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Novel constructions for closed convex cones through inequalities and support functions

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Abstract Two novel ways to generate closed convex cones, the main ingredient of conic optimization, are proposed in this study. The first way is constructing closed convex cones via inequalities, whereas the second one is through support functions. The contribution of this article is twofold. One is opening up new ideas for looking into structures of closed convex cones. The other one is providing novel approaches and mediums for investigating conic optimization.

Keywords Closed convex cone · Superadditivity · Subadditivity · Homogeneity · Support function

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1 Introduction

Conic optimization is a mathematical framework that holds a pivotal role in optimization problems where the feasible set can be represented as a cone. This framework extends beyond traditional linear programming and convex optimization, allowing for the handling of more intricate structures. In particular, cone structure is the main ingredient in tackling cone convexity, cone monotonicity, and cone decomposition etc., which are needed for analysis in conic optimization. Needless to say, conic optimization problems are noteworthy because they offer a versatile and efficient approach to solving a diverse array of optimization problems across various fields, see [14] and references therein.

Cones play a fundamental role in convex optimization, offering a geometric structure for defining convex sets. A cone formula refers to an expression that can be represented as a cone in a vector space. In the literature, several useful cones have been proposed and applied, including the second-order cone, circular cone, p-order cone, geometric cone, exponential cone and power cone. Recently, Morshed, Vogiatzis, and Noor-E-Alam introduced a new second-order cone, termed the type-2 second-order cone, in 2021 [18]. It is defined as

$$Y^n = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; (x_1 + x_2)^2 \geqslant 2 \sum_{i=3}^{n} x_i^2,\ x_1 \geqslant x_2,\ x_1 + x_2 \geqslant 0 \right\}. \tag{1.1}$$

This modified cone can be regarded as a conventional second-order cone with one less dimension than the original one, achieved through an algebraic transformation. In fact, this cone will degenerate to the polyhedral cone in $\mathbb{R}^3$, see Figure 1. For further details, please refer to [18, Remark 1]. Furthermore, they have also introduced the generalized formula for the so-called type-k cone involving more complicated variables.

$$\Omega^n = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \left(\sum_{i=1}^{k} x_i \right)^2 \geqslant \xi_k \sum_{j=k+1}^{n} x_j^2,\ g_l(x_{1:k}) \geqslant 0, x_r \geqslant 0,\ r, l \in (1, 2, \cdots, k) \right\}, \tag{1.2}$$

where ξ_k is a constant dependent on k and g_l represent additional constraints similar to those in (1.1).

Examining expressions (1.1) and (1.2), if we take the square root on both sides of the cone formula inequalities, it leads to a linear combination of k variables on the left side and a regular 2-norm of an $(n-k)$-dimensional vector on the right side in the main cone inequality. This observation prompts us to consider the cone formula as two real-valued functions connected by an inequality. In this paper, our emphasis will be on exploring cone formula generated by an inequality of functions that adhere to closed convex cone properties, as illustrated in the below expression:

$$K = \{ (\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2}) \},$$

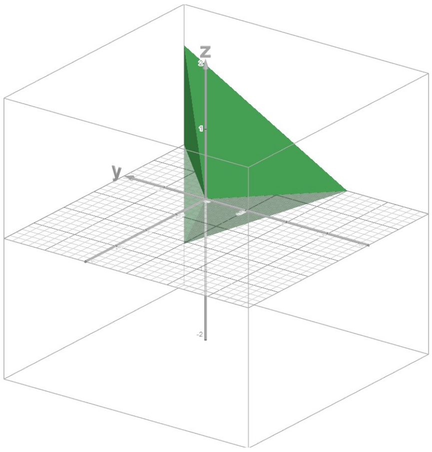


Fig. 1: Degenerated type-2 second-order cone in $\mathbb{R}^3$

where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}$ and $G : E \subseteq \mathbb{R}^n \to \mathbb{R}$ are two real-valued functions, with the sets D and E being subsets of the domains of F and G, respectively.

To proceed, we review several cone formulas that have been elucidated in the literature. It is noteworthy that we consistently represent the inequality using the greater than or equal sign. The definitions of these cones can be found in [1,3,6,8,10,13,16,17,26,29].

- second-order cone:
$$\mathcal{K}^n = \left\{(x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geqslant \|\mathbf{x_2}\|\right\}, \tag{1.3}$$
where $\|\cdot\|$ means the Euclidean norm or 2-norm $\|\cdot\|_2$.
- circular cone:
$$\mathcal{L}_\theta = \left\{(x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geqslant \|\mathbf{x_2}\| \cot\theta\right\}, \tag{1.4}$$
where $\theta \in (0, \pi/2)$ is its half-aperture angle.
- p-order cone:
$$\mathcal{K}_p = \left\{(x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geqslant \|\mathbf{x_2}\|_p\right\} \ (p \geqslant 1), \tag{1.5}$$
where $\|\cdot\|_p$ means the l_p-norm.
- geometric cone:
$$\mathcal{G}^n = \mathrm{cl}\left\{(\mathbf{x}, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_{++} \,\middle|\, 1 \geqslant \sum_{i=1}^{n} e^{-\frac{x_i}{\theta}}\right\}, \tag{1.6}$$
where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n_+$ and $\mathrm{cl}\{\cdot\}$ means the closure of the set.
- exponential cone:
$$\mathcal{K}_e = \mathrm{cl}\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,\middle|\, x_3 \geqslant x_2 e^{\frac{x_1}{x_2}},\ x_2 > 0\right\}. \tag{1.7}$$

– power cone:

$$\mathcal{K}^{\alpha}_{m,n} = \left\{ (\mathbf{x}, \mathbf{z}) \in \mathbb{R}^m_+ \times \mathbb{R}^n \;\middle|\; \prod_{i=1}^{m} x_i^{\alpha_i} \geqslant \|\mathbf{z}\| \right\}, \tag{1.8}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{m} \alpha_i = 1$.

In this paper, we attribute properties which form a closed convex cone to positively homogeneous sub/super-additive functions as a means to articulate the representation of the formula by an inequality. Besides, we explore the characteristics of the cone and investigate the cone formula generated by the support/dual support functions corresponding to closed convex sets. Nice results about the formula of dual/polar cone associated to convex cones generated by the inequality are established. Through this paper, we aim to conduct a more comprehensive and diverse study on the generating methods of cones. In summary, the contribution of this article is twofold. One is opening up new ideas for looking into structures of closed convex cones. The other one is providing novel approaches and mediums for investigating conic optimization.

2 Closed convex cones by inequalities

To comprehend the structure of the cone formula, let us begin by looking into the definition of a convex cone, which is available in various textbook sources, including [12, 24].

Definition 2.1 Let K be a nonempty set in a vector space V, K is called a convex cone if the following two properties hold:

(1) For all $\mathbf{x} \in K$ and all $\lambda > 0$, there holds $\lambda \mathbf{x} \in K$.
(2) For all $\mathbf{x}, \mathbf{y} \in K$, there holds $\mathbf{x} + \mathbf{y} \in K$.

The property (1) ensures that K is a cone, while the property (2) with the help of property (1) guarantees convexity of K. We shall consider two real-valued functions linked by an inequality, which satisfies the property (1) to form a cone and plug them into our cone formula (2.1) in the first step, then deal with the convexity property (2) in the subsequence discussion. More specifically, we consider a nonempty set K defined by

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\}, \tag{2.1}$$

where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}$ and $G : E \subseteq \mathbb{R}^n \to \mathbb{R}$ are two real-valued functions, with the sets D and E being subsets of the domains of F and G, respectively.

Lemma 2.1 *Let K be the set defined as in (2.1). If K satisfies condition:*

(C1) $\quad F(\lambda \mathbf{x_1}) \geqslant G(\lambda \mathbf{x_2})$ *for all* $\lambda > 0$ *and* $(\mathbf{x_1}, \mathbf{x_2}) \in K$, *then K is a cone.*

Proof The condition (C1) ensures that if $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$ and $\lambda > 0$, then $\lambda\mathbf{x} = (\lambda\mathbf{x_1}, \lambda\mathbf{x_2}) \in K$ which means K is a cone by Definition 2.1. □

Example 2.1 Consider the nonnegative closed half space in $\mathbb{R}^n$, which is described as

$$\{(x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geqslant 0\}.$$

We define $F : \mathbb{R}_+ \to \mathbb{R}$ and $G : \mathbb{R}^{n-1} \to \mathbb{R}$ as $F(x_1) = x_1$ and $G(\mathbf{x_2}) = 0$ in above inequality of formula, i.e., $F(x_1) \geqslant G(\mathbf{x_2})$. Accordingly, these two real valued functions satisfy (C1), that is,

$$F(\lambda x_1) = \lambda x_1 \geqslant \lambda \cdot 0 = 0 = G(\lambda\mathbf{x_2}),$$

for all $\lambda > 0$ and $(x_1, \mathbf{x_2})$ in the nonnegative closed half space. Hence, it is a cone by Lemma 2.1 for sure. In fact, almost all cones appeared in the literature including (1.3) to (1.8) satisfy Lemma 2.1.

However, using condition (C1) in Lemma 2.1 directly to find new functions and to generate a cone is not so intuitive. We propose another approach and idea by employing homogeneous functions.

Definition 2.2 A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a homogeneous function of degree k (k is an integer) if it satisfies

$$f(\lambda\mathbf{x}) = \lambda^k f(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and all $\lambda \neq 0$. If no specifically indicated, f is homogeneous means that f is homogeneous of degree 1. In this paper, we call f is positively homogeneous if $\lambda > 0$. We also call functions F and G are positively homogeneous of degree k in K given as in (2.1) means if $(\mathbf{x_1}, \mathbf{x_2}) \in K$, there hold $F(\lambda\mathbf{x_1}) = \lambda^k F(\mathbf{x_1})$ and $G(\lambda\mathbf{x_2}) = \lambda^k G(\mathbf{x_2})$.

The positively homogeneous functions of the same degree implies condition (C1) in Lemma 2.1, hence we build up another lemma as below.

Lemma 2.2 *Let K be the set defined as in (2.1). If K satisfies condition:*

(C2) *F and G are positively homogeneous of the same degree k in K,*

then K is a cone.

Proof If $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$ and $\lambda > 0$, then we have

$$F(\lambda\mathbf{x_1}) = \lambda^k F(\mathbf{x_1}) \geqslant \lambda^k G(\mathbf{x_2}) = G(\lambda\mathbf{x_2}),$$

where the equalities are due to condition (C2) and the inequality is from definition of K. This indicates that $\lambda\mathbf{x}$ is also in K, hence K is a cone. □

If we regard known cones listed from (1.3) to (1.8) in the literature as the expression of functions inequality (2.1), most of the functions on both sides of inequality sign are positively homogeneous of degree 1 except the geometric cone (1.6). In fact, the formula in the geometric cone (1.6) is written as the special formula with positively homogeneous functions of degree $k = 0$. If we rewrite it as

$$\mathcal{G}^n = \mathrm{cl}\left\{(\mathbf{x}, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_{++} \,\middle|\, x_1 \geqslant -\theta \ln(1 - \sum_{i=2}^{n} e^{-\frac{x_i}{\theta}})\right\}, \tag{2.2}$$

then the functions on both sides of inequality are also positively homogeneous of degree 1.

Example 2.2 The cone defined by

$$\{(x, y, z) \in \mathbb{R} \times \mathbb{R}^2 \mid x^2 \geqslant y^2 + z^2\}$$

satisfies condition (C2) in Lemma 2.2 with $F(x) = x^2$ and $G(y, z) = y^2 + z^2$, where F and G are both positively homogeneous of degree 2, see Figure 2 below. Furthermore, following up this cone, we can define a generalized cone by

$$\left\{(x, \mathbf{z}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,\middle|\, x^k \geqslant \sum_{i=1}^{n-1} |z_i|^k\right\},$$

where $\mathbf{z} = (z_1, \cdots, z_{n-1})^T \in \mathbb{R}^{n-1}$, k is a positive integer and $F(x) = x^k$, $G(\mathbf{z}) = \sum_{i=1}^{n-1} |z_i|^k$. Then, F and G are both positively homogeneous of degree k. Note that this generalized cone is not convex if k is even; and it becomes the k-order cone (1.5) if k is odd.

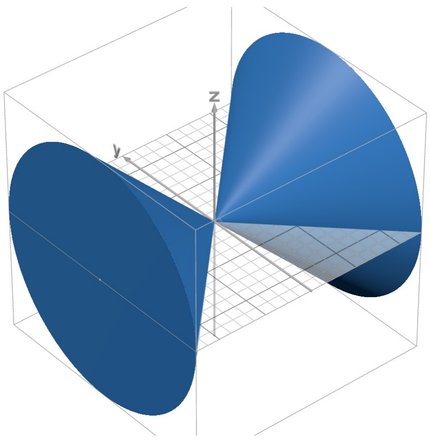


Fig. 2: $\{(x, y, z) \in \mathbb{R} \times \mathbb{R}^2 \mid x^2 \geqslant y^2 + z^2\}$

In view of Example 2.2, it is possible to establish convex cones formulas with functions as in (2.1), which are also positively homogeneous of degree $k > 1$. The next lemma reflects and confirms this idea.

Lemma 2.3 *Suppose that the set K (2.1) satisfy the following conditions:*
(C2) F and G are positively homogeneous of the same degree k in K;
(C3) $F(\mathbf{x_1} + \mathbf{y_1}) \geqslant G(\mathbf{x_2} + \mathbf{y_2})$ for all $(\mathbf{x_1}, \mathbf{x_2}), (\mathbf{y_1}, \mathbf{y_2}) \in K$.
Then, K is a convex cone.

Proof The cone property (1) in Definition 2.1 holds due to Lemma 2.2. On the other hand, condition (C3) directly leads to the property (2) in Definition 2.1. Hence, K is a convex cone. □

Example 2.3 The toppled second-order cone in $\mathbb{R}^3$ is defined by

$$\mathcal{K}_{\text{toppled}} = \left\{(x, y, z) \in \mathbb{R}^2 \times \mathbb{R} \mid xy \geqslant z^2,\ x \geqslant 0,\ y \geqslant 0\right\}. \tag{2.3}$$

Consider $F : \mathbb{R}^2_+ \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ by $F(x, y) = xy$ and $G(z) = z^2$ in above formula. The positive homogeneity of degree 2 of F and G are obvious. It remains to check condition (C3) in Lemma 2.3. To see this, for any $(x, y, z), (x', y', z') \in \mathcal{K}_{\text{toppled}}$, we have

$$\begin{aligned} F(x + x', y + y') &= (x + x')(y + y') \\ &= xy + x'y' + xy' + x'y \\ &\geqslant z^2 + z'^2 + 2\sqrt{xy'x'y} \\ &\geqslant z^2 + z'^2 + 2|z||z'| \\ &\geqslant z^2 + z'^2 + 2zz' \\ &= (z + z')^2 \\ &= G(z + z'). \end{aligned}$$

Hence, the condition (C3) in Lemma 2.3 holds, which says that $\mathcal{K}_{\text{toppled}}$ is a convex cone. The graph of this cone is depicted in Figure 3. Moreover, this cone can be generalized by the following extension:

$$\left\{(x, y, \mathbf{z}) \in \mathbb{R}^2 \times \mathbb{R}^n \mid xy \geqslant \|\mathbf{z}\|^2,\ x \geqslant 0,\ y \geqslant 0\right\}.$$

For more detailed discussion about its algebraic structure, please refer to [2].

Overall, it is still challenging to seek two functions directly satisfying the conditions (C2) and (C3) in Lemma 2.3. To overcome this, we introduce subadditivity and superadditivity of real-valued functions.

Definition 2.3 A function $f : D \subseteq \mathbb{R}^m \to \mathbb{R}$ is called superadditive if it satisfies

$$f(\mathbf{x} + \mathbf{y}) \geqslant f(\mathbf{x}) + f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y}$ in D. A function $g : E \subseteq \mathbb{R}^n \to \mathbb{R}$ is called subadditive if it satisfies

$$g(\mathbf{x} + \mathbf{y}) \leqslant g(\mathbf{x}) + g(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y}$ in E. In this paper, we call F is superadditive and G is subadditive in K (2.1) means if $(\mathbf{x_1}, \mathbf{x_2}), (\mathbf{y_1}, \mathbf{y_2}) \in K$, then we have $F(\mathbf{x_1} + \mathbf{y_1}) \geqslant F(\mathbf{x_1}) + F(\mathbf{y_1})$ and $G(\mathbf{x_2} + \mathbf{y_2}) \leqslant G(\mathbf{x_2}) + G(\mathbf{y_2})$.

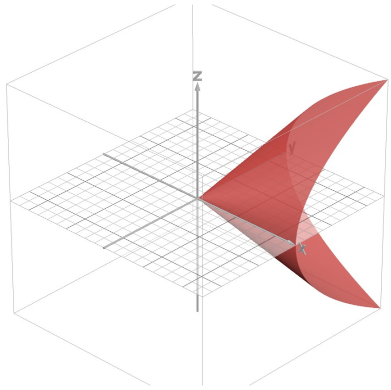


Fig. 3: $\mathcal{K}_{\text{toppled}} = \{(x, y, z) \in \mathbb{R}^3 \mid xy \geqslant z^2,\ x \geqslant 0,\ y \geqslant 0\}$

Building upon the aforementioned concepts, we modify the conditions of Lemma 2.3 to the following reasonable lemma.

Lemma 2.4 *Let K be the set defined as in (2.1). Suppose that K satisfies conditions:*

(C2) F and G are positively homogeneous of the same degree k in K;
(C4) F is superadditive and G is subadditive in K.

Then, K is a convex cone.

Proof Again, Condition (C2) leads to the cone property (1) by Lemma 2.2. As to convexity, since F is superadditive and G is subadditive in K, we have

$$F(\mathbf{x_1} + \mathbf{y_1}) \geqslant F(\mathbf{x_1}) + F(\mathbf{y_1}) \geqslant G(\mathbf{x_2}) + G(\mathbf{y_2}) \geqslant G(\mathbf{x_2} + \mathbf{y_2}),$$

for all $(\mathbf{x_1}, \mathbf{x_2}), (\mathbf{y_1}, \mathbf{y_2}) \in K$. The property (2) in Definition 2.1 holds. □

Unfortunately, we do not have a nontrivial example of a convex cone which satisfies conditions in Lemma 2.4 if the homogeneity degree $k \neq 1$. Part of the reason is, if F and G are both positively homogeneous of degree $k > 1$, they normally can't keep the original super/sub-additivity at the same time. Please see Example 1.1.2 and Example 1.1.6 in [25] for more details. If $k = 1$, several known cones such as the second-order cone (1.3), circular cone (1.4) and p-order cone (1.5) can be examples because the L_p norm is positively homogeneous subadditive, and the identity function is clearly positively homogeneous superadditive. Additionally, the power cone (1.8) also satisfies Lemma 2.4.

Remark 2.1 The nontrivial example we mentioned in previous paragraph is a cone which does not like the trivial cone as follows:

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 \geqslant 0, x \geqslant 0\}.$$

Define $F : \mathbb{R}_+ \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ by $F(x) = x^2$ and $G(y) = 0$. In this context, F and G are both positively homogeneous of degree 2. Additionally, F

is superadditive when $x \geqslant 0$, while G can be considered a subadditive function. This cone is the trivial nonnegative orthant cone in $\mathbb{R}^2$. Another trivial case arises from the toppled second-order cone:

$$\{(x, y, z) \in \mathbb{R}^2 \times \mathbb{R} \mid xy \geqslant -z^2,\ x \geqslant 0, y \geqslant 0, z \geqslant 0\}.$$

Define $F : \mathbb{R}^2_+ \to \mathbb{R}$ and $G : \mathbb{R}_+ \to \mathbb{R}$ by $F(x, y) = xy$ and $G(z) = -z^2$. Both of them are positively homogeneous of degree 2 and F is superadditive when $x \geqslant 0, y \geqslant 0$; and G is subadditive when $z \geqslant 0$. However, this cone is the nonnegative orthant cone of $\mathbb{R}^3$ space. From this point of view we can create many trivial cone of any homogeneity degree $k > 1$, but it may be meaningless.

Several properties related to homogeneity and super/sub-additivity have been explored in previous studies. Now, we leverage these properties to develop additional concepts concerning our cone formula. Most of the proofs for these properties can be found in [24,25,15].

Property 2.1

(1) If $f : \mathbb{R}^n \to \mathbb{R}$ is superadditive, then $f(\mathbf{0}) \leqslant 0$. If $g : \mathbb{R}^n \to \mathbb{R}$ is subadditive, then $g(\mathbf{0}) \geqslant 0$.
(2) If $f : \mathbb{R}^n \to \mathbb{R}$ is superadditive, then $f(-\mathbf{x}) \leqslant -f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. If $g : \mathbb{R}^n \to \mathbb{R}$ is subadditive, then $g(-\mathbf{x}) \geqslant -g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.
(3) If $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are superadditive (respectively, subadditive) functions and $c_1, c_2 \geqslant 0$, then $f = c_1 f_1 + c_2 f_2$ is superadditive (respectively, subadditive).
(4) If $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $k \neq 0$, then $f(\mathbf{0}) = 0$.
(5) If $f : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous of degree 1, then f is superadditive if and only if f is concave. If $g : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous of degree 1, then g is subadditive if and only if g is convex.
(6) If $f : \mathbb{R}^n \to \mathbb{R}_+$ is superadditive, then f^k is also superadditive where k is a positive integer; if $g : \mathbb{R}^n \to \mathbb{R}_+$ is subadditive, then $g^{1/k}$ is also subadditive where k is a positive integer.

Proof (1) It is from [15, Lemma 16.1.3].

(2) See [15, Lemma 16.1.5].

(3) See [25, Theorem 1.3.1(α)].

(4) Choosing $\lambda > 1$ yields $f(\mathbf{0}) = f(\lambda \cdot \mathbf{0}) = \lambda^k f(\mathbf{0})$. Hence, $f(\mathbf{0}) = 0$.

(5) See [24, Theorem 4.7].

(6) Since f is superadditive and $f(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$, we have

$$f^k(\mathbf{x} + \mathbf{y}) \geqslant (f(\mathbf{x}) + f(\mathbf{y}))^k \geqslant f(\mathbf{x})^k + f(\mathbf{y})^k = f^k(\mathbf{x}) + f^k(\mathbf{y}),$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $k \geqslant 1$. Hence, f^k is superadditive. Moreover, g is subadditive and $g(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$. In addition, because $k \geqslant 1$, we obtain

$$g^{1/k}(\mathbf{x} + \mathbf{y}) \leqslant (g(\mathbf{x}) + g(\mathbf{y}))^{1/k} \leqslant g(\mathbf{x})^{1/k} + g(\mathbf{y})^{1/k} = g^{1/k}(\mathbf{x}) + g^{1/k}(\mathbf{y}),$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Thus, $g^{1/k}$ is subadditive. □

In fact, homogeneity of degree 1 will result in the necessary and sufficient condition for subadditivity and convexity (or superadditivity and concavity), as indicated in Property 2.1(5). The next theorem provides a more comprehensive understanding of the cone formula.

Theorem 2.1 *Suppose that the set K (2.1) satisfies conditions:*

(i) F and G are positively homogeneous (of degree 1) in K,
(ii) F is superadditive (concave) and G is subadditive (convex) in K.

Then, the following hold.

(1) K is a convex cone.
(2) $\mathrm{cl}(K)$ is a closed convex cone.
(3) If both F and G are continuous and both D and E are closed, then $K = \mathrm{cl}(K)$ is a closed convex cone.

Proof (1) Applying Lemma 2.4 and Property 2.1(5), the proof is straightforward and we omit it here. (2) The closure of K is closed by definition. It is straightforward to verify that the closure of K preserves both the conicity and convexity of K. Therefore, $\mathrm{cl}(K)$ is a closed convex cone. (3) If the functions F and G are both continuous and D and E are both closed, we assert that K is closed. For every limit point $(\mathbf{z_1}, \mathbf{z_2})$ of K, there exists a sequence $\{(\mathbf{x_n^{(1)}}, \mathbf{x_n^{(2)}})\} \subseteq K \backslash \{(\mathbf{z_1}, \mathbf{z_2})\}$ such that

$$\lim_{n\to\infty} (\mathbf{x_n^{(1)}}, \mathbf{x_n^{(2)}}) = (\mathbf{z_1}, \mathbf{z_2}),$$

which mean $\lim_{n\to\infty} \mathbf{x_n^{(1)}} = \mathbf{z_1}$ and $\lim_{n\to\infty} \mathbf{x_n^{(2)}} = \mathbf{z_2}$, where $\mathbf{x_n^{(1)}} \in D$ and $\mathbf{x_n^{(2)}} \in E$. Based on the assumption that D and E are closed, we have $\mathbf{z_1} \in D$ and $\mathbf{z_2} \in E$. Now because $(\mathbf{x_n^{(1)}}, \mathbf{x_n^{(2)}}) \in K$, we have

$$F(\mathbf{x_n^{(1)}}) \geqslant G(\mathbf{x_n^{(2)}}), \ \forall n = 1, 2, 3, \cdots .$$

By the continuity of F and G, it follows that F is continuous at $\mathbf{z_1}$ and G is continuous at $\mathbf{z_2}$, and

$$\lim_{n\to\infty} F(\mathbf{x_n^{(1)}}) = F(\mathbf{z_1}) \geqslant G(\mathbf{z_2}) = \lim_{n\to\infty} G(\mathbf{x_n^{(2)}}).$$

This implies that every limit point $(\mathbf{z_1}, \mathbf{z_2})$ of K is in K, then K is closed. Hence, $K = \mathrm{cl}(K)$ is a closed convex cone. □

Notice that from Property 2.1(4), the cones described in Theorem 2.1 must satisfy $F(\mathbf{0}) = G(\mathbf{0}) = 0$ if $\mathbf{0} \in D$ and $\mathbf{0} \in E$, since F and G are both positively homogeneous. A positively homogeneous subadditive function is also called a sublinear function in some materials. We point out that there have results in the literature concerning Theorem 2.1 already. More specifically, the epigraph of any sublinear function from $\mathbb{R}^n$ to $\mathbb{R}$ is a nonempty cone in $\mathbb{R}^n \times \mathbb{R}$, please see Proposition 11.1 in [27] or [4]. Indeed, the aforementioned result is a special case in Theorem 2.1.

The toppled second-order cone in Example 2.3 is a special cone which we will use it again as an example below.

Example 2.4 The toppled second-order cone (2.3) can be reformulated as

$$\mathcal{K}_{\text{toppled}} = \{(x, y, z) \in \mathbb{R}^2 \times \mathbb{R} \mid \sqrt{xy} \geqslant |z|,\ x \geqslant 0,\ y \geqslant 0\}. \tag{2.4}$$

If we define $F : \mathbb{R}^2_+ \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ by $F(x, y) = \sqrt{xy}$ and $G(z) = |z|$, then F and G are positively homogeneous and G is subadditive clearly. For any $(x, y, z), (x', y', z') \in \mathcal{K}_{\text{toppled}}$, we have

$$\begin{aligned} F(x + x', y + y') &= \sqrt{(x + x')(y + y')} \\ &= \sqrt{xy + x'y' + xy' + x'y} \\ &\geqslant \sqrt{\sqrt{xy}^2 + \sqrt{x'y'}^2 + 2\sqrt{xy'x'y}} \\ &= \sqrt{xy} + \sqrt{x'y'} \\ &= F(x, y) + F(x', y'). \end{aligned}$$

Hence, F is superadditive in $\mathcal{K}_{\text{toppled}}$. As the same point of view in Example 2.3, we can generalize this cone (2.4) as the following cone:

$$\mathcal{K}^n_{\text{toppled}} = \{(x, y, \mathbf{z}) \in \mathbb{R}^2 \times \mathbb{R}^n \mid \sqrt{xy} \geqslant ||\mathbf{z}||,\ x \geqslant 0,\ y \geqslant 0\},$$

which still satisfies Theorem 2.1.

Remark 2.2 We point out something by reviewing the formula of two toppled second-order cones (2.3), (2.4) and the Property 2.1(6). In particular, $F(x, y) = \sqrt{xy}$ is superadditive when $x, y \geqslant 0$ and its square function $F^2(x, y) = xy$ is also superadditive when $x, y \geqslant 0$ according to Property 2.1(6). However, we cannot derive from $G(z) = |z|$ is subadditive that its square function $G^2(z) = z^2$ is subadditive. In fact, G^2 is superadditive when $z \geqslant 0$, while G^2 is convex for $z \in \mathbb{R}$. This is why it is hard to find a nontrivial example to satisfy conditions in Lemma 2.4 if the homogeneity degree $k \neq 1$.

By Property 2.1(3), it is clear to see that the positive combinations of positively homogeneous super/sub-additive functions are also positively homogeneous super/sub-additive. We illustrate the formulas of several extended cones derived from simple super/sub-additive functions or well-known convex cones through the combination of them.

Example 2.5 From Example 2.4, $H(y, z) = \sqrt{yz}$ is superadditive when $y, z \geqslant 0$, it follows that $-H(y, z) = -\sqrt{yz}$ is subadditive when $y, z \geqslant 0$ by definition. We obtain $G(y, z) = y + z - 2\sqrt{yz} = (\sqrt{y} - \sqrt{z})^2$ is also subadditive when $y, z \geqslant 0$. Hence, the set

$$\{(x, y, z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant (\sqrt{y} - \sqrt{z})^2,\ y \geqslant 0,\ z \geqslant 0\}$$

is a convex cone by Theorem 2.1. However, the continuity of both F and G in their closed domains leads to the closeness of this cone. Please refer to Figure 4 for the graph of this cone.

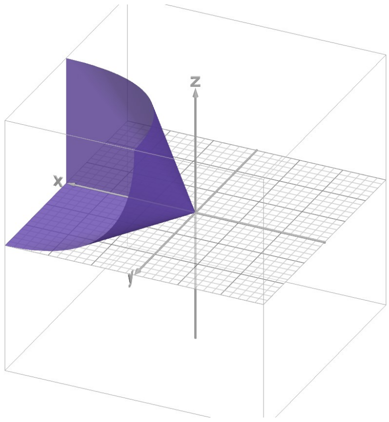


Fig. 4: $\{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant (\sqrt{y} - \sqrt{z})^2,\ y \geqslant 0, z \geqslant 0\}$

Example 2.6 The well-known Fischer–Burmeister NCP-function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\text{FB}}(a,b) = \sqrt{a^2+b^2} - (a+b).$$

This function is generalized to $\phi_p : \mathbb{R}^2 \to \mathbb{R}$ in [7] as follows:

$$\phi_p(a,b) = \|(a,b)\|_p - (a+b),\ p \geqslant 1.$$

In fact, ϕ_p is a subadditive function minus an additive function which is also subadditive. Hence, the set defined by

$$\mathcal{K}_{\text{FB}} = \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant \|(y,z)\|_p - (y+z)\} \tag{2.5}$$

is a convex cone by Theorem 2.1. In fact, it is also closed according to the continuity of their associated functions. The geometric views of ϕ_p function or the generalized Fischer–Burmeister cone $\mathcal{K}_{\text{FB}}$ is fully revealed and depicted in [28].

Example 2.7 Let $F : D \subseteq \mathbb{R}^m \to \mathbb{R}$, $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_s} \to \mathbb{R}$ be defined by

$$F(\mathbf{x}) = \sum_{i=1}^{m} x_i\,, \quad G(\mathbf{z}) = \sum_{j=1}^{s} \|\mathbf{z_j}\|_{p_j},\ p_j \geqslant 1,$$

where $\|\cdot\|_{p_j}$ is the p_j-norm in $\mathbb{R}^{n_j}, j = 1, 2, \cdots, s$, and $\mathbf{x} = (x_1, \cdots, x_m)^T \in \mathbb{R}^m$, $\mathbf{z} = (\mathbf{z_1}, \cdots, \mathbf{z_s})^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_s}$, and $D = \{\mathbf{x} \in \mathbb{R}^m \mid \sum_{i=1}^m x_i \geqslant 0\}$. It is easy to see that F is additive, and the positive linear combination of subadditive functions G (norms) is subadditive, both of them are positively homogeneous. Then, the set

$$\mathcal{K}_{ms} = \left\{ (\mathbf{x}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^{n_1 + \cdots + n_s} \;\middle|\; \sum_{i=1}^{m} x_i \geqslant \sum_{j=1}^{s} \|\mathbf{z_j}\|_{p_j},\ p_j \geqslant 1 \right\}$$

is a closed convex cone in $\mathbb{R}^{m+n_1+\cdots+n_s}$ by the continuity of F, G in their closed domains and Theorem 2.1. However, we can identify nonzero vectors $(\mathbf{x}, \mathbf{0})$ where $x_1 = -\sum_{i=2}^m x_i \neq 0$ and its negative vector $(-\mathbf{x}, \mathbf{0}) \neq \mathbf{0}$ is also in this cone. This implies that the cone is not pointed unless additional constraints, similar to those in the type-2 (1.1) or type-k second-order cone, are imposed.

The cone $\mathcal{K}_{ms}$ is essentially a trivial extension of the second-order or p-order cone. Specifically, they correspond some special cases:

- If $m = 1$ and $s = 1$, then $\mathcal{K}_{ms}$ is the p_1-order cone in $\mathbb{R}^{n_1+1}$.
- If $m = 2$, $s = 1$, and $p_1 = 2$, then $\mathcal{K}_{ms}$ is akin to the type-2 second order cone (1.1) which appeared in [18]. They differ only by a constant multiple $\sqrt{2}$ and the additional constraint $x_1 \geqslant x_2$.

Another example below is modified from the formula of the exponential cone (1.7) to a higher dimensional form.

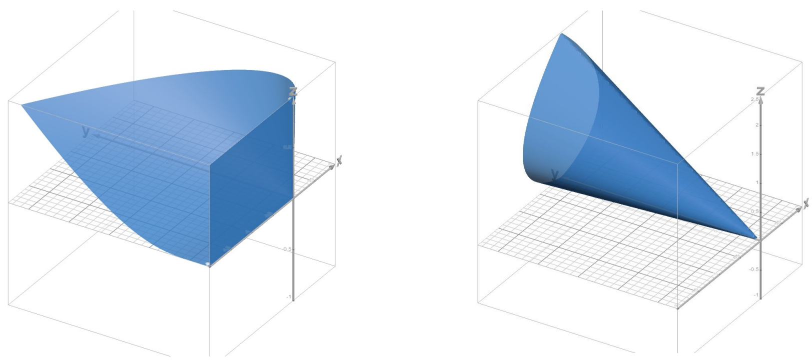
Example 2.8 Let $F : \mathbb{R}^2_{++} \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$F(y, z) = y \ln \frac{z}{y}, \; y > 0, \; z > 0, \quad G(\mathbf{x}) = \|\mathbf{x}\|_p, \; p \geqslant 1.$$

Because F is concave, G is subadditive and both of them are positively homogeneous, then the set

$$\widetilde{\mathcal{K}_e} = \mathrm{cl}\, \{(y, z, \mathbf{x}) \in \mathbb{R}^2 \times \mathbb{R}^n \mid y \ln \frac{z}{y} \geqslant ||\mathbf{x}||_p, \; y > 0, \; z > 0\}$$

is a closed convex cone in $\mathbb{R}^{n+2}$ by Theorem 2.1. If $n = 1$, then $\widetilde{\mathcal{K}_e}$ will degenerate to a variant type of the exponential cone in $\mathbb{R}^3$. Please refer to Figure 5(a) for the original exponential cone and Figure 5(b) for this specific cone to appreciate their distinctions.



(a) The exponential cone (b) $\widetilde{\mathcal{K}_e}$ with $n = 1$

Fig. 5: Comparison of the exponential cone and $\widetilde{\mathcal{K}_e}$ with $n = 1$

Remark 2.3 For reference in this example, if we substitute the L_p-norm value with the sum of elements $\sum_{i=1}^{n} x_i$ on the right side of the inequality, we create another type of extension cone. This cone will degenerate to the original exponential cone when $n = 1$.

As below, we outline sufficient conditions for a convex cone is pointed. A convex cone K is termed pointed if $K \bigcap (-K) = \{\mathbf{0}\}$.

Theorem 2.2 *If the convex cone*

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\},$$

where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}$ is positively homogeneous and superadditive (or concave) in K and $G : E \subseteq \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous and subadditive (or convex) in K and satisfy the following two conditions:

(a) $G(\mathbf{x_2}) \geqslant 0$ for any $(\mathbf{x_1}, \mathbf{x_2}) \in K$.
(b) If $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, then $F(\mathbf{x_1}) = 0$ implies $\mathbf{x_1} = \mathbf{0}$ and $G(\mathbf{x_2}) = 0$ implies $\mathbf{x_2} = \mathbf{0}$.

Then, K is pointed.

Proof Suppose $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$ and $-\mathbf{x} \in K$, we claim that $\mathbf{x} = \mathbf{0}$. Since F is superadditive, $F(-\mathbf{x_1}) \leqslant -F(\mathbf{x_1})$ from Property 2.1(2). By condition (a) we have $F(\mathbf{x_1}) \geqslant G(\mathbf{x_2}) \geqslant 0$. Again $-\mathbf{x} \in K$, we have $-F(\mathbf{x_1}) \geqslant F(-\mathbf{x_1}) \geqslant G(-\mathbf{x_2}) \geqslant 0$. These imply $F(\mathbf{x_1}) = G(\mathbf{x_2}) = 0$, then we have $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) = \mathbf{0}$ from condition (b). Hence, K is pointed. □

We declare that Theorem 2.2 only provides sufficient conditions for a convex cone being pointed. Most of cones in the literature satisfy them except the power cone (1.8), the toppled second-order cone (2.3), (2.4) and the generalized Fischer–Burmeister cone (2.5). In fact, the toppled second-order cone is the degenerated case of the generalized power cone. In the toppled second-order cone, if we consider point $\mathbf{x} = (\mathbf{x_1}, x_2) = ((1,0), 0)$, it is clear to see that $F(1,0) = G(0) = 0$ but $\mathbf{x_1} \neq \mathbf{0}$. It does not satisfy condition (b) of Theorem 2.2, however it's indeed a pointed cone.

To end this section, we provide a list of examples of homogeneous and super/sub-additive functions for reference, more examples can be found in [25]. Additionally, we summarize the detailing well-known closed convex cones mentioned in previous of this section in Table 1.

Let $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $s, t, a_i \in \mathbb{R}$.

- $f(x_1, \cdots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + ... + a_n x_n$, $a_i \in \mathbb{R}$, is subadditive, superadditive and homogeneous.

Examples of subadditive functions:

- L_p norm $\|\mathbf{x}\|_p$ with $p \geqslant 1$.
- $f(t) = \sqrt{t}$ for $t \geqslant 0$.

- $f(t) = \ln(1+t)$ for $t \geqslant 0$.
- the support function of a closed convex set in $\mathbb{R}^n$ is subadditive and homogeneous.

Examples of superadditive functions:

- $f(t) = t^2$ for $t \geqslant 0$.
- $f(s,t) = st$ for $s, t \geqslant 0$.

Examples of homogeneous functions:

- L_p norm $\|\mathbf{x}\|_p$ with $p \geqslant 1$ (positively homogeneous).
- homogeneous polynomial function of degree k.

F	G	cone
x_1	$\|\mathbf{x_2}\|$	the second-order cone
	$\|\mathbf{x_2}\|_p$	the p-order cone
$\sqrt{xy}$	$\|\mathbf{z}\|$	the toppled second-order cone
x_1	$\|\mathbf{x_2}\| \cot\theta$	the circular cone
x	$\|(y,z)\|_p - (y+z)$	the Fischer–Burmeister cone
x_1	$-\theta \ln(1 - \sum_{i=2}^n e^{-\frac{x_i}{\theta}})$	the geometric cone
x_3	$x_2 e^{\frac{x_1}{x_2}}$	the exponential cone
$\prod_{i=1}^m x_i^{\alpha_i}$	$\|\mathbf{z}\|$	the power cone

Table 1: Some known cones through inequality $F \geqslant G$.

Remark 2.4 The corresponding functions F and G of the geometric cone in Table 1 has been rewritten to fit the requirements of Theorem 2.1. The original formula of the geometric cone is $1 \geqslant \sum_{i=1}^n e^{-\frac{x_i}{\theta}}$.

Remark 2.5 The support function of any closed convex set in $\mathbb{R}^n$ is a suitable function to be incorporated in the right side of cone formula (2.1) in the capacity of the function G. This is because the support function is both subadditive and positively homogeneous. A detailed discussion of this types of cones will be presented in the next section.

3 Generating cones by support functions

3.1 Support function cones

Definition 3.1 The support function $h_A : E \subseteq \mathbb{R}^n \to \mathbb{R}$ of a nonempty closed convex set A in $\mathbb{R}^n$ is given by

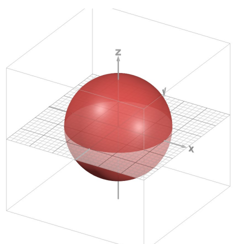
$$h_A(\mathbf{x}) = \sup\{\langle \mathbf{x}, \mathbf{a} \rangle \mid \mathbf{a} \in A\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^n$.

The support function h_A describes the (signed) distances of supporting hyperplanes of A from the origin with the unit direction $\mathbf{x}$. The value of a support function may become infinity if the associated closed convex set is unbounded, so we only consider the domain that h_A becomes not infinite for convenience in this paper. Here are some examples of support functions:

- The support function of a singleton $A = \{\mathbf{a}\}$ is $h_A(\mathbf{x}) = \langle\mathbf{x}, \mathbf{a}\rangle$.
- The support function of the Euclidean unit ball $B = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y}\|_2 \leqslant 1\}$ is $h_B(\mathbf{x}) = \|\mathbf{x}\|_2$.
- If C is a line segment through the origin with endpoints $-\mathbf{a}$ and $\mathbf{a}$, then its corresponding support function is $h_C(\mathbf{x}) = |\langle\mathbf{x}, \mathbf{a}\rangle|$.

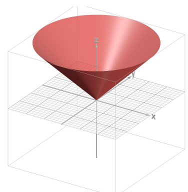
It is straightforward to verify that the support function is positively homogeneous and subadditive. The continuity of the support function associated to a bounded closed convex set can be found in [12, Example 11.2]. Therefore, we can place the associated support function on the right side of the inequality in (2.1), assuming the role of the function G, to generate the corresponding convex cone as illustrated in Figure 6.



(a) a closed convex set $A = \{\mathbf{z} \in \mathbb{R}^{n-1} \mid \|\mathbf{z}\| \leqslant 1\}$

$$A \xrightarrow{h_A} K_A$$

(b) associated support function $h_A(\mathbf{z}) = \|\mathbf{z}\|$



(c) a closed convex cone $\mathcal{K}^n = \{(x, \mathbf{z}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x \geqslant \|\mathbf{z}\|\}$

Fig. 6: K_A cone generated by support function

Now, we present the main result regarding cones generated by support functions.

Theorem 3.1 *Let A be a nonempty closed convex set in $\mathbb{R}^n$ and $h_A : E \subseteq \mathbb{R}^n \to \mathbb{R}$ is the associated support function with A, that is,*

$$h_A(\mathbf{x}) = \sup\{\langle\mathbf{x}, \mathbf{a}\rangle \mid \mathbf{a} \in A\}.$$

Suppose that $F : D \subseteq \mathbb{R}^m \to \mathbb{R}$ is a positively homogeneous superadditive (concave) function and the set K_A is defined by

$$K_A := \{(\mathbf{z}, \mathbf{x}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{z}) \geqslant h_A(\mathbf{x})\}, \tag{3.1}$$

then the following hold.

(1) K_A *is a convex cone.*
(2) $\mathrm{cl}(K_A)$ *is a closed convex cone.*
(3) If F *is continuous and* D *is closed, and* h_A *is continuous and* E *is closed, then* $K_A = \mathrm{cl}(K_A)$ *is a closed convex cone.*

Proof (1) Since A is a nonempty closed convex set, its corresponding support function h_A is positively homogeneous and subadditive. By assumption, F is a positively homogeneous superadditive (concave) function, applying Theorem 2.1 yields that K_A is a convex cone. The reasons for (2) and (3) are the same as the proofs provided in Theorem 2.1. □

For illustrative purposes, we showcase the p-order cones in $\mathbb{R}^3$ as familiar examples and another 3-dimensional cone generated by a 2-dimensional convex set, which can be visualized by using graphic software. In Example 3.1 and Example 3.2, the identity function $F(z) = z$ will be employed as the role F in Theorem 3.1. We have included the calculations of all the support functions presented in this paper in the Appendix section for reference.

Example 3.1 We demonstrate the p-order cones in $\mathbb{R}^3$.

- Given $A = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\|_1 = |x|+|y| \leqslant 1\}$, $h_A(x, y) = \max\{|x|, |y|\} = \|(x, y)\|_\infty$. They generate the maximal-order cone $\{(z, x, y) \in \mathbb{R}^3 \mid z \geqslant \|(x, y)\|_\infty\}$.
- Given $A = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\|_2 = \sqrt{x^2 + y^2} \leqslant 1\}$, $h_A(x, y) = \sqrt{x^2 + y^2} = \|(x, y)\|_2$. They generate the second-order cone $\{(z, x, y) \in \mathbb{R}^3 \mid z \geqslant \|(x, y)\|_2\}$.
- Given $A = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\|_p \leqslant 1\}$, $h_A(x, y) = ||(x, y)||_q$, where $p \geqslant 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. They generate the q-order cone $\{(z, x, y) \in \mathbb{R}^3 \mid z \geqslant \|(x, y)\|_q\}$.
- Given $A = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\|_\infty = \max\{|x|, |y|\} \leqslant 1\}$, $h_A(x, y) = |x| + |y| = \|(x, y)\|_1$. They generate the 1-order cone $\{(z, x, y) \in \mathbb{R}^3 \mid z \geqslant \|(x, y)\|_1\}$.

Considering the p-order cones discussed in Example 3.1, we are intrigued by the possibility that the cone generated by the support function might be the dual cone of some related cone. This constitutes another aspect that we will delve into as part of this paper later.

Example 3.2 Consider the closed convex set

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y \leqslant 1\},$$

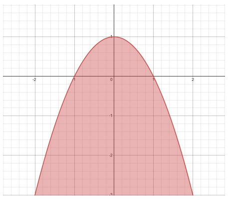
shown as in Figure 7(a) and its associated support function

$$h_A(x, y) = y + \frac{x^2}{4y},\ y > 0.$$

Then, the set

$$K_A := \mathrm{cl}\left\{(z, x, y) \in \mathbb{R}^3 \mid z \geqslant y + \frac{x^2}{4y},\ y > 0\right\}$$

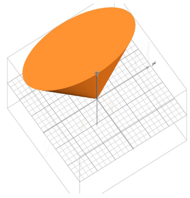
forms a closed convex cone in $\mathbb{R}^3$, as illustrated in Figure 7(c). It is worth mentioning that this cone includes its limit points $\{(x,y,z) \in \mathbb{R}^3 \mid x = y = 0, z \geqslant 0\}$, i.e., the nonnegative z-axis.



(a) a closed convex set $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y \leqslant 1\}$

$$A \xrightarrow{h_A} K_A$$

(b) associated support function $h_A(x,y) = y + \frac{x^2}{4y},\ y > 0$



(c) K_A in $\mathbb{R}^3$

Fig. 7: K_A cone generated by support function

In the preceding example within the $\mathbb{R}^2$ space, it is feasible to represent any closed convex set numerically through the utilization of computer graphic tools such as Matlab to ascertain the associated values of the support function. Building upon this conceptual framework, we can further devise computational algorithms to generate 3-dimensional cone representations based on closed convex set data within the scope of further research.

3.2 Dual support function and dual cone

As mentioned in Section 3.1, we aim to develop the similar support function, which can be placed on the left side of cone inequality (2.1). In view of notations, we name it the dual support function as follows.

Definition 3.2 If A is a nonempty closed convex set in $\mathbb{R}^n$, the corresponding dual support function $\sigma_A : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\sigma_A(\mathbf{x}) := \inf\{\langle \mathbf{x}, \mathbf{a} \rangle \mid \mathbf{a} \in A\},$$

where $\langle \cdot, \cdot \rangle$ represents the inner product in $\mathbb{R}^n$.

As the same reason to support functions, the value of a dual support function may become negative infinity for the associated closed convex set, so we only consider the domain that σ_A becomes not negative infinite in this paper. It is clear to see that such a dual support function is superadditive and positively homogeneous. We notice that $\sigma_A(\mathbf{x}) = -h_A(-\mathbf{x})$, and this can lead to the continuity of the dual support function if A is bounded. Below are some examples of dual support functions.

– In $\mathbb{R}^1$, the dual support function of $A = \{x \in \mathbb{R} \mid x \geqslant 1\}$ is $\sigma_A(x) = x$, where $x \geqslant 0$.
– The dual support function of the Euclidean unit ball $B = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y}\|_2 \leqslant 1\}$ is $\sigma_B(\mathbf{x}) = -\|\mathbf{x}\|_2$.
– The dual support function of the set $A = \{(x, y) \in \mathbb{R}^2 \mid 4xy \geqslant 1,\ x, y > 0\}$ is $\sigma_A(x, y) = \sqrt{xy}$, where $x, y \geqslant 0$.

Next, we place the suitable dual support function into the left side of cone inequality (2.1). Another main result comes out shown as in the following theorem.

Theorem 3.2 *Let A be a nonempty closed convex set in $\mathbb{R}^m$ and $\sigma_A : D \subseteq \mathbb{R}^m \to \mathbb{R}$ is the associated dual support function with A, that is,*

$$\sigma_A(\mathbf{z}) = \inf\{\langle \mathbf{z}, \mathbf{a}\rangle \mid \mathbf{a} \in A\}.$$

Let B be a nonempty closed convex set in $\mathbb{R}^n$ and $h_B : E \subseteq \mathbb{R}^n \to \mathbb{R}$ is the associated support function with B, that is,

$$h_B(\mathbf{z}) = \sup\{\langle \mathbf{z}, \mathbf{b}\rangle \mid \mathbf{b} \in B\}.$$

The set $K_{A,B}$ is defined by

$$K_{A,B} := \{(\mathbf{z_1}, \mathbf{z_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_A(\mathbf{z_1}) \geqslant h_B(\mathbf{z_2})\}.$$

Then, the following hold.

(1) $K_{A,B}$ is a convex cone.
(2) $\mathrm{cl}(K_{A,B})$ is a closed convex cone.
(3) If both σ_A and h_B are continuous and both D and E are closed, then $K_{A,B} = \mathrm{cl}(K_{A,B})$ is a closed convex cone.

Proof (1) Note that the support function is positively homogeneous and subadditive; and the dual support function is positively homogeneous and superadditive. Accordingly, the set $K_{A,B}$ is certainly a convex cone by Theorem 2.1. The reasons for (2) and (3) are the same as the proofs provided in Theorem 2.1. □

Example 3.3 Consider the closed convex set

$$A = \{(x, y) \in \mathbb{R}^2 \mid 4xy \geqslant 1,\ x, y > 0\}$$

with its associated dual support function

$$\sigma_A(x, y) = \sqrt{xy},\ x, y \geqslant 0.$$

In addition, consider the set $B = \{z \in \mathbb{R} \mid |z| \leqslant 1\}$ with its support function $h_B(z) = |z|$. Then, the generated set

$$K_{A,B} := \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{xy} \geqslant |z|,\ x, y \geqslant 0\}$$

is exactly the toppled second-order cone $\mathcal{K}_{\text{toppled}}$ (2.4) in $\mathbb{R}^3$ as illustrated in Figure 8.

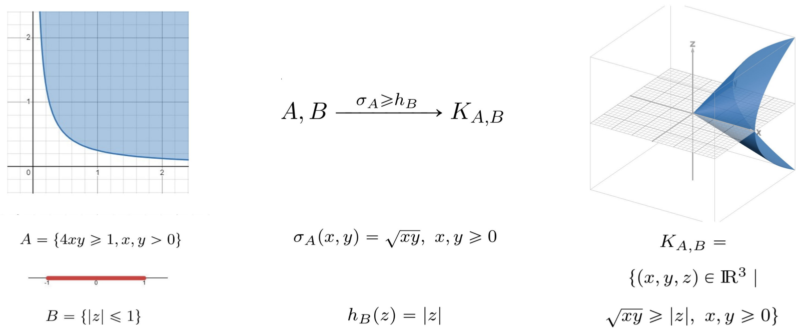


Fig. 8: $K_{A,B}$ cone generated by dual support and support functions

Before we discuss further about the dual cone formula, some properties regarding support and dual support functions will be presented as below.

Lemma 3.1 *If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is concave, $g : E \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex, A is a nonempty closed convex set in $\mathbb{R}^n$, then the following properties hold.*

(a) $\mathbf{0} \in A$ if and only if $h_A(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
(b) $\mathbf{0} \in A$ if and only if $\sigma_A(\mathbf{x}) \leqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
(c) For any $r \in \mathbb{R}$, the sets $\{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \geqslant r\}$ and $\{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) \leqslant r\}$ are convex.

Proof (a) ($\Rightarrow$) Since $\mathbf{0} \in A$, the supremum of $\langle \mathbf{x}, \mathbf{a} \rangle$ must be greater or equal to the value at the $\mathbf{0}$ point. Hence, we have

$$h_A(\mathbf{x}) \geqslant \langle \mathbf{x}, \mathbf{0} \rangle = 0, \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

($\Leftarrow$) Assume, for the sake of contradiction, that $\mathbf{0} \notin A$. Given that A is a nonempty, closed and convex set in $\mathbb{R}^n$ and $\mathbf{0}$ is not an element of A, according to Theorem 3.14 in [12], there exists a unique projection point $\mathbf{p}$ of $\mathbf{0}$. For all $\mathbf{x} \in A$, we have

$$\begin{aligned} &\langle \mathbf{0} - \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle \leqslant 0 \\ \Longrightarrow\ &\langle -\mathbf{p}, \mathbf{x} \rangle \leqslant -\|\mathbf{p}\|^2 < 0 \\ \Longrightarrow\ &h_A(-\mathbf{p}) \leqslant -\|\mathbf{p}\|^2 < 0. \end{aligned}$$

Apparently, this leads to a contradiction.

(b) For all $\mathbf{x} \in \mathbb{R}^n$, we have

$$\sigma_A(\mathbf{x}) = -h_A(-\mathbf{x}) \leqslant 0$$

if and only if $\mathbf{0} \in A$ by applying part (a).

(c) The set $\{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \geqslant r\}$ can be viewed as $\{\mathbf{x} \in \mathbb{R}^n \mid -f(\mathbf{x}) \leqslant -r\}$. Since f is concave, $-f$ is convex, making this set the level set of the convex function $-f$. Therefore, it is a convex set. Moreover, the set $\{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) \leqslant r\}$ is the level set of the convex function g, it is convex as well. □

Using Lemma 3.1, the connection between a pointed convex cone and its dual/polar cone will be presented in below theorem.

Theorem 3.3 *Let K be the pointed convex cone defined by*

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\},$$

where functions F and G satisfy the sufficient conditions in Theorem 2.2. Consider $A = \{\mathbf{x_1} \in \mathbb{R}^m \mid F(\mathbf{x_1}) \geqslant 1\}$, $B = \{\mathbf{x_2} \in \mathbb{R}^n \mid G(\mathbf{x_2}) \leqslant 1\}$. Then, the cone

$$K^D_{A,B} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_A(\mathbf{y_1}) \geqslant h_B(-\mathbf{y_2})\}$$

is the dual cone of K and the cone

$$K^P_{A,B} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_A(-\mathbf{y_1}) \geqslant h_B(\mathbf{y_2})\}$$

is the polar cone of K, where σ_A is the dual support function of A and h_B is the support function of B.

Proof For the sake of analysis need, we write out the dual cone of K as

$$K^* = \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \langle(\mathbf{y_1}, \mathbf{y_2}), (\mathbf{x_1}, \mathbf{x_2})\rangle \geqslant 0,\ \forall (\mathbf{x_1}, \mathbf{x_2}) \in K\}.$$

First, we claim that $K^D_{A,B} \subseteq K^*$. If $(\mathbf{y_1}, \mathbf{y_2}) \in K^D_{A,B}$, for any $(\mathbf{x_1}, \mathbf{x_2}) \in K$, we have $F(\mathbf{x_1}) \geqslant G(\mathbf{x_2}) \geqslant 0$ by condition (a) of K in Theorem 2.2. We will prove $\langle(\mathbf{y_1}, \mathbf{y_2}), (\mathbf{x_1}, \mathbf{x_2})\rangle \geqslant 0$ by discussing two cases.

Case I: $G(\mathbf{x_2}) = 0$. If $F(\mathbf{x_1}) = G(\mathbf{x_2}) = 0$, then $\mathbf{x_1} = \mathbf{x_2} = 0$ by condition (b) of K in Theorem 2.2. This is the trivial case.
If $F(\mathbf{x_1}) > G(\mathbf{x_2}) = 0$, then $\mathbf{x_2} = 0$ by condition (b) of K in Theorem 2.2. Let $\lambda = F(\mathbf{x_1}) > 0$, then $F(\frac{\mathbf{x_1}}{\lambda}) = 1$ due to the homogeneity of F, that is, $\frac{\mathbf{x_1}}{\lambda}$ is in the set A. Hence, we have

$$\begin{aligned}
\langle(\mathbf{y_1}, \mathbf{y_2}), (\mathbf{x_1}, \mathbf{x_2})\rangle &= \langle \mathbf{y_1}, \mathbf{x_1}\rangle + \langle \mathbf{y_2}, \mathbf{x_2}\rangle \\
&= \lambda \langle \mathbf{y_1}, \tfrac{\mathbf{x_1}}{\lambda}\rangle \\
&\geqslant \lambda \sigma_A(\mathbf{y_1}) \\
&\geqslant \lambda h_B(-\mathbf{y_2}),\ \text{since } (\mathbf{y_1}, \mathbf{y_2}) \in K^D_{A,B} \\
&\geqslant 0.
\end{aligned}$$

The last inequality is because $\mathbf{0} \in B$ and by Lemma 3.1 (a).

Case II: $G(\mathbf{x_2}) > 0$. Let $G(\mathbf{x_2}) = k > 0$. Then, $F(\frac{\mathbf{x_1}}{k}) \geqslant G(\frac{\mathbf{x_2}}{k}) = 1$ due to the homogeneity of F and G, which means $\frac{\mathbf{x_1}}{k} \in A$, $\frac{\mathbf{x_2}}{k} \in B$ and

$$\begin{aligned}
\langle(\mathbf{y_1}, \mathbf{y_2}), (\mathbf{x_1}, \mathbf{x_2})\rangle &= \langle \mathbf{y_1}, \mathbf{x_1}\rangle + \langle \mathbf{y_2}, \mathbf{x_2}\rangle \\
&= k\langle \mathbf{y_1}, \tfrac{\mathbf{x_1}}{k}\rangle - k\langle -\mathbf{y_2}, \tfrac{\mathbf{x_2}}{k}\rangle \\
&\geqslant k(\sigma_A(\mathbf{y_1}) - h_B(-\mathbf{y_2})) \\
&\geqslant 0.
\end{aligned}$$

The last inequality comes from $(\mathbf{y_1}, \mathbf{y_2}) \in K^D_{A,B}$.

From the above discussions, we have shown that $K^D_{A,B} \subseteq K^*$.

It remains to claim that $K^* \subseteq K^D_{A,B}$. For any $(\mathbf{y_1}, \mathbf{y_2}) \in K^*$ and $\mathbf{a} \in A$, $\mathbf{b} \in B$, we have $F(\mathbf{a}) \geqslant 1 \geqslant G(\mathbf{b})$, which means $(\mathbf{a}, \mathbf{b}) \in K$. From the definition of dual cone, we know

$$\langle (\mathbf{y_1}, \mathbf{y_2}), (\mathbf{a}, \mathbf{b}) \rangle = \langle \mathbf{y_1}, \mathbf{a} \rangle + \langle \mathbf{y_2}, \mathbf{b} \rangle \geqslant 0$$

which implies

$$\langle \mathbf{y_1}, \mathbf{a} \rangle \geqslant \langle -\mathbf{y_2}, \mathbf{b} \rangle, \ \forall\ \mathbf{a} \in A, \ \forall\ \mathbf{b} \in B.$$

Hence, we obtain $\sigma_A(\mathbf{y_1}) \geqslant h_B(-\mathbf{y_2})$. This says $(\mathbf{y_1}, \mathbf{y_2}) \in K^D_{A,B}$, and hence $K^* \subseteq K^D_{A,B}$.

To sum up, by the above discussion, we have proved that $K^D_{A,B}$ is the dual cone of K.

Moreover, if K° is the polar cone of K, then $(\mathbf{y_1}, \mathbf{y_2}) \in K^\circ = -K^* = -K^D_{A,B}$ if and only if $\sigma_A(-\mathbf{y_1}) \geqslant h_B(-(-\mathbf{y_2})) = h_B(\mathbf{y_2})$. It is equivalent to $(\mathbf{y_1}, \mathbf{y_2}) \in K^P_{A,B}$; hence $K^P_{A,B}$ is the polar cone of K. □

Example 3.4 Consider the circular cone in $\mathbb{R}^n$ defined by

$$\mathcal{L}_\theta = \left\{ (x_1, \mathbf{x_2}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geqslant \|\mathbf{x_2}\| \cot\theta \right\},$$

where $\theta \in (0, \pi/2)$ is its half-aperture angle. Let $A = \{x_1 \in \mathbb{R} \mid x_1 \geqslant 1\}$ and $B = \{\mathbf{x_2} \in \mathbb{R}^{n-1} \mid \|\mathbf{x_2}\| \cot\theta \leqslant 1\}$, the corresponding dual support and support functions are respectively $\sigma_A(y_1) = y_1$ where $y_1 \geqslant 0$ and $h_B(-\mathbf{y_2}) = \|\mathbf{y_2}\| \tan\theta$ where $\mathbf{y_2} \in \mathbb{R}^{n-1}$. By Theorem 3.3, we have the dual cone $\mathcal{L}^*_\theta$ as

$$\mathcal{L}^*_\theta = \left\{ (y_1, \mathbf{y_2}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid y_1 \geqslant \|\mathbf{y_2}\| \tan\theta \right\}.$$

Likewise, $\sigma_A(-y_1) = -y_1$ where $y_1 \leqslant 0$ and $h_B(\mathbf{y_2}) = \|\mathbf{y_2}\| \tan\theta$ where $\mathbf{y_2} \in \mathbb{R}^{n-1}$. Then, as also depicted in Figure 7, the polar cone of $\mathcal{L}_\theta$ is

$$\mathcal{L}^\circ_\theta = \left\{ (y_1, \mathbf{y_2}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid -y_1 \geqslant \|\mathbf{y_2}\| \tan\theta \right\} = -\mathcal{L}^*_\theta.$$

Those cones described in Theorem 3.3 require sufficient conditions as in Theorem 2.2, which does not include all of the convex cones. However, the contribution of Theorem 3.3 is that we can easily employ functions F and G in a cone formula to define convex sets A and B, and then obtain the associated dual support and support function to formulate the dual/polar cone. In the next theorem, we drop the sufficient conditions of the pointed property to find the associated dual/polar cone, even though the process is not so practical as in Theorem 3.3.

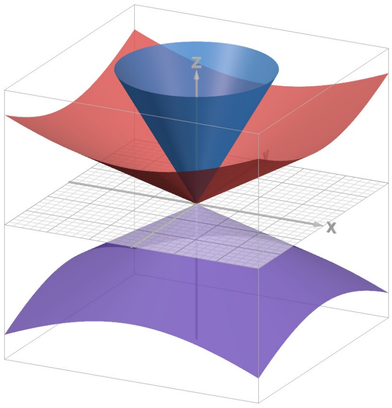


Fig. 9: $\mathcal{L}_\theta, \mathcal{L}_\theta^*$ and $\mathcal{L}_\theta^\circ$ with $\theta = \pi/6$

Theorem 3.4 *Let K be the convex cone defined by*

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\},$$

where functions F and G satisfy conditions in Theorem 2.1. Consider the sets $A(r) \subseteq \mathbb{R}^m$, $B(s) \subseteq \mathbb{R}^n$ defined by

$$\begin{aligned} A(r) &= \{\mathbf{x_1} \in \mathbb{R}^m \mid F(\mathbf{x_1}) \geqslant r\}, \\ B(s) &= \{\mathbf{x_2} \in \mathbb{R}^n \mid G(\mathbf{x_2}) \leqslant s\}, \end{aligned}$$

where r, s are real numbers. For any $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, by choosing a real number $c_\mathbf{x}$ such that $F(\mathbf{x_1}) \geqslant c_\mathbf{x} \geqslant G(\mathbf{x_2})$ and define convex cones $K^D_{c_\mathbf{x}}$ as

$$K^D_{c_\mathbf{x}} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(c_\mathbf{x})}(\mathbf{y_1}) \geqslant h_{B(c_\mathbf{x})}(-\mathbf{y_2})\}.$$

Then, the set $K^D = \bigcap_{\mathbf{x} \in K} K^D_{c_\mathbf{x}}$ is the dual cone of K. Moreover, define convex cones $K^P_{c_\mathbf{x}}$ as

$$K^P_{c_\mathbf{x}} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(c_\mathbf{x})}(-\mathbf{y_1}) \geqslant h_{B(c_\mathbf{x})}(\mathbf{y_2})\}.$$

Then, the set $K^P = \bigcap_{\mathbf{x} \in K} K^P_{c_\mathbf{x}}$ is the polar cone of K.

Proof Since K^D and K^P are the intersection of nonempty convex cones, they are both convex cones. For convenience, we denote K^* be the dual cone of K.

First, we will claim $K^D \subseteq K^*$. For any $(\mathbf{y_1}, \mathbf{y_2}) \in K^D$ and $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, we have

$$\begin{aligned} \langle (\mathbf{y_1}, \mathbf{y_2}), (\mathbf{x_1}, \mathbf{x_2}) \rangle &= \langle \mathbf{y_1}, \mathbf{x_1} \rangle - \langle -\mathbf{y_2}, \mathbf{x_2} \rangle \\ &\geqslant \sigma_{A(c_\mathbf{x})}(\mathbf{y_1}) - h_{B(c_\mathbf{x})}(-\mathbf{y_2}) \\ &\geqslant 0. \end{aligned}$$

The first inequality comes from $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$ and $\mathbf{x_1} \in A(c_\mathbf{x})$, $\mathbf{x_2} \in B(c_\mathbf{x})$; whereas the last inequality is due to $(\mathbf{y_1}, \mathbf{y_2}) \in K^D \subseteq K^D_{c_\mathbf{x}}$. Hence, we prove $K^D \subseteq K^*$.

It remains to verify $K^* \subseteq K^D$. For any $(\mathbf{y_1}, \mathbf{y_2}) \in K^*$ and $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, for all $\mathbf{a} \in A(c_\mathbf{x})$, $\mathbf{b} \in B(c_\mathbf{x})$, we have $F(\mathbf{a}) \geqslant c_\mathbf{x} \geqslant G(\mathbf{b})$ which means $(\mathbf{a}, \mathbf{b}) \in K$. From the definition of dual cone, it yields

$$
\begin{aligned}
&\langle(\mathbf{y_1}, \mathbf{y_2}), (\mathbf{a}, \mathbf{b})\rangle = \langle \mathbf{y_1}, \mathbf{a}\rangle - \langle -\mathbf{y_2}, \mathbf{b}\rangle \geqslant 0 \\
&\Rightarrow \langle \mathbf{y_1}, \mathbf{a}\rangle \geqslant \langle -\mathbf{y_2}, \mathbf{b}\rangle,\ \forall \mathbf{a} \in A(c_\mathbf{x}),\ \forall \mathbf{b} \in B(c_\mathbf{x}) \\
&\Rightarrow \sigma_{A(c_\mathbf{x})}(\mathbf{y_1}) \geqslant h_{B(c_\mathbf{x})}(-\mathbf{y_2}) \\
&\Rightarrow (\mathbf{y_1}, \mathbf{y_2}) \in K^D_{c_\mathbf{x}},\ \forall \mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K \\
&\Rightarrow (\mathbf{y_1}, \mathbf{y_2}) \in K^D
\end{aligned}
$$

Then, we obtain $K^* \subseteq K^D$; hence K^D is the dual cone of K.

Moreover, if K° is the polar cone of K, then $(\mathbf{y_1}, \mathbf{y_2}) \in K^\circ = -K^* = -K^D = \bigcap_{\mathbf{x} \in K} -K^D_{c_\mathbf{x}}$ if and only if

$$
\begin{aligned}
&\sigma_{A(c_\mathbf{x})}(-\mathbf{y_1}) \geqslant h_{B(c_\mathbf{x})}(-(-\mathbf{y_2})),\ \forall \mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K \\
\Longleftrightarrow\ &\sigma_{A(c_\mathbf{x})}(-\mathbf{y_1}) \geqslant h_{B(c_\mathbf{x})}(\mathbf{y_2}),\ \forall \mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K.
\end{aligned}
$$

The last inequality is equivalent to $(\mathbf{y_1}, \mathbf{y_2}) \in K^P$, which says that K^P is the polar cone of K. □

For a practical derivation process to determine the dual cone, we can consider $c_\mathbf{x}$ as an uncertain real number to find the associated dual support and support functions. As we mentioned before, the generalized toppled second-order cone (3.2) does not satisfy Theorem 2.2, so we cannot apply Theorem 3.3 to find its dual cone. Instead, we will do it by using Theorem 3.4 as below.

Example 3.5 Consider the generalized toppled second-order cone defined by

$$
\mathcal{K}^n_{\text{toppled}} = \{(x, y, \mathbf{z}) \in \mathbb{R}^2 \times \mathbb{R}^n \mid \sqrt{xy} \geqslant \|\mathbf{z}\|,\ x \geqslant 0,\ y \geqslant 0\}. \tag{3.2}
$$

For any $\mathbf{x} = (x, y, \mathbf{z}) \in \mathcal{K}^n_{\text{toppled}}$, there exists real number $c_\mathbf{x}$ such that

$$
\sqrt{xy} \geqslant c_\mathbf{x} \geqslant \|\mathbf{z}\|,
$$

where $c_\mathbf{x} \geqslant 0$. Let the sets $A(c_\mathbf{x}) \subseteq \mathbb{R}^2$, $B(c_\mathbf{x}) \subseteq \mathbb{R}^n$ be defined by

$$
A(c_\mathbf{x}) = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{xy} \geqslant c_\mathbf{x},\ x, y \geqslant 0\},\ B(c_\mathbf{x}) = \{\mathbf{z} \in \mathbb{R}^n \mid \|\mathbf{z}\| \leqslant c_\mathbf{x}\}.
$$

Then, we have

$$
\sigma_{A(c_\mathbf{x})}(x, y) = 2c_\mathbf{x}\sqrt{xy} \text{ when } x, y \geqslant 0,\ h_{B(c_\mathbf{x})}(-\mathbf{z}) = c_\mathbf{x}\|\mathbf{z}\|.
$$

Dividing by $c_\mathbf{x}$ on both sides of the following inequality

$$
2c_\mathbf{x}\sqrt{xy} \geqslant c_\mathbf{x}\|\mathbf{z}\|,\ x, y \geqslant 0
$$

yields

$$2\sqrt{xy} \geqslant \|\mathbf{z}\|, \ \forall \mathbf{x} \in \mathcal{K}^n_{\text{toppled}}$$

From Theorem 3.4, every $K^D_{c_\mathbf{x}}$ has the same formula as above. Hence, we obtain the dual cone formula as the following expression.

$$\mathcal{K}^{n*}_{\text{toppled}} = \{(x, y, \mathbf{z}) \in \mathbb{R}^2 \times \mathbb{R}^n \mid 2\sqrt{xy} \geqslant \|\mathbf{z}\|, \ x \geqslant 0, \ y \geqslant 0\}. \tag{3.3}$$

Please refer to the graph (Figure 10) of the dual cone (3.3) below. Apparently, the dual cone (3.3) is not identical to (3.2), it is not self-dual under the standard inner product of the $\mathbb{R}^{n+2}$ space. However, the authors in paper [2] provided another version of inner product to make it to be self-dual. That is another perspective.

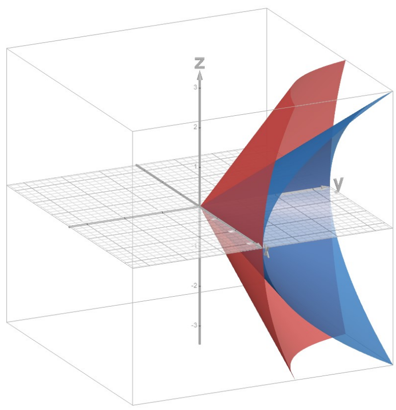


Fig. 10: The toppled second-order cone and its dual cone in $\mathbb{R}^3$

4 Future Works

Apparently, there are a few research directions worthy of further investigation as long as we have new ways to construct cones. We list some possible future works based on the bricks of this article.

- Explore cone decomposition, cone convexity, cone monotonicity, and Jordan product associated with the constructed cones. These will be foundations for analyzing the corresponding conic optimization problems.
- Look for real applications which fit in the cone structures generated by the proposed ways in this study.
- Analyze the KKT conditions and design solution methods for the generated conic optimization problems.

– Discover key features and establish important inequalities from the cone structures.
– Establish error bounds for conic feasibility problems based on various conic structures.

5 Appendix

5.1 The calculations of support functions and dual support functions

There are two approaches to achieving the associated support or dual support function for a closed convex set. In fact, it is essentially an optimization problem. To obtain the support function $h_A(\mathbf{x})$, we consider a fixed vector $\mathbf{x}$ in $\mathbb{R}^n$ and the following supremum optimization problem:

$$\begin{aligned} \sup\ &\langle \mathbf{x}, \mathbf{a} \rangle \\ \text{subject to } &\mathbf{a} \in A \end{aligned} \tag{5.1}$$

The solution of (5.1) will yield the value of $h_A(\mathbf{x})$. For example, if the set A is defined by a real-valued convex function f as $A = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \leqslant 1\}$, then the constraint becomes $f(\mathbf{a}) \leqslant 1$ to evaluate the supremum of $\langle \mathbf{x}, \mathbf{a} \rangle$. The first approach is employing algebraic or analytical calculation techniques to derive the result. If this approach proves challenging, an alternative method involves considering the problem from a geometric perspective. First, $\langle \mathbf{x}, \mathbf{a} \rangle = \|\mathbf{x}\| \cdot \|\mathbf{a}\| \cos\theta$ where θ is the angle between vectors $\mathbf{x}$ and $\mathbf{a}$. Because $\mathbf{x}$ is fixed and $\|\mathbf{a}\| \cos\theta$ is the length of projection from $\mathbf{a}$ in A to the vector $\mathbf{x}$, the supremum of $\langle \mathbf{x}, \mathbf{a} \rangle$ will depend on the supremum of $\|\mathbf{a}\| \cos\theta$ where $\mathbf{a} \in A$. Geometrically, as an example, refer to Figure 11 below. The supremum of $\|\mathbf{a}\| \cos\theta$ is attained when $\mathbf{a}$ lies on the boundary of A, and the supporting hyperplane at $\mathbf{a}$ is perpendicular to $\mathbf{x}$. Thus, we can obtain the supremum of $\langle \mathbf{x}, \mathbf{a} \rangle$ by identifying such a vector $\mathbf{a}$. The same approaches can be applied on the dual support function by replacing supremum with infimum.

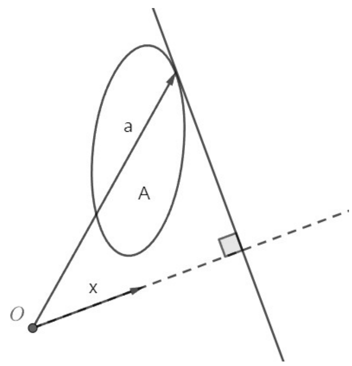


Fig. 11: A fixed vector $\mathbf{x}$ and the supremum vector $\mathbf{a}$

.

Next, we will derive the support functions and dual support functions presented in this paper as follows.

1. Examples following Definition 3.1.
   - $A = \{\mathbf{a}\}$ a singleton. $h_A(\mathbf{x}) = \sup\{\langle \mathbf{x}, \mathbf{a}\rangle\} = \langle \mathbf{x}, \mathbf{a}\rangle$ clearly.
   - The Euclidean ball $B = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y}\|_2 \leqslant 1\}$. For any $\mathbf{y} \in B$, $\langle \mathbf{x}, \mathbf{y}\rangle \leqslant \|\mathbf{x}\|\|\mathbf{y}\| \leqslant \|\mathbf{x}\|$. Hence $h_B(\mathbf{x}) = \|\mathbf{x}\|_2$.
   - If C is a line segment through the origin with endpoints $-\mathbf{a}$ and $\mathbf{a}$, we observe that if $\mathbf{x}$ is closer to $\mathbf{a}$, then $h_C(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a}\rangle$; otherwise, $h_C(\mathbf{x}) = \langle \mathbf{x}, -\mathbf{a}\rangle = -\langle \mathbf{x}, \mathbf{a}\rangle$. Hence $h_C(\mathbf{x}) = |\langle \mathbf{x}, \mathbf{a}\rangle|$. Please refer to Figure 12.

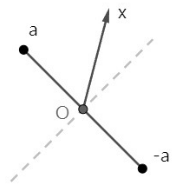


Fig. 12: A line segment with $\mathbf{x}$ is closer to $\mathbf{a}$

.

2. Example 3.1, the p-order cones in $\mathbb{R}^3$. Let's take a shortcut. The boundary of the set A in the following cases is symmetric with respect to the x-axis, the y-axis and the origin. It is evident that if $(x, y) \in \mathbb{R}^2$ is in one of the four quadrants, the point (a, b) that satisfies $h_A(x, y) = \langle (x, y), (a, b)\rangle$ lies in the same quadrant. Additionally, the role of the point (a, b) maintains symmetry with respect to the x-axis, the y-axis and the origin. For example, considering symmetry with respect to the y-axis, if (a, b) is the point that satisfies $h_A(x, y) = \langle (x, y), (a, b)\rangle$, then $(-a, b)$ will be the point that satisfies $h_A(-x, y) = \langle (-x, y), (-a, b)\rangle = h_A(x, y)$. It turns out that we can replace x with $|x|$ in the function $h_A(x, y)$ due to symmetry. The same rule applies to symmetry with respect to the x-axis and the origin. Therefore, considering the first quadrant case is sufficient in this example by symmetry.

   - Given $A = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\|_1 = |x| + |y| \leqslant 1\}$, consider any $(a, b) \in A$ in the first quadrant, $a + b \leqslant 1, 0 \leqslant a \leqslant 1, 0 \leqslant b \leqslant 1$. Now fix a vector $(x, y),\ x \geqslant 0, y \geqslant 0$, we have

$$\begin{aligned}\langle (x, y), (a, b)\rangle &= xa + yb \leqslant xa + y(1 - a) = (x - y)a + y \\ &\leqslant \begin{cases} (x - y) \cdot 1 + y = x, \text{ if } x \geqslant y \\ (x - y) \cdot 0 + y = y, \text{ if } x \leqslant y \end{cases} = \max\{x, y\}.\end{aligned}$$

     By symmetry, we have $h_A(x, y) = \max\{|x|, |y|\} = \|(x, y)\|_\infty$.
   - Given $A = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\|_2 = \sqrt{x^2 + y^2} \leqslant 1\}$, this is a special case of the Euclidean ball. $h_A(x, y) = \|(x, y)\|_2$.
   - Given $A = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\|_p \leqslant 1\}$, we consider the first quadrant case and use the second approach from the geometric point of view.

Suppose (a,b) is the point that satisfies $h_A(x,y) = \langle (x,y),(a,b)\rangle$ for a fixed (x,y) in the first quadrant, we have

$$a^p + b^p = 1 \tag{5.2}$$

The tangent line at (a,b), where $b \neq 0$, with the slope $-\frac{a^{p-1}}{b^{p-1}}$, is perpendicular to the vector (x,y), where $x \neq 0$. It leads to

$$-\frac{a^{p-1}}{b^{p-1}} \cdot \frac{y}{x} = -1 \tag{5.3}$$

From (5.2), $b = (1-a^p)^{\frac{1}{p}}$, $a = (1-b^p)^{\frac{1}{p}}$. By taking b, a into (5.3) respectively, and using the relation $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$a = \frac{x^{\frac{q}{p}}}{(x^q + y^q)^{\frac{1}{p}}}, \; b = \frac{y^{\frac{q}{p}}}{(x^q + y^q)^{\frac{1}{p}}}.$$

Then,

$$h_A(x,y) = xa + yb = \frac{x^{\frac{q}{p}+1} + y^{\frac{q}{p}+1}}{(x^q + y^q)^{\frac{1}{p}}} = \frac{x^q + y^q}{(x^q + y^q)^{\frac{1}{p}}} = (x^q + y^q)^{\frac{1}{q}}.$$

Notice that $b = 0$ is a special case of $h_A(x,0) = x$, while $x = 0$ is the special case of $h_A(0,y) = y$. By symmetry, $h_A(x,y) = (|x|^q + |y|^q)^{\frac{1}{q}} = \|(x,y)\|_q$.

– Given $A = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\|_\infty = \max\{|x|,|y|\} \leqslant 1\}$. Consider the first quadrant case, for any $(a,b) \in A$, $\max\{a,b\} \leqslant 1$ and $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$, and a fixed $(x,y) \in \mathbb{R}^2_+$, it is easy to see that

$$\langle (x,y),(a,b)\rangle = xa + yb \leqslant x + y.$$

By symmetry, $h_A(x,y) = |x| + |y| = \|(x,y)\|_1$.

3. Example 3.2, $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y \leqslant 1\}$. For any $(a,b) \in A$ with $a^2 + b \leqslant 1$, and a fixed vector (x,y) with $y \neq 0$, we have

$$\begin{aligned}\langle (x.y),(a,b)\rangle &= xa + yb \leqslant xa + y(1-a^2) = -ya^2 + xa + y \\ &= -y(a - \tfrac{x}{2y})^2 + y + \tfrac{x^2}{4y} \\ &\leqslant y + \tfrac{x^2}{4y}, \text{ if } y > 0.\end{aligned} \tag{5.4}$$

If $y < 0$, the supremum value of above equation will approach infinity. Hence we have $h_A(x,y) = y + \frac{x^2}{4y}$, $y > 0$. It is evident from (5.4) that the supremum occurs when $a = \frac{x}{2y}$, where the slope is $-2a = -\frac{x}{y}$. This is consistent with the geometric approach, where the tangent line passing through the supremum point is perpendicular to the vector (x,y).

4. Examples following Definition 3.2.
   – $A = \{x \in \mathbb{R} \mid x \geqslant 1\}$. For any $a \in A$ with $a \geqslant 1$, it is evident that $\langle x,a\rangle = xa \geqslant x$ if $x \geqslant 0$; the infimum approaches negative infinity if $x < 0$. Hence $\sigma_A(x) = x$, $x \geqslant 0$.

– The Euclidean unit ball $B = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y}\|_2 \leqslant 1\}$. For any $\mathbf{y} \in B$ with $\|\mathbf{y}\| \leqslant 1$, we have $\langle \mathbf{x}, \mathbf{y} \rangle \geqslant -\|\mathbf{x}\|\|\mathbf{y}\| \geqslant -\|\mathbf{x}\|$. Hence $\sigma_B(\mathbf{x}) = -\|\mathbf{x}\|$.
– $A = \{(x, y) \in \mathbb{R}^2 \mid 4xy \geqslant 1,\ x, y > 0\}$. For any $(a, b) \in A$ with $4ab \geqslant 1$ and $a, b > 0$, and a fixed (x, y) with $x, y > 0$, we have

$$\begin{aligned}\langle (x,y),(a,b)\rangle &= xa + yb \geqslant xa + \tfrac{y}{4a}\\ &= \frac{x(a - \frac{\sqrt{y}}{2\sqrt{x}})^2}{a} + \sqrt{xy}.\end{aligned}$$

The infimum of the above equation occurs when $a = \frac{\sqrt{y}}{2\sqrt{x}}$ and the slope at this point is $\frac{-1}{4a^2} = -\frac{x}{y}$. Notice that $x = 0$ is the special case where $\langle (0, y), (a, b)\rangle = yb \geqslant 0$ for $y \geqslant 0$ and $a, b > 0$, while $y = 0$ is the special cases where $\langle (x, 0), (a, b)\rangle = xa \geqslant 0$ for $x \geqslant 0$ and $a, b > 0$. Hence, $\sigma_A(x, y) = \sqrt{xy},\ x, y \geqslant 0$.

5. Example 3.3. The dual support function of A and support function of B in this example have been included in item 4 and item 1 above.

6. Example 3.4. $A = \{x_1 \in \mathbb{R} \mid x_1 \geqslant 1\}$. The dual support function of A is the same as in item 4, where $\sigma_A(y_1) = y_1,\ y_1 \geqslant 0$. As to the set $B = \{\mathbf{x_2} \in \mathbb{R}^{n-1} \mid \|\mathbf{x_2}\| \cot\theta \leqslant 1\}$, for $\mathbf{b} \in B$ with $\|\mathbf{b}\| \cot\theta \leqslant 1$, and a fixed $\mathbf{y_2} \in \mathbb{R}^{n-1}$, we have

$$\langle \mathbf{y_2}, \mathbf{b}\rangle \leqslant \|\mathbf{y_2}\|\|\mathbf{b}\| \leqslant \|\mathbf{y_2}\| \tan\theta.$$

Hence, $h_B(\mathbf{y_2}) = \|\mathbf{y_2}\| \tan\theta,\ \mathbf{y_2} \in \mathbb{R}^{n-1}$. It is evident that $h_B(-\mathbf{y_2}) = \|-\mathbf{y_2}\| \tan\theta = \|\mathbf{y_2}\| \tan\theta$, where $-\mathbf{y_2} \in \mathbb{R}^{n-1}$, i.e., $\mathbf{y_2} \in \mathbb{R}^{n-1}$. Additionally, $h_A(-y_1) = -y_1$ where $-y_1 \geqslant 0$, i.e, $y_1 \leqslant 0$.

7. Example 3.5. The sets in this example are defined by

$$A(c_{\mathbf{x}}) = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{xy} \geqslant c_{\mathbf{x}},\ x, y \geqslant 0\}, \quad B(c_{\mathbf{x}}) = \{\mathbf{z} \in \mathbb{R}^n \mid \|\mathbf{z}\| \leqslant c_{\mathbf{x}}\},$$

where $c_{\mathbf{x}} \geqslant 0$. For any $(a, b) \in A(c_{\mathbf{x}})$ with $\sqrt{ab} \geqslant c_{\mathbf{x}},\ a > 0, b \geqslant 0$, and a fixed vector (x, y), where $x > 0, y \geqslant 0$, we have

$$\langle (x,y),(a,b)\rangle = xa + yb \geqslant xa + y \cdot \frac{c_{\mathbf{x}}^2}{a} = \frac{x(a - \frac{c_{\mathbf{x}}\sqrt{y}}{\sqrt{x}})^2}{a} + 2c_{\mathbf{x}}\sqrt{xy}.$$

The infimum of above equation occurs when $a = \frac{c_{\mathbf{x}}\sqrt{y}}{\sqrt{x}}$, and the value is $\sigma_{A(c_{\mathbf{x}})}(x, y) = 2c_{\mathbf{x}}\sqrt{xy}$, where $x > 0$ and $y \geqslant 0$. Notice that $a = 0$ is the special case when $c_{\mathbf{x}} = 0$, where $\langle (x, y), (0, b)\rangle = yb \geqslant 0,\ x, y \geqslant 0$ and $b \geqslant 0$; while $x = 0$ is the special case where $\langle (0, y), (a, b)\rangle = yb \geqslant 0$ when $y \geqslant 0$ and $a, b \geqslant 0$. Hence $\sigma_{A(c_{\mathbf{x}})}(x, y) = 2c_{\mathbf{x}}\sqrt{xy},\ x, y \geqslant 0$. As to $B(c_{\mathbf{x}})$, for any $\mathbf{b} \in B(c_{\mathbf{x}})$ with $\|\mathbf{b}\| \leqslant c_{\mathbf{x}}$, and a fixed $\mathbf{z} \in \mathbb{R}^n$, we have

$$\langle \mathbf{z}, \mathbf{b}\rangle \leqslant \|\mathbf{z}\|\|\mathbf{b}\| \leqslant c_{\mathbf{x}}\|\mathbf{z}\|.$$

Hence $h_{B(c_{\mathbf{x}})}(\mathbf{z}) = c_{\mathbf{x}}\|\mathbf{z}\|$. It is evident that $h_{B(c_{\mathbf{x}})}(-\mathbf{z}) = c_{\mathbf{x}}\| - \mathbf{z}\| = c_{\mathbf{x}}\|\mathbf{z}\|$.

5.2 Two extended cones

We point out that there are additional cones mentioned in the literature that are not included in this paper. Some of these cones fit our settings, while others may not. In this subsection, we look into some of these cones for reference.

The monotone extended second-order cone $\mathcal{L}_{p,q}$ was first introduced by O.P. Ferreira, Y. Gao and S. Z. Németh in 2022 [9], it also appeared in [11]. It is defined by

$$\mathcal{L}_{p,q} = \{(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^p \times \mathbb{R}^q \mid x_1 \geqslant x_2 \geqslant \cdots \geqslant x_p \geqslant \|\mathbf{u}\| \}, \tag{5.5}$$

where $p, q \geqslant 1$ and $\mathbf{x} = (x_1, x_2, \cdots .x_p)^T \in \mathbb{R}^p$. If $p = 1$, it degenerates to the second-order cone in $\mathbb{R}^{q+1}$. If $q = 0$, the cone $\mathcal{L}_{p,q}$ becomes the monotone nonnegative cone in [11,5] as below:

$$\mathcal{C}_p = \{\mathbf{x} \in \mathbb{R}^p \mid x_1 \geqslant x_2 \geqslant \cdots \geqslant x_p \geqslant 0 \}.$$

In our setting, we define $F : D \subseteq \mathbb{R}^p \to \mathbb{R}$ and $G : \mathbb{R}^q \to \mathbb{R}$ by

$$F(\mathbf{x}) = x_p, \ \ G(\mathbf{u}) = \|\mathbf{u}\|,$$

where $D = \{\mathbf{x} \in \mathbb{R}^p \mid x_1 \geqslant x_2 \geqslant \cdots \geqslant x_p \geqslant 0 \}$. Obviously, F,G are positively homogeneous, while F is additive and G is subadditive. Additionally, D is closed and F, G are continuous. Hence, by applying Theorem 2.1, $\mathcal{L}_{p,q}$ is a closed convex cone. Moreover, we can apply Theorem 3.4 to obtain its dual cone by considering closed convex sets

$$A(c_{\mathbf{z}}) = \{\mathbf{x} \in \mathbb{R}^p \mid x_p \geqslant c_{\mathbf{z}}, \ x_1 \geqslant x_2 \geqslant \cdots \geqslant x_p\}, \quad B(c_{\mathbf{z}}) = \{\mathbf{u} \in \mathbb{R}^q \mid \|\mathbf{u}\| \leqslant c_{\mathbf{z}}\},$$

where $\mathbf{z} = (\mathbf{x}, \mathbf{u})$, $c_{\mathbf{z}} \geqslant 0$. For any $\mathbf{a} = (a_1, a_2, \cdots, a_p)^T \in A(c_{\mathbf{z}})$, and a fixed $\mathbf{y} = (y_1, y_2, \cdots, y_p)^T \in \mathbb{R}^p$, we have

$$\begin{aligned}
\langle \mathbf{y}, \mathbf{a} \rangle &= \textstyle\sum_{i=1}^p y_i a_i \\
&\geqslant \textstyle\sum_{i=1}^2 y_i a_2 + \sum_{i=3}^p y_i a_i, \ \text{if } y_1 \geqslant 0 \\
&\geqslant \textstyle\sum_{i=1}^3 y_i a_3 + \sum_{i=4}^p y_i a_i, \ \text{if } y_1 + y_2 \geqslant 0 \\
&\geqslant \cdots \\
&\geqslant \textstyle\sum_{i=1}^{p-1} y_i a_{p-1} + y_p a_p, \ \text{if } \sum_{i=1}^{p-2} y_i \geqslant 0 \\
&\geqslant \textstyle\sum_{i=1}^p y_i a_p, \ \text{if } \sum_{i=1}^{p-1} y_i \geqslant 0 \\
&\geqslant c_{\mathbf{z}} \textstyle\sum_{i=1}^p y_i
\end{aligned}$$

It follows that $\sigma_{A(c_{\mathbf{z}})}(\mathbf{y}) = c_{\mathbf{z}} \sum_{i=1}^p y_i$ when $\sum_{i=1}^j y_i \geqslant 0, \ j = 1, 2, \cdots, p-1$. From Section 5.1, item 7, we have $h_{B(c_{\mathbf{z}})}(-\mathbf{v}) = c_{\mathbf{z}}\|\mathbf{v}\|, \ \mathbf{v} \in \mathbb{R}^q$. Hence, by Theorem 3.4, the dual cone of $\mathcal{L}_{p,q}$ becomes

$$\mathcal{L}^*_{p,q} = \{(\mathbf{y}, \mathbf{v}) \in \mathbb{R}^p \times \mathbb{R}^q \mid \sum_{i=1}^p y_i \geqslant \|\mathbf{v}\|, \ \sum_{i=1}^j y_i \geqslant 0, \ j = 1, 2, \cdots, p-1\}.$$

The above dual cone formula is the same as in [9, Proposition 2].

Another kind of extended second-order cone, which was named the extended Lorentz cone, was first revealed by S.Z. Németh and G. Zhang in 2015 [20]. This cone has been extensively studied in subsequent research papers [19, 21–23]. The extended Lorentz cone is defined by

$$\mathcal{L}(p,q) = \{(\mathbf{x},\mathbf{u}) \in \mathbb{R}^p \times \mathbb{R}^q \mid \mathbf{x} \geqslant \|\mathbf{u}\|\mathbf{e}\}, \tag{5.6}$$

where $\mathbf{e} = (1,1,\cdots,1)^T \in \mathbb{R}^p$. In equation (5.6), the authors employ the componentwise inequality. If $p = 1$, $\mathcal{L}(p,q)$ is the $(q+1)$-dimensional second-order cone. It appears that this cone is incompatible with our settings, as the corresponding functions adjacent to the inequality in (5.6) are not real-valued. However, we can approach the problem from this perspective. Let $\mathbf{x} = (x_1, x_2, \cdots, x_p)^T \in \mathbb{R}^p$, the cone $\mathcal{L}(p,q)$ (5.6) is equivalent to

$$\mathcal{L}(p,q) = \{(\mathbf{x},\mathbf{u}) \in \mathbb{R}^p \times \mathbb{R}^q \mid \min_i\{x_i\} \geqslant \|\mathbf{u}\|\}.$$

We define $F : \mathbb{R}^p \to \mathbb{R}$ and $G : \mathbb{R}^q \to \mathbb{R}$ by $F(\mathbf{x}) = \min_i\{x_i\}$ and $G(\mathbf{u}) = \|\mathbf{u}\|$, then F is concave, G is convex and both F and G are continuous, satisfying the conditions in Theorem 2.1. Therefore, $\mathcal{L}(p,q)$ is a closed convex cone. The same approach can be used to obtain its dual cone by considering the following closed convex sets:

$$A(c_{\mathbf{z}}) = \{\mathbf{x} \in \mathbb{R}^p \mid \min_i\{x_i\} \geqslant c_{\mathbf{z}}\},$$

$$B(c_{\mathbf{z}}) = \{\mathbf{u} \in \mathbb{R}^q \mid \|\mathbf{u}\| \leqslant c_{\mathbf{z}}\},$$

where $\mathbf{z} = (\mathbf{x},\mathbf{u})$, $c_{\mathbf{z}} \geqslant 0$. Using the same technique, for any $\mathbf{a} = (a_1, a_2, \cdots, a_p)^T \in A(c_{\mathbf{z}})$ and a fixed $\mathbf{x} = (x_1, x_2, \cdots, x_p)^T \in \mathbb{R}^p$, we have

$$\langle \mathbf{x}, \mathbf{a} \rangle = \sum_{i=1}^{p} x_i a_i \geqslant c_{\mathbf{z}} \sum_{i=1}^{p} x_i, \text{ if } x_i \geqslant 0, \ i = 1, 2, \cdots, p.$$

Hence, $\sigma_{A(c_{\mathbf{z}})}(\mathbf{x}) = c_{\mathbf{z}} \sum_{i=1}^{p} x_i, \ x_i \geqslant 0, \ i = 1, 2, \cdots, p$. Again, $h_{B(c_{\mathbf{z}})}(-\mathbf{u}) = c_{\mathbf{z}}\|\mathbf{u}\|, \mathbf{u} \in \mathbb{R}^q$. By Theorem 3.4, the dual cone of $\mathcal{L}(p,q)$ becomes

$$\mathcal{M}(p,q) = \{(\mathbf{x},\mathbf{u}) \in \mathbb{R}^p \times \mathbb{R}^q \mid \sum_{i=1}^{p} x_i \geqslant \|\mathbf{u}\|, \ x_i \geqslant 0, \ i = 1, 2, \cdots, p\}.$$

The above formula is equivalent to

$$\mathcal{M}(p,q) = \{(\mathbf{x},\mathbf{u}) \in \mathbb{R}^p \times \mathbb{R}^q \mid \langle \mathbf{x}, \mathbf{e} \rangle \geqslant \|\mathbf{u}\|, \ \mathbf{x} \geqslant 0\},$$

which is the same as in [20, (6)].

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