathcal{K}^n} \mathbf{x} \wedge \mathbf{y}.$$

However, for circular cones, we were unable to find a counterpart to the definition of $P(\mathbf{x})$ as seen in the SOC setting. Therefore, we can not find similar inequalities analogous to those in the SOC setting. We leave them as future works.

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