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Improved approach for obtaining dual cones and extended construction for convex cones

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Abstract In our previous work [8], we proposed two novel mechanisms for generating closed convex cones: one based on systems of inequalities and the other on support functions, together with explicit characterizations of the associated dual and polar cones. Building upon these results, the present paper develops a strengthened and more systematic approach for deriving dual and polar cones from given convex cones. In addition, we establish a significantly extended and unified framework for the construction of convex cones, which broadens the scope of applicability of existing methods and provides greater structural flexibility.

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1 Introduction

This is a continued work for the previous paper titled as "Novel constructions for closed convex cones through inequalities and support functions" [8]. In our previous work, we established the novel constructions for closed convex cones through inequalities by two real-valued functions with homogeneity and super/sub-additivity. We also obtained the dual/polar cones of convex cones by using their associated dual support/support functions of associated closed convex sets. The dual cone is not just a theoretical construct; it is the backbone of several mathematical disciplines. For example, in conic programming the dual cone is used to formulate the dual problem [1,4,5]. In machine learning, dual cones are often used to characterize gradient directions or support vectors under constraints [3,9]. In the topics of variational inequalities and equilibrium, dual cones can be used to define KKT conditions on manifolds, or transform the complementarity condition into a smooth function, to solve the optimization or complementarity problem [6,7].

This paper establishes two main advances on this topic. We first develop an improved method for deriving the dual and polar cones associated with a given convex cone. We then extend our construction framework, significantly enlarging the class of convex cones that can be generated within this setting.

The remainder of this paper is organized as follows. Section 2 establishes the foundational theory for constructing closed convex cones via inequalities and support functions, and presents the key results that underpin all subsequent developments. In Section 3, we significantly strengthen the characterization of dual and polar cones by exploiting the homogeneity of their associated real-valued functions, and substantiate the theoretical findings with illustrative examples. Finally, Section 4 proposes a substantially extended framework for constructing closed convex cones, which generalizes and unifies our earlier constructions and is directly motivated by the insights obtained in Section 3.

2 Preliminaries

This section recalls the fundamental concepts and results developed in our previous work [8], which are indispensable for the subsequent analysis and extensions introduced in this paper.

As defined in [8], let K be the nonempty set given by

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\}, \tag{2.1}$$

where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}$ and $G : E \subseteq \mathbb{R}^n \to \mathbb{R}$ are two real-valued functions, with the sets D and E being subsets of the domains of F and G, respectively.

Definition 2.1 A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a homogeneous function of degree k (k is an integer) if it satisfies

$$f(\lambda \mathbf{x}) = \lambda^k f(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and all $\lambda \neq 0$. If not specifically indicated, f is homogeneous means that f is homogeneous of degree 1. In this paper, we call f is positively homogeneous if $\lambda > 0$. We also say that functions F and G are positively homogeneous of degree k in K given as in (2.1) if $(\mathbf{x_1}, \mathbf{x_2}) \in K$, there hold $F(\lambda \mathbf{x_1}) = \lambda^k F(\mathbf{x_1})$ and $G(\lambda \mathbf{x_2}) = \lambda^k G(\mathbf{x_2})$.

Definition 2.2 A function $f : D \subseteq \mathbb{R}^m \to \mathbb{R}$ is called superadditive if it satisfies

$$f(\mathbf{x} + \mathbf{y}) \geqslant f(\mathbf{x}) + f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y}$ in D. A function $g : E \subseteq \mathbb{R}^n \to \mathbb{R}$ is called subadditive if it satisfies

$$g(\mathbf{x} + \mathbf{y}) \leqslant g(\mathbf{x}) + g(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y}$ in E. In this paper, we say that F is superadditive and G is subadditive in K (2.1) if, for any $(\mathbf{x_1}, \mathbf{x_2}), (\mathbf{y_1}, \mathbf{y_2}) \in K$, we have $F(\mathbf{x_1} + \mathbf{y_1}) \geqslant F(\mathbf{x_1}) + F(\mathbf{y_1})$ and $G(\mathbf{x_2} + \mathbf{y_2}) \leqslant G(\mathbf{x_2}) + G(\mathbf{y_2})$.

Theorem 2.1 *Suppose that the set K (2.1) satisfies conditions:*

(i) F and G are positively homogeneous (of degree 1) in K,
(ii) F is superadditive (concave) and G is subadditive (convex) in K.

Then, the following hold.

(1) K is a convex cone.
(2) $\mathrm{cl}(K)$ is a closed convex cone.
(3) If both F and G are continuous and both D and E are closed, then $K = \mathrm{cl}(K)$ is a closed convex cone.

Theorem 2.2 *Let the convex cone*

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\},$$

where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}$ is positively homogeneous and superadditive (or concave) in K and $G : E \subseteq \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous and subadditive (or convex) in K and satisfy the following two conditions:

(a) $G(\mathbf{x_2}) \geqslant 0$ for any $(\mathbf{x_1}, \mathbf{x_2}) \in K$.
(b) If $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, then $F(\mathbf{x_1}) = 0$ implies $\mathbf{x_1} = \mathbf{0}$ and $G(\mathbf{x_2}) = 0$ implies $\mathbf{x_2} = \mathbf{0}$.

Then, K is pointed.

Definition 2.3 The support function $h_A : E \subseteq \mathbb{R}^n \to \mathbb{R}$ of a nonempty closed convex set A in $\mathbb{R}^n$ is given by

$$h_A(\mathbf{x}) = \sup\{\langle \mathbf{x}, \mathbf{a} \rangle \mid \mathbf{a} \in A\},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$. Note that, for simplicity, we only consider the domain where h_A is finite.

Definition 2.4 If A is a nonempty closed convex set in $\mathbb{R}^n$, the corresponding dual support function $\sigma_A : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\sigma_A(\mathbf{x}) := \inf\{\langle \mathbf{x}, \mathbf{a}\rangle \mid \mathbf{a} \in A\},$$

where $\langle \cdot, \cdot \rangle$ represents the inner product in $\mathbb{R}^n$. As with the support function, for simplicity, we only consider the domain where σ_A is finite.

Theorem 2.3 *Let A be a nonempty closed convex set in $\mathbb{R}^m$ and $\sigma_A : D \subseteq \mathbb{R}^m \to \mathbb{R}$ be the associated dual support function with A, that is,*

$$\sigma_A(\mathbf{z}) = \inf\{\langle \mathbf{z}, \mathbf{a}\rangle \mid \mathbf{a} \in A\}.$$

Let B be a nonempty closed convex set in $\mathbb{R}^n$ and $h_B : E \subseteq \mathbb{R}^n \to \mathbb{R}$ be the associated support function with B, that is,

$$h_B(\mathbf{z}) = \sup\{\langle \mathbf{z}, \mathbf{b}\rangle \mid \mathbf{b} \in B\}.$$

The set $K_{A,B}$ is defined by

$$K_{A,B} := \{(\mathbf{z_1}, \mathbf{z_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_A(\mathbf{z_1}) \geqslant h_B(\mathbf{z_2})\}.$$

Then, the following hold.

(1) $K_{A,B}$ is a convex cone.
(2) $\mathrm{cl}(K_{A,B})$ is a closed convex cone.
(3) If both σ_A and h_B are continuous and both D and E are closed, then $K_{A,B} = \mathrm{cl}(K_{A,B})$ is a closed convex cone.

Theorem 2.4 *Let K be the pointed convex cone defined by*

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\},$$

where functions F and G satisfy the sufficient conditions in Theorem 2.2. Consider $A = \{\mathbf{x_1} \in \mathbb{R}^m \mid F(\mathbf{x_1}) \geqslant 1\}$, $B = \{\mathbf{x_2} \in \mathbb{R}^n \mid G(\mathbf{x_2}) \leqslant 1\}$. Then, the cone

$$K^D_{A,B} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_A(\mathbf{y_1}) \geqslant h_B(-\mathbf{y_2})\}$$

is the dual cone of K and the cone

$$K^P_{A,B} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_A(-\mathbf{y_1}) \geqslant h_B(\mathbf{y_2})\}$$

is the polar cone of K, where σ_A is the dual support function of A and h_B is the support function of B.

Theorem 2.5 *Let K be the convex cone defined by*

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\},$$

where functions F and G satisfy conditions in Theorem 2.1. Consider the sets $A(r) \subseteq \mathbb{R}^m$, $B(s) \subseteq \mathbb{R}^n$ defined by

$$A(r) = \{\mathbf{x_1} \in \mathbb{R}^m \mid F(\mathbf{x_1}) \geqslant r\},$$
$$B(s) = \{\mathbf{x_2} \in \mathbb{R}^n \mid G(\mathbf{x_2}) \leqslant s\},$$

where r, s are real numbers. For any $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, by choosing a real number $c_{\mathbf{x}}$ such that $F(\mathbf{x_1}) \geqslant c_{\mathbf{x}} \geqslant G(\mathbf{x_2})$ and defining convex cones $K^D_{c_{\mathbf{x}}}$ as

$$K^D_{c_{\mathbf{x}}} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(c_{\mathbf{x}})}(\mathbf{y_1}) \geqslant h_{B(c_{\mathbf{x}})}(-\mathbf{y_2})\}.$$

Then, the set $K^D = \bigcap_{\mathbf{x} \in K} K^D_{c_{\mathbf{x}}}$ is the dual cone of K. Moreover, define convex cones $K^P_{c_{\mathbf{x}}}$ as

$$K^P_{c_{\mathbf{x}}} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(c_{\mathbf{x}})}(-\mathbf{y_1}) \geqslant h_{B(c_{\mathbf{x}})}(\mathbf{y_2})\}.$$

Then, the set $K^P = \bigcap_{\mathbf{x} \in K} K^P_{c_{\mathbf{x}}}$ is the polar cone of K.

As demonstrated in Theorem 2.5, characterizing the dual cone requires an uncertain constant $c_{\mathbf{x}}$ to construct the associated convex sets. However, the geometric nature of these sets varies significantly as $c_{\mathbf{x}}$ ranges over the real line, specifically across positive, zero, and negative values. This sensitivity introduces substantial complexity to the analytical derivation. Consequently, we aim to simplify the influence of $c_{\mathbf{x}}$ to ensure a more robust and tractable approach.

3 Improved approach for obtaining dual cones

This section begins by establishing a lemma that leverages the positive homogeneity of F and G to simplify the expressions for the dual support and support functions associated with the closed convex sets defined in Theorem 2.5. Subsequently, this lemma is employed to derive a refined characterization of the dual cones in Theorem 3.1.

Lemma 3.1 *Let K be the convex cone defined by*

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\},$$

where functions F and G satisfy conditions in Theorem 2.1. Consider the sets $A(r) \subseteq \mathbb{R}^m$, $B(s) \subseteq \mathbb{R}^n$ defined by

$$A(r) = \{\mathbf{a} \in \mathbb{R}^m \mid F(\mathbf{a}) \geqslant r\}, \quad B(s) = \{\mathbf{b} \in \mathbb{R}^n \mid G(\mathbf{b}) \leqslant s\},$$

where r, s are real numbers, then for all $(\mathbf{x_1}, \mathbf{x_2}) \in K$ we have

(1) If $c > 0$, then $\sigma_{A(c)}(\mathbf{x_1}) = c\sigma_{A(1)}(\mathbf{x_1})$, $h_{B(c)}(\mathbf{x_2}) = ch_{B(1)}(\mathbf{x_2})$;
(2) If $c < 0$, then $\sigma_{A(c)}(\mathbf{x_1}) = -c\sigma_{A(-1)}(\mathbf{x_1})$, $h_{B(c)}(\mathbf{x_2}) = -ch_{B(-1)}(\mathbf{x_2})$,

where $\sigma_{A(c)}$ is the dual support function of the set $A(c)$, and $h_{B(c)}$ is the support function of the set $B(c)$.

Proof

(1) If $c > 0$, by the positive homogeneity of F we have

$$\begin{aligned}\sigma_{A(c)}(\mathbf{x_1}) &= \inf\{\langle \mathbf{x_1}, \mathbf{a}\rangle \mid \mathbf{a} \in A(c)\}\\ &= \inf\{\langle \mathbf{x_1}, \mathbf{a}\rangle \mid F(\mathbf{a}) \geqslant c\}\\ &= \inf\{\langle \mathbf{x_1}, \mathbf{a}\rangle \mid F(\tfrac{\mathbf{a}}{c}) \geqslant 1\}\\ &= \inf\{c\,\langle \mathbf{x_1}, \tfrac{\mathbf{a}}{c}\rangle \mid F(\tfrac{\mathbf{a}}{c}) \geqslant 1\}\\ &= c\inf\{\langle \mathbf{x_1}, \tfrac{\mathbf{a}}{c}\rangle \mid \tfrac{\mathbf{a}}{c} \in A(1)\}\\ &= c\sigma_{A(1)}(\mathbf{x_1}).\end{aligned}$$

Hence $\sigma_{A(c)}(\mathbf{x_1}) = c\sigma_{A(1)}(\mathbf{x_1})$. Similarly, by the positive homogeneity of G,

$$\begin{aligned}h_{B(c)}(\mathbf{x_2}) &= \sup\{\langle \mathbf{x_2}, \mathbf{b}\rangle \mid \mathbf{b} \in B(c)\}\\ &= \sup\{\langle \mathbf{x_2}, \mathbf{b}\rangle \mid G(\mathbf{b}) \leqslant c\}\\ &= \sup\{\langle \mathbf{x_2}, \mathbf{b}\rangle \mid G(\tfrac{\mathbf{b}}{c}) \leqslant 1\}\\ &= \sup\{c\,\langle \mathbf{x_2}, \tfrac{\mathbf{b}}{c}\rangle \mid G(\tfrac{\mathbf{b}}{c}) \leqslant 1\}\\ &= c\sup\{\langle \mathbf{x_2}, \tfrac{\mathbf{b}}{c}\rangle \mid \tfrac{\mathbf{b}}{c} \in B(1)\}\\ &= ch_{B(1)}(\mathbf{x_2}).\end{aligned}$$

Hence $h_{B(c)}(\mathbf{x_2}) = ch_{B(1)}(\mathbf{x_2})$.

(2) If $c < 0$, by the positive homogeneity of F we have

$$\begin{aligned}\sigma_{A(c)}(\mathbf{x_1}) &= \inf\{\langle \mathbf{x_1}, \mathbf{a}\rangle \mid \mathbf{a} \in A(c)\}\\ &= \inf\{\langle \mathbf{x_1}, \mathbf{a}\rangle \mid F(\mathbf{a}) \geqslant c\}\\ &= \inf\{\langle \mathbf{x_1}, \mathbf{a}\rangle \mid F(\tfrac{\mathbf{a}}{-c}) \geqslant -1\}\\ &= \inf\{-c\,\langle \mathbf{x_1}, \tfrac{\mathbf{a}}{-c}\rangle \mid F(\tfrac{\mathbf{a}}{-c}) \geqslant -1\}\\ &= -c\inf\{\langle \mathbf{x_1}, \tfrac{\mathbf{a}}{-c}\rangle \mid \tfrac{\mathbf{a}}{-c} \in A(-1)\}\\ &= -c\sigma_{A(-1)}(\mathbf{x_1}).\end{aligned}$$

Hence $\sigma_{A(c)}(\mathbf{x_1}) = -c\sigma_{A(-1)}(\mathbf{x_1})$. Similarly, by the positive homogeneity of G,

$$\begin{aligned}h_{B(c)}(\mathbf{x_2}) &= \sup\{\langle \mathbf{x_2}, \mathbf{b}\rangle \mid \mathbf{b} \in B(c)\}\\ &= \sup\{\langle \mathbf{x_2}, \mathbf{b}\rangle \mid G(\mathbf{b}) \leqslant c\}\\ &= \sup\{\langle \mathbf{x_2}, \mathbf{b}\rangle \mid G(\tfrac{\mathbf{b}}{-c}) \leqslant -1\}\\ &= \sup\{-c\,\langle \mathbf{x_2}, \tfrac{\mathbf{b}}{-c}\rangle \mid G(\tfrac{\mathbf{b}}{-c}) \leqslant -1\}\\ &= -c\sup\{\langle \mathbf{x_2}, \tfrac{\mathbf{b}}{-c}\rangle \mid \tfrac{\mathbf{b}}{-c} \in B(-1)\}\\ &= -ch_{B(-1)}(\mathbf{x_2}).\end{aligned}$$

Hence $h_{B(c)}(\mathbf{x_2}) = -ch_{B(-1)}(\mathbf{x_2})$.

□

Since the support and dual support functions exhibit homogeneity with respect to $c_{\mathbf{x}}$, their behavior is essentially determined by the cases where $c_{\mathbf{x}} \in \{1, 0, -1\}$. This allows us to simplify the dependency on the uncertain constant $c_{\mathbf{x}}$ and establish a more direct version of Theorem 2.5 as follows.

Theorem 3.1 *Let K be the convex cone defined by*

$$K = \{(\mathbf{x_1}, \mathbf{x_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\},$$

where functions F and G satisfy conditions in Theorem 2.1. Consider the sets $A(r) \subseteq \mathbb{R}^m$, $B(s) \subseteq \mathbb{R}^n$ defined by

$$\begin{aligned} A(r) &= \{\mathbf{x_1} \in \mathbb{R}^m \mid F(\mathbf{x_1}) \geqslant r\}, \\ B(s) &= \{\mathbf{x_2} \in \mathbb{R}^n \mid G(\mathbf{x_2}) \leqslant s\}, \end{aligned}$$

where r, s are real numbers. For any $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, by choosing a real number $c_\mathbf{x}$ such that $F(\mathbf{x_1}) \geqslant c_\mathbf{x} \geqslant G(\mathbf{x_2})$ and defining convex cones by

$$K^D_{c_\mathbf{x}} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(c_\mathbf{x})}(\mathbf{y_1}) \geqslant h_{B(c_\mathbf{x})}(-\mathbf{y_2})\}.$$

and

$$K^P_{c_\mathbf{x}} := \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(c_\mathbf{x})}(-\mathbf{y_1}) \geqslant h_{B(c_\mathbf{x})}(\mathbf{y_2})\}.$$

Then we have

(1) If $G(\mathbf{x_2}) \geqslant 0$ for all $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, then $K^D = K^D_1 \bigcap K^D_0$ is the dual cone of K, while $K^P = K^P_1 \bigcap K^P_0$ is the polar cone of K.
(2) If $F(\mathbf{x_1}) \leqslant 0$ for all $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, then $K^D = K^D_{-1} \bigcap K^D_0$ is the dual cone of K, while $K^P = K^P_{-1} \bigcap K^P_0$ is the polar cone of K.
(3) Otherwise, $K^D = K^D_1 \bigcap K^D_0 \bigcap K^D_{-1}$ is the dual cone of K, while $K^P = K^P_1 \bigcap K^P_0 \bigcap K^P_{-1}$ is the polar cone of K.

Proof

(1) For any $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, $F(\mathbf{x_1}) \geqslant c_\mathbf{x} \geqslant G(\mathbf{x_2}) \geqslant 0$. Since functions F is concave, G is convex, the corresponding $A(c_\mathbf{x})$ and $B(c_\mathbf{x})$ are both closed and convex by [8, Lemma 3.1(c)]. If $c_\mathbf{x} > 0$, by using Lemma 3.1 (1) we have

$$\sigma_{A(c_\mathbf{x})}(\mathbf{y_1}) = c_\mathbf{x} \sigma_{A(1)}(\mathbf{y_1}) \quad \text{and} \quad h_{B(c_\mathbf{x})}(-\mathbf{y_2}) = c_\mathbf{x} h_{B(1)}(-\mathbf{y_2}),$$

for all $(\mathbf{y_1}, \mathbf{y_2}) \in K^D_{c_\mathbf{x}}$. It leads to

$$\begin{aligned} K^D_{c_\mathbf{x}} &= \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(c_\mathbf{x})}(\mathbf{y_1}) \geqslant h_{B(c_\mathbf{x})}(-\mathbf{y_2})\} \\ &= \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid c_\mathbf{x} \sigma_{A(1)}(\mathbf{y_1}) \geqslant c_\mathbf{x} h_{B(1)}(-\mathbf{y_2})\} \\ &= \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(1)}(\mathbf{y_1}) \geqslant h_{B(1)}(-\mathbf{y_2})\} \\ &= K^D_1, \end{aligned}$$

for all $c_\mathbf{x} > 0$. From Theorem 2.5, K^D is the intersection of all $K^D_{c_\mathbf{x}}$. Hence $K^D = K^D_1 \bigcap K^D_0$ is the dual cone of K. With the same argument, we can prove that $K^P = K^P_1 \bigcap K^P_0$ is the polar cone of K.

(2) In this case, for any $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}) \in K$, $0 \geqslant F(\mathbf{x_1}) \geqslant c_{\mathbf{x}} \geqslant G(\mathbf{x_2})$. If $c_{\mathbf{x}} < 0$, by using Lemma 3.1 (2) we have

$$\sigma_{A(c_{\mathbf{x}})}(\mathbf{y_1}) = -c_{\mathbf{x}}\sigma_{A(-1)}(\mathbf{y_1}) \quad \text{and} \quad h_{B(c_{\mathbf{x}})}(-\mathbf{y_2}) = -c_{\mathbf{x}}h_{B(-1)}(-\mathbf{y_2}),$$

for all $(\mathbf{y_1}, \mathbf{y_2}) \in K^D_{c_{\mathbf{x}}}$. It leads to

$$\begin{aligned} K^D_{c_{\mathbf{x}}} &= \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(c_{\mathbf{x}})}(\mathbf{y_1}) \geqslant h_{B(c_{\mathbf{x}})}(-\mathbf{y_2})\} \\ &= \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid -c_{\mathbf{x}}\sigma_{A(-1)}(\mathbf{y_1}) \geqslant -c_{\mathbf{x}}h_{B(-1)}(-\mathbf{y_2})\} \\ &= \{(\mathbf{y_1}, \mathbf{y_2}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \sigma_{A(-1)}(\mathbf{y_1}) \geqslant h_{B(-1)}(-\mathbf{y_2})\} \\ &= K^D_{-1}, \end{aligned}$$

for all $c_{\mathbf{x}} < 0$. From Theorem 2.5, K^D is the intersection of all $K^D_{c_{\mathbf{x}}}$. Hence $K^D = K^D_{-1} \bigcap K^D_0$ is the dual cone of K. Similar argument will prove that $K^P = K^P_{-1} \bigcap K^P_0$ is the polar cone of K.

(3) Otherwise, the possible values of $c_{\mathbf{x}}$ must range over positive, zero, and negative real numbers. By the result of (1) and (2), we have $K^D = K^D_1 \bigcap K^D_0 \bigcap K^D_{-1}$ is the dual cone of K, while $K^P = K^P_1 \bigcap K^P_0 \bigcap K^P_{-1}$ is the polar cone of K.

□

We provide three examples to illustrate the scenarios (1)–(3) outlined in the aforementioned Theorem. For brevity, we focus on the explicit formulae for dual cones, as the corresponding polar cones can be readily derived once the dual cones are established.

Example 3.1 The set

$$\{(x, y, z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant (\sqrt{y} - \sqrt{z})^2,\ y \geqslant 0,\ z \geqslant 0\}$$

is a closed convex cone by Theorem 2.1. This cone appears in Example 2.5 of

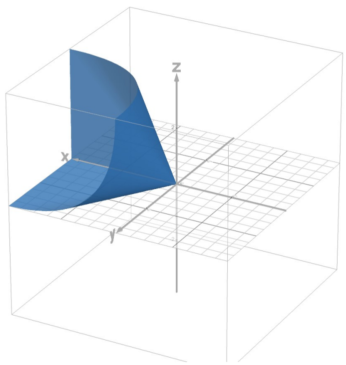


Fig. 1: $\{(x, y, z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant (\sqrt{y} - \sqrt{z})^2,\ y \geqslant 0, z \geqslant 0\}$

paper [8], please see Fig. 1 for its graph. We observe that $(\sqrt{y}-\sqrt{z})^2 \geqslant 0$ for all

$y \geqslant 0,\ z \geqslant 0$, we apply Theorem 3.1 (1) to obtain its dual cone. Furthermore, the detailed derivations of the support functions associated with this example are provided in Appendix 6.1 for completeness.

(1) Let $A(1) = \{a \in \mathbb{R} \mid a \geqslant 1\}$ and $B(1) = \{(b,c) \in \mathbb{R}^2_+ \mid (\sqrt{b} - \sqrt{c})^2 \leqslant 1\}$. Then, $\sigma_{A(1)}(x) = x, x \geqslant 0$ and

$$h_{B(1)}(y,z) = \begin{cases} \frac{yz}{y+z} & \text{, if } y+z<0,\ y>0 \text{ or } z>0; \\ 0 & \text{, if } y \leqslant 0 \text{ and } \ z \leqslant 0. \end{cases}$$

Hence,

$$\begin{aligned} K_1^D = \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant -\tfrac{yz}{y+z},\ y+z>0,\ y<0 \text{ or } z<0\} \\ \bigcup \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant 0,\ y \geqslant 0 \text{ and } z \geqslant 0\} \end{aligned} \tag{3.1}$$

(2) Let $A(0) = \{a \in \mathbb{R} \mid a \geqslant 0\}$ and $B(0) = \{(b,c) \in \mathbb{R}^2_+ \mid (\sqrt{b} - \sqrt{c})^2 \leqslant 0\}$. Then, $\sigma_{A(0)}(x) = 0,\ x \geqslant 0$ and $h_{B(0)}(y,z) = 0,\ y+z \leqslant 0$ and

$$K_0^D = \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid 0 \geqslant 0,\ x \geqslant 0,\ y+z \geqslant 0\} \tag{3.2}$$

By Theorem 3.1, equations (3.1) and (3.2), we have the dual cone of Example 3.1 as

$$\begin{aligned} K^* = K_1^D \bigcap K_0^D = \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant -\tfrac{yz}{y+z},\ y+z>0,\ y<0 \text{ or } z<0\} \\ \bigcup \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant 0,\ y \geqslant 0 \text{ and } z \geqslant 0\}. \end{aligned} \tag{3.3}$$

The geometric representations of the cone and its corresponding dual cone are illustrated in Fig. 2.

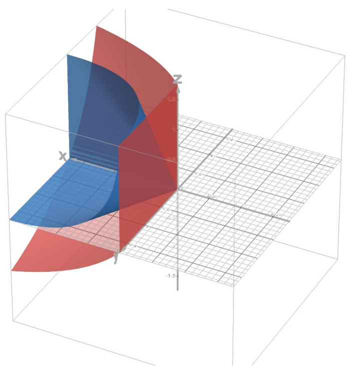


Fig. 2: The cone and dual cone of $\{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant (\sqrt{y} - \sqrt{z})^2,\ y \geqslant 0, z \geqslant 0\}$

Before the next example, let's clarify that, generally speaking, it's rare for a cone's expression to intentionally make $F(\mathbf{x_1}) \leqslant 0$. We make the following simple example just for presenting on purpose.

Example 3.2 The negative second-order cone is defined by

$$\{(\boldsymbol{x_1}, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid -||\boldsymbol{x_1}|| \geqslant x_2\}.$$

The geometric representation of this cone in $\mathbb{R}^3$ is illustrated in Fig. 3. In

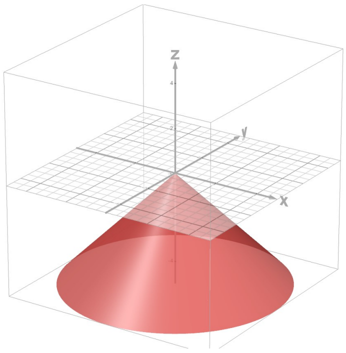


Fig. 3: The negative second-order cone in $\mathbb{R}^3$

this example, the value of $F(\mathbf{x_1}) = -||\boldsymbol{x_1}||$ is non-positive, so we consider $K^D = K^D_{-1} \bigcap K^D_0$ by Theorem 3.1 (2).

(1) Let $A(-1) = \{\mathbf{a} \in \mathbb{R}^{n-1} \mid -||\mathbf{a}|| \geqslant -1\}$ and $B(-1) = \{b \in \mathbb{R} \mid b \leqslant -1\}$. Then, $\sigma_{A(-1)}(\mathbf{x_1}) = -||\mathbf{x_1}||,\ \mathbf{x_1} \in \mathbb{R}^{n-1}$ and $h_{B(-1)} = -x_2,\ x_2 \geqslant 0$. Hence,

$$K^D_{-1} = \{(\mathbf{x_1}, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid -||\mathbf{x_1}|| \geqslant x_2,\ x_2 \leqslant 0\} \tag{3.4}$$

(2) Let $A(0) = \{\mathbf{a} \in \mathbb{R}^{n-1} \mid -||\mathbf{a}|| \geqslant 0\}$ and $B(0) = \{b \in \mathbb{R} \mid b \leqslant 0\}$. Then, $\sigma_{A(0)}(\mathbf{x_1}) = 0,\ \mathbf{x_1} \in \mathbb{R}^{n-1}$ and $h_{B(0)}(x_2) = 0,\ x_2 \geqslant 0$ and

$$K^D_0 = \{(\mathbf{x_1}, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid 0 \geqslant 0,\ x_2 \leqslant 0\} \tag{3.5}$$

By Theorem 3.1, equations (3.4) and (3.5), we have the dual cone of the negative second-order cone as

$$K^* = K^D_{-1} \bigcap K^D_0 = \{(\mathbf{x_1}, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid -||\mathbf{x_1}|| \geqslant x_2,\ x_2 \leqslant 0\}, \tag{3.6}$$

which is equivalent to the negative second-order cone, a structure that is characterized by its self-duality.

The next example is adapted from Example 2.6 in [8], which is the so-called Fischer–Burmeister cone. This designation originates from the famous Fischer–Burmeister NCP function [2].

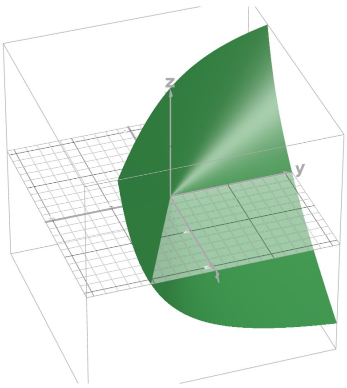


Fig. 4: Fischer–Burmeister cone in $\mathbb{R}^3$

Example 3.3 The Fischer–Burmeister cone is defined by

$$\mathcal{K}_{\text{FB}} = \{(x, y, z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant \sqrt{y^2 + z^2} - (y + z)\}. \tag{3.7}$$

Its geometric representation is depicted in Fig. 4.

We can see that the value of $\sqrt{y^2 + z^2} - (y + z)$ can be positive, negative or zero, hence it belongs to the third case in Theorem 3.1. Furthermore, the detailed derivations of the support functions associated with this example are provided in Appendix 6.2 for completeness.

(1) $A(1) = \{a \in \mathbb{R} \mid a \geqslant 1\}$, $B(1) = \{(b, c) \in \mathbb{R}^2_+ \mid \sqrt{b^2 + c^2} - (b + c) \leqslant 1\}$. Then, $\sigma_{A(1)}(x) = x,\ x \geqslant 0$ and

$$h_{B(1)}(y, z) = -\sqrt{2yz} - (y + z),\ y \leqslant 0,\ z \leqslant 0.$$

Hence,

$$K_1^D = \{(x, y, z) \in \mathbb{R}^3 \mid x \geqslant -\sqrt{2yz} + (y + z),\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\} \tag{3.8}$$

(2) $A(0) = \{a \in \mathbb{R} \mid a \geqslant 0\}$, $B(0) = \{(b, c) \in \mathbb{R}^2_+ \mid \sqrt{b^2 + c^2} - (b + c) \leqslant 0\}$. Then, $\sigma_{A(0)}(x) = 0, x \geqslant 0$ and $h_{B(0)}(y, z) = 0,\ y \leqslant 0,\ z \leqslant 0$.

$$K_0^D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \geqslant 0,\ x \geqslant 0, y \geqslant 0,\ z \geqslant 0\}. \tag{3.9}$$

(3) $A(-1) = \{a \in \mathbb{R} \mid a \geqslant -1\}$, $B(-1) = \{(b, c) \in \mathbb{R}^2_+ \mid \sqrt{b^2 + c^2} - (b + c) \leqslant -1\}$. Then $\sigma_{A(-1)}(x) = -x, x \geqslant 0$ and

$$h_{B(-1)}(y, z) = -\sqrt{2yz} + (y + z),\ y \leqslant 0,\ z \leqslant 0.$$

Hence,

$$K_{-1}^D = \{(x, y, z) \in \mathbb{R}^3 \mid -x \geqslant -\sqrt{2yz} - (y + z),\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\} \tag{3.10}$$

By Theorem 3.1, equations (3.8), (3.9) and (3.10), we have the dual cone of the Fischer–Burmeister cone as

$$\begin{aligned} K_{FB}^{*} &= K_1^D \bigcap K_0^D \bigcap K_{-1}^D \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geqslant -\sqrt{2yz} + (y+z),\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\} \\ &\bigcap \{(x,y,z) \in \mathbb{R}^3 \mid -x \geqslant -\sqrt{2yz} - (y+z),\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}. \end{aligned} \tag{3.11}$$

The above dual cone formula (3.11) can be reformulated as the following simpler formula.

$$K_{FB}^{*} = \left\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{2yz} \geqslant |y+z-x|,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\right\}. \tag{3.12}$$

Fig. 5 depicts the resulting dual cone derived from the aforementioned framework.

It is observed that the aforementioned equivalent representation of the dual cone (3.12) fails to satisfy the non-emptiness requirement for set K as stipulated in Definition (2.1). This discrepancy motivates a generalization of the construction for closed convex cones within our framework, which will be detailed in the subsequent section.

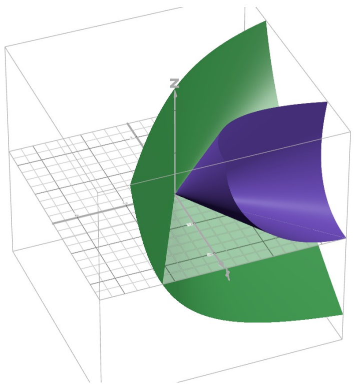


Fig. 5: The cone and dual cone of Fischer–Burmeister cone

4 Extended construction for convex cones

As noted at the conclusion of the previous section, the reformulated dual cone expression (3.12) for the Fischer–Burmeister cone appears to deviate from the standard form required by our original setting (2.1). To clarify this apparent discrepancy, let us perform a more detailed examination of the Fischer–Burmeister cone's structure.

$$\mathcal{K}_{\text{FB}} = \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \geqslant \sqrt{y^2+z^2} - (y+z)\}.$$

Apparently, it can be reformulated as the following equivalent form:

$$\mathcal{K}_{\text{FB}} = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z \geqslant \sqrt{y^2 + z^2}\}.$$

The aforementioned insights prompt a re-evaluation of our initial assumptions, leading to the extended construction framework presented in this work.

Considering the following generalization of nonempty set K defined by

$$K = \{\mathbf{x} \in \mathbb{R}^p \mid F(\mathbf{x_1}) \geqslant G(\mathbf{x_2})\}, \tag{4.1}$$

where $\mathbf{x_1} = \rho_1(\mathbf{x})$, $\mathbf{x_2} = \rho_2(\mathbf{x})$, $\rho_1 : \mathbb{R}^p \to \mathbb{R}^m$ and $\rho_2 : \mathbb{R}^p \to \mathbb{R}^n$ are projections. $\mathbb{R}^m, \mathbb{R}^n$ are subspaces of $\mathbb{R}^p$, $p \geqslant \max\{m, n\}$ and $F : D \subseteq \mathbb{R}^m \to \mathbb{R}$ and $G : E \subseteq \mathbb{R}^n \to \mathbb{R}$ are two real-valued functions, with the sets D and E being subsets of the domains of F and G, respectively. If $p = m + n$, that is, $\mathbb{R}^p = \mathbb{R}^m \times \mathbb{R}^n$, then this extended definition will degenerate to the original definition of K in (2.1).

Building upon this extension, we generalize the fundamental properties of homogeneity and super/sub-additivity for F and G to the following framework:

Definition 4.1 For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$ and $\mathbf{x_1} = \rho_1(\mathbf{x})$, $\mathbf{x_2} = \rho_2(\mathbf{x})$, $\mathbf{y_1} = \rho_1(\mathbf{y})$, $\mathbf{y_2} = \rho_2(\mathbf{y})$, and $\lambda > 0$, we say that

(1) F and G are positively homogeneous (of degree 1) in K if

$$F(\lambda\mathbf{x_1}) = \lambda F(\mathbf{x_1}),\ G(\lambda\mathbf{x_2}) = \lambda G(\mathbf{x_2}).$$

(2) F is superadditive in K if

$$F(\mathbf{x_1} + \mathbf{y_1}) \geqslant F(\mathbf{x_1}) + F(\mathbf{y_1}),$$

and G is subadditive in K if

$$G(\mathbf{x_2} + \mathbf{y_2}) \leqslant G(\mathbf{x_2}) + G(\mathbf{y_2}).$$

To ensure that the extended set K in (4.1) constitutes a convex cone, we represent F and G as compositions with the projections ρ_1 and ρ_2, i.e., $F(\mathbf{x_1}) = F(\rho_1(\mathbf{x}))$ and $G(\mathbf{x_2}) = G(\rho_2(\mathbf{x}))$. Since projections preserve both positive homogeneity and additivity, the set K satisfies the definition of a convex cone, provided that F is superadditive, G is subadditive, and both functions are positively homogeneous.

Theorem 4.1 *Suppose that the closed set K (4.1) satisfies conditions:*

(a) F and G are positively homogeneous (of degree 1) in K,
(b) F is superadditive (concave) and G is subadditive (convex) in K.

Then, the following hold.

(1) K is a convex cone.
(2) $\mathrm{cl}(K)$ is a closed convex cone.

(3) If both F and G are continuous and both D and E are closed, then $K = \mathrm{cl}(K)$ is a closed convex cone.

Proof

(1) It is obvious that a projection in $\mathbb{R}^p$ is a linear transformation. We are going to check that the set K satisfies the equivalent conditions of a convex cone in [8, Definition 2.1] under conditions (a) and (b). Since ρ_1 and ρ_2 are projections, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$ and $\lambda > 0$ we have

$$\rho_1(\lambda \mathbf{x}) = \lambda \rho_1(\mathbf{x}) = \lambda \mathbf{x_1}, \;\; \rho_2(\lambda \mathbf{x}) = \lambda \rho_2(\mathbf{x}) = \lambda \mathbf{x_2}$$

By condition (a),

$$F(\lambda \mathbf{x_1}) = \lambda F(\mathbf{x_1}) \geqslant \lambda G(\mathbf{x_2}) = G(\lambda \mathbf{x_2}),$$

which means $\lambda \mathbf{x}$ is also in K. In addition, it is clear to see that

$$\rho_1(\mathbf{x}+\mathbf{y}) = \rho_1(\mathbf{x}) + \rho_1(\mathbf{y}) = \mathbf{x_1} + \mathbf{y_1}, \;\; \rho_2(\mathbf{x}+\mathbf{y}) = \rho_2(\mathbf{x}) + \rho_2(\mathbf{y}) = \mathbf{x_2} + \mathbf{y_2}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$. With condition (b), we can see that

$$F(\mathbf{x_1} + \mathbf{y_1}) \geqslant F(\mathbf{x_1}) + F(\mathbf{y_1}) \geqslant G(\mathbf{x_2}) + G(\mathbf{y_2}) \geqslant G(\mathbf{x_2} + \mathbf{y_2}),$$

which means $\mathbf{x} + \mathbf{y}$ is also in K for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$. Based on the above discussion, we have shown that K is a convex cone.

(2) The reason is the same as in the proof of [8, Theorem 2.1(2)], we omit it here.

(3) The proof is analogous to that of [8, Theorem 2.1(3)], with the exception of the projection argument. For every limit point $\mathbf{z}$ of K with $\mathbf{z_1} = \rho_1(\mathbf{z})$, $\mathbf{z_2} = \rho_2(\mathbf{z})$, there exists a sequence $\{\mathbf{x}_n\} \subseteq K \backslash \{\mathbf{z}\}$ with $\mathbf{x}_n^{(1)} = \rho_1(\mathbf{x}_n)$, $\mathbf{x}_n^{(2)} = \rho_2(\mathbf{x}_n)$ such that

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{z},$$

which mean $\lim_{n\to\infty} \mathbf{x}_n^{(1)} = \mathbf{z_1}$ and $\lim_{n\to\infty} \mathbf{x}_n^{(2)} = \mathbf{z_2}$, where $\mathbf{x}_n^{(1)} \in D$ and $\mathbf{x}_n^{(2)} \in E$. Based on the assumption that D and E are closed, we have $\mathbf{z_1} \in D$ and $\mathbf{z_2} \in E$. Now because $\mathbf{x}_n \in K$, we have

$$F(\mathbf{x}_n^{(1)}) \geqslant G(\mathbf{x}_n^{(2)}), \;\; \forall n = 1, 2, 3, \cdots.$$

By the continuity of F and G, it follows that F is continuous at $\mathbf{z_1}$ and G is continuous at $\mathbf{z_2}$, and

$$\lim_{n\to\infty} F(\mathbf{x}_n^{(1)}) = F(\mathbf{z_1}) \geqslant G(\mathbf{z_2}) = \lim_{n\to\infty} G(\mathbf{x}_n^{(2)}).$$

This implies that every limit point $\mathbf{z}$ of K is in K, then K is closed. Hence, $K = \mathrm{cl}(K)$ is a closed convex cone.

□

We construct several illustrative examples to validate the applicability and effectiveness of this extension.

Example 4.1 The following cones are variant types of second-order cones in $\mathbb{R}^3$.

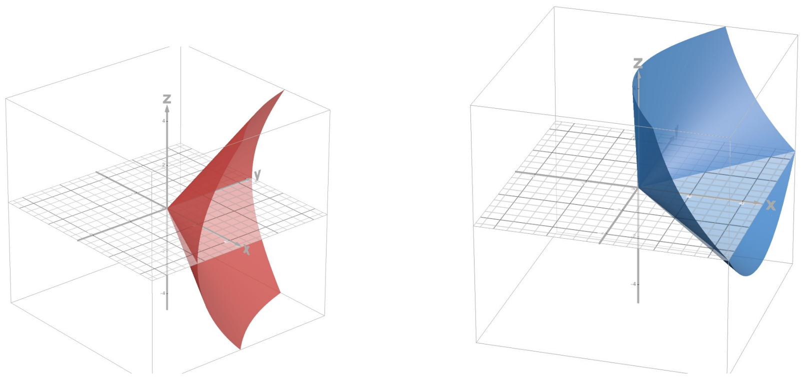
(a) $\widetilde{K_{xy}^{yz}} = \left\{(x, y, z) \in \mathbb{R}^3 \mid x + y \geqslant \sqrt{y^2 + z^2}\right\}$.

(b) $\widetilde{K_{xz}^{yz}} = \left\{(x, y, z) \in \mathbb{R}^3 \mid x + z \geqslant \sqrt{y^2 + z^2}\right\}$.

In above equations, for example 4.1(a), $\rho_1(x, y, z) = (x, y)$ and $\rho_2(x, y, z) = (y, z)$, while $F(x, y) = x+y$ and $G(y, z) = \sqrt{y^2 + z^2}$. The respective geometric representations are illustrated in Fig. 6(a) and 6(b). Moreover, it's easy to see that the Fischer–Burmeister cone in $\mathbb{R}^3$ becomes

(c) $\widetilde{K_{xyz}^{yz}} = \left\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z \geqslant \sqrt{y^2 + z^2}\right\}$.

In this case, $\rho_1(x, y, z) = (x, y, z)$ and $\rho_2(x, y, z) = (y, z)$, while $F(x, y, z) = x + y + z$ and $G(y, z) = \sqrt{y^2 + z^2}$.



(a) $\widetilde{K_{xy}^{yz}}$ (b) $\widetilde{K_{xz}^{yz}}$

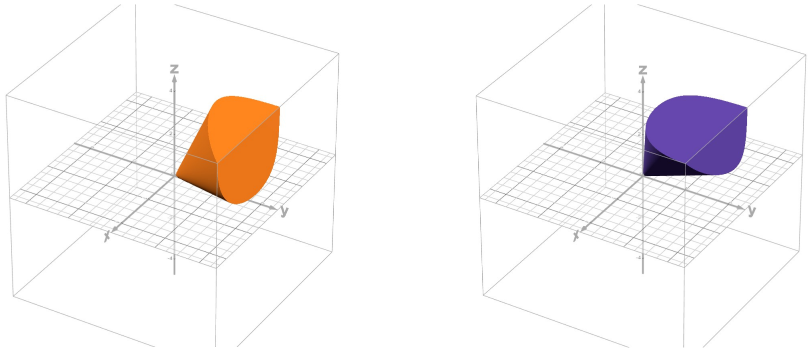
Fig. 6: The variant second-order cones $\widetilde{K_{xy}^{yz}}$ and $\widetilde{K_{xz}^{yz}}$

We can formulate a three-dimensional convex cone by coupling two appropriate two-dimensional functions on both sides of a defining inequality. The following example demonstrates this construction methodology.

Example 4.2

$$\widetilde{\mathcal{K}_{xy}^{e}} = \mathrm{cl}\{(x, y, z) \in \mathbb{R}^3_{++} \mid x \ln \frac{y}{x} \geqslant (\sqrt{y} - \sqrt{z})^2\}, \tag{4.2}$$

where the function $F(x, y) = x \ln \frac{y}{x}$ comes from example 2.8 of paper [8], while the function $G(y, z) = (\sqrt{y} - \sqrt{z})^2$ comes from example 2.5 of paper [8]. Fig. 7(a) illustrates the profile of the cone constructed in this example.



(a) $\widetilde{\mathcal{K}^e_{xy}}$ (b) $\widetilde{\mathcal{K}^e_{xz}}$

Fig. 7: The variant cones $\widetilde{\mathcal{K}^e_{xy}}$ and $\widetilde{\mathcal{K}^e_{xz}}$

Remark 4.1 If we let function $F(x, z) = x \ln \frac{z}{x}$, there exists a variant type of cone similar to (4.2) as follows.

$$\widetilde{\mathcal{K}^e_{xz}} = \mathrm{cl}\{(x, y, z) \in \mathbb{R}^3_{++} \mid x \ln \frac{z}{x} \geqslant (\sqrt{y} - \sqrt{z})^2\} \tag{4.3}$$

Fig. 7(b) illustrates the profile of this cone.

By incorporating the concepts of support functions and dual support functions, Theorem 2.3 can be generalized to the following broader form.

Theorem 4.2 *Let A be a nonempty closed convex set in $\mathbb{R}^m$ and $\sigma_A : D \subseteq \mathbb{R}^m \to \mathbb{R}$ be the associated dual support function with A, that is,*

$$\sigma_A(\mathbf{x}) = \inf\{\langle \mathbf{x}, \mathbf{a}\rangle \mid \mathbf{a} \in A\}.$$

Let B be a nonempty closed convex set in $\mathbb{R}^n$ and $h_B : E \subseteq \mathbb{R}^n \to \mathbb{R}$ be the associated support function with B, that is,

$$h_B(\mathbf{x}) = \sup\{\langle \mathbf{x}, \mathbf{b}\rangle \mid \mathbf{b} \in B\}.$$

The set $K_{A,B}$ is defined by

$$K_{A,B} := \{\mathbf{z} \in \mathbb{R}^p \mid \sigma_A(\mathbf{z_1}) \geqslant h_B(\mathbf{z_2})\},$$

where $\mathbf{z_1} = \rho_1(\mathbf{z})$, $\mathbf{z_2} = \rho_2(\mathbf{z})$, $\rho_1 : \mathbb{R}^p \to \mathbb{R}^m$ and $\rho_2 : \mathbb{R}^p \to \mathbb{R}^n$ are projections. $\mathbb{R}^m, \mathbb{R}^n$ are subspaces of $\mathbb{R}^p$, $p \geqslant \max\{m, n\}$. Then, the following hold.

(1) $K_{A,B}$ is a convex cone.
(2) $\mathrm{cl}(K_{A,B})$ is a closed convex cone.
(3) If both σ_A and h_B are continuous and both D and E are closed, then $K_{A,B} = \mathrm{cl}(K_{A,B})$ is a closed convex cone.

Proof (1) Note that the support function is positively homogeneous and subadditive; and the dual support function is positively homogeneous and superadditive. Accordingly, the set $K_{A,B}$ is certainly a convex cone by Theorem 4.1. The reasons for (2) and (3) are the same as the proofs provided in Theorem 4.1. $\square$

We now construct new classes of cones by characterizing two-dimensional closed convex sets whose boundaries are defined by specific trigonometric functions.

Example 4.3 Consider the following closed convex set B, see Fig. 8.

$$B = \{(a,b) \in \mathbb{R}^2 \mid -\cos a \leqslant b \leqslant \cos a,\ -\frac{\pi}{2} \leqslant a \leqslant \frac{\pi}{2}\}$$

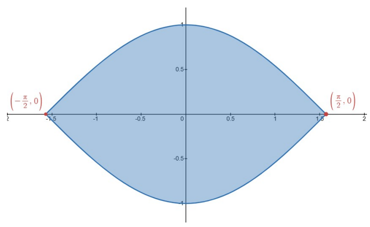


Fig. 8: The closed convex set B

Since the set B is symmetric with respect to the x-axis, the y-axis and the origin, we consider the first quadrant case is sufficient by symmetry. Here, we adopt the geometric approach described in [8, Appendix]. For any fixed vector $(x, y) \in \mathbb{R}^2$ where $x > 0, y > 0$, suppose vector (a, b) in the set B within the first quadrant is the one which makes

$$h_B(x, y) = \langle (x, y), (a, b) \rangle.$$

The vector (a, b) must lie on the boundary of set B satisfying $b = \cos a$, and the tangent line passing through (a, b) is perpendicular to (x, y), then

$$\begin{aligned} & (-\sin a) \cdot \tfrac{y}{x} = -1 \\ \Rightarrow \quad & \sin a = \tfrac{x}{y} \\ \Rightarrow \quad & a = \arcsin \tfrac{x}{y},\ \text{if } y \geqslant x. \end{aligned} \tag{4.4}$$

Meanwhile,

$$b = \cos a = \frac{\sqrt{y^2 - x^2}}{y} \tag{4.5}$$

By equations (4.4) and (4.5),

$$h_B(x, y) = xa + yb = x \arcsin \frac{x}{y} + \sqrt{y^2 - x^2},\ y \geqslant x. \tag{4.6}$$

For the case of (x, y) in the first quadrant and $y \leqslant x$,

$$\langle (x, y), (a, b) \rangle = xa + yb \leqslant xa + x \cos a = x(a + \cos a), \ 0 \leqslant a \leqslant \frac{\pi}{2},$$

where the supremum of $a + \cos a$ occurs when $a = \frac{\pi}{2}$. In this case,

$$h_B(x, y) = \frac{\pi}{2} x, \ y \leqslant x. \tag{4.7}$$

Combine equations (4.6), (4.7) and the symmetry, we have

$$h_B(x, y) = \begin{cases} |x| \arcsin |\frac{x}{y}| + \sqrt{y^2 - x^2}, & |y| \geqslant |x|, \\ \frac{\pi}{2}|x|, & |y| \leqslant |x|. \end{cases} \tag{4.8}$$

Notice that $x = 0$ is the special case of $\langle (0, |y|), (a, b) \rangle = |y| b \leqslant |y|$ when $|b| \leqslant 1$, while $y = 0$ is the special case of $\langle (|x|, 0), (a, b) \rangle = a|x| \leqslant \frac{\pi}{2}|x|$ when $|a| \leqslant \frac{\pi}{2}$.

If we use the associated dual support function $\sigma_A(z) = z, \ z \geqslant 0$ of the set $A = \{z \in \mathbb{R} \mid z \geqslant 1\}$, by applying Theorem 2.3, the following set will be a closed convex cone in $\mathbb{R}^3$.

$$\begin{aligned} \mathcal{K}_z^{\text{arcsin}} = & \{(z, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid z \geqslant |x| \arcsin |\tfrac{x}{y}| + \sqrt{y^2 - x^2}, \ |y| \geqslant |x|\} \\ & \cup \{(z, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid z \geqslant \tfrac{\pi}{2}|x|, \ |y| \leqslant |x|\} \end{aligned} \tag{4.9}$$

Fig. 9 depicts the visualization of the cone, highlighting the boundary curvature induced by the arcsine functional representation.

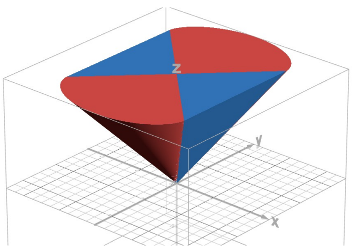


Fig. 9: The arcsine cone $\mathcal{K}$

Following the construction of the toppled second-order cone (4.10) as in [8, Example 3.3], we adopt the dual support function $\sqrt{xz}$ associated with the closed convex set $\{(x, z) \in \mathbb{R}^2 \mid 4xz \geqslant 1, x, z > 0\}$.

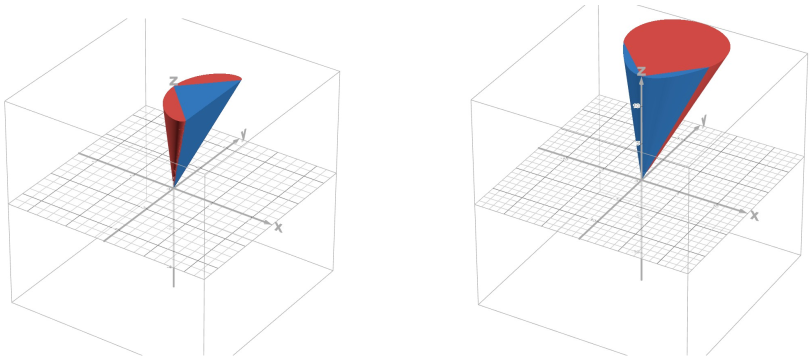
$$\mathcal{K}_{\text{toppled}} = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{xz} \geqslant |y|, \ x \geqslant 0, \ z \geqslant 0\}, \tag{4.10}$$

and combine with the support function (4.8), we have the following closed convex cone which meets the requirement of Theorem 4.2.

$$\begin{aligned} \mathcal{K}_{\sqrt{xz}}^{\text{arcsin}} = & \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{xz} \geqslant |x| \arcsin |\tfrac{x}{y}| + \sqrt{y^2 - x^2}, \ |y| \geqslant |x|, \ x, z \geqslant 0\} \\ & \cup \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{xz} \geqslant \tfrac{\pi}{2}|x|, \ |y| \leqslant |x|, \ x, z \geqslant 0\} \end{aligned} \tag{4.11}$$

With the same idea, we can have the following variant type of cone similar to above one. Their respective geometric representations are illustrated in Figs. 10(a) and (b).

$$\begin{aligned}\mathcal{K}^{\text{arcsin}}_{\sqrt{yz}} &= \{(x,y,z)\in\mathbb{R}^3 \mid \sqrt{yz} \geqslant |x|\arcsin|\tfrac{x}{y}| + \sqrt{y^2-x^2},\ |y|\geqslant|x|,\ y,z\geqslant 0\} \\ &\quad \cup\{(x,y,z)\in\mathbb{R}^3 \mid \sqrt{yz}\geqslant \tfrac{\pi}{2}|x|,\ |y|\leqslant|x|,\ y,z\geqslant 0\}\end{aligned} \tag{4.12}$$



(a) The $\mathcal{K}^{\text{arcsin}}_{\sqrt{xz}}$ cone (b) The $\mathcal{K}^{\text{arcsin}}_{\sqrt{yz}}$ cone

Fig. 10: The variant arcsine cones $\mathcal{K}^{\text{arcsin}}_{\sqrt{xz}}$ and $\mathcal{K}^{\text{arcsin}}_{\sqrt{yz}}$

The next example is another trigonometric cone which we want to make the formula more concise.

Example 4.4 Consider the following closed convex set B, see Fig. 11.

$$B=\{(a,b)\in\mathbb{R}^2 \mid -(\tfrac{\pi}{2}-a\arcsin a-\sqrt{1-a^2})\leqslant b\leqslant \tfrac{\pi}{2}-a\arcsin a-\sqrt{1-a^2},\ -1\leqslant a\leqslant 1\}$$

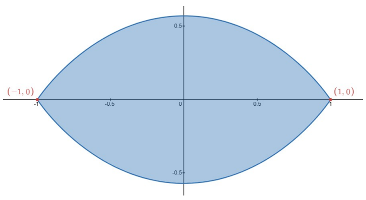


Fig. 11: The closed convex set B

Since the set B is symmetric with respect to the x-axis, the y-axis and the origin, we consider the first quadrant case is sufficient by symmetry. Here, we adopt the geometric approach described in [8, Appendix]. For any fixed vector $(x, y) \in \mathbb{R}^2$ where $x > 0, y > 0$, suppose vector (a, b) in the set B within the first quadrant is the one which makes

$$h_B(x, y) = \langle (x, y), (a, b) \rangle.$$

The vector (a, b) must lie on the boundary of set B satisfying $b = \frac{\pi}{2} - a \arcsin a - \sqrt{1 - a^2}$, and the tangent line passing through (a, b) is perpendicular to (x, y), where $b' = -\arcsin a$. Then we have

$$\begin{array}{rl} & (-\arcsin a) \cdot \frac{y}{x} = -1 \\ \Rightarrow & \arcsin a = \frac{x}{y} \\ \Rightarrow & a = \sin \frac{x}{y}, \text{ if } 0 < \frac{x}{y} \leqslant \frac{\pi}{2} \end{array} \tag{4.13}$$

Meanwhile,

$$b = \frac{\pi}{2} - a \arcsin a - \sqrt{1 - a^2} = \frac{\pi}{2} - \frac{x}{y} \sin \frac{x}{y} - \cos \frac{x}{y} \tag{4.14}$$

By equations (4.13) and (4.14), we have the one part of support function of B

$$h_B(x, y) = xa + yb = y \left(\frac{\pi}{2} - \cos \frac{x}{y} \right), \ y \geqslant \frac{2}{\pi} x. \tag{4.15}$$

For the case of (x, y) in the first quadrant and $y \leqslant \frac{2}{\pi} x$,

$$\begin{aligned} \langle (x, y), (a, b) \rangle = xa + yb &\leqslant xa + \frac{2}{\pi} x \left(\frac{\pi}{2} - a \arcsin a - \sqrt{1 - a^2} \right) \\ &= x \left(a + 1 - \frac{2}{\pi} a \arcsin a - \frac{2}{\pi} \sqrt{1 - a^2} \right), \end{aligned}$$

where the supremum of the above expression is attained at $a = 1$. In this case,

$$h_B(x, y) = x, \ y \leqslant \frac{2}{\pi} x. \tag{4.16}$$

Combining equations (4.15), (4.16) and the symmetry, we have

$$h_B(x, y) = \begin{cases} |y| \left(\frac{\pi}{2} - \cos |\frac{x}{y}| \right), & |y| \geqslant \frac{2}{\pi} |x|, \\ |x|, & |y| \leqslant \frac{2}{\pi} |x|. \end{cases} \tag{4.17}$$

Notice that $x = 0$ is the special case of $\langle (0, |y|), (a, b) \rangle = |y| b \leqslant \frac{\pi}{2} |y|$ when $|b| \leqslant \frac{\pi}{2}$, while $y = 0$ is the special case of $\langle (|x|, 0), (a, b) \rangle = a|x| \leqslant \frac{\pi}{2} |x|$ when $|a| \leqslant 1$. If we choose the associated dual support function $\sigma_A(z) = z, \ z \geqslant 0$ of the set $A = \{z \in \mathbb{R} \mid z \geqslant 1\}$, by applying Theorem 2.3, the following set will be a closed convex cone in $\mathbb{R}^3$. Its visualization is depicted in Fig. 12.

$$\begin{aligned} \mathcal{K}_z^{\cos} = &\{(z, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid z \geqslant |y| \left(\tfrac{\pi}{2} - \cos |\tfrac{x}{y}| \right), \ |y| \geqslant \tfrac{2}{\pi} |x|\} \\ &\cup \{(z, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid z \geqslant |x|, \ |y| \leqslant \tfrac{2}{\pi} |x|\} \end{aligned} \tag{4.18}$$

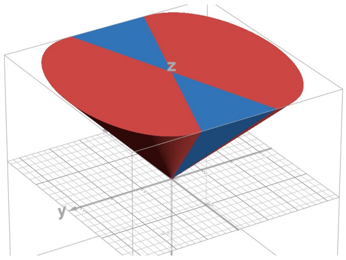


Fig. 12: The cosine cone $\mathcal{K}_z^{\cos}$

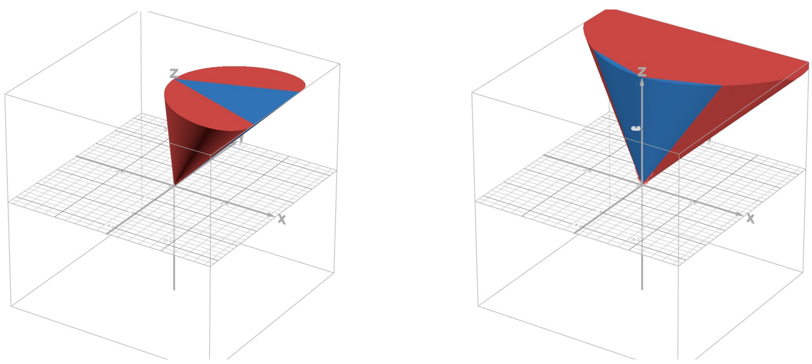
Following the construction of the toppled second-order cone (4.10) as in [8, Example 3.3], we adopt the dual support function $\sqrt{xz}$ associated with the closed convex set $\{(x,z) \in \mathbb{R}^2 \mid 4xz \geqslant 1, x, z > 0\}$. By combining this with the support function (4.17), we obtain the following closed convex cone, which satisfies the conditions of Theorem 4.2 and is more concise than the formulation in the previous example.

$$\begin{aligned}\mathcal{K}_{\sqrt{xz}}^{\cos} = &\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{xz} \geqslant |y| \left(\tfrac{\pi}{2} - \cos\left|\tfrac{x}{y}\right|\right),\ |y| \geqslant \tfrac{2}{\pi}|x|,\ x,z \geqslant 0\} \\ &\cup \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{xz} \geqslant |x|,\ |y| \leqslant \tfrac{2}{\pi}|x|,\ x,z \geqslant 0\}\end{aligned} \tag{4.19}$$

The similar yz variant type of cosine cone will be presented as follows.

$$\begin{aligned}\mathcal{K}_{\sqrt{yz}}^{\cos} = &\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{yz} \geqslant |y| \left(\tfrac{\pi}{2} - \cos\left|\tfrac{x}{y}\right|\right),\ |y| \geqslant \tfrac{2}{\pi}|x|\ y,z \geqslant 0\} \\ &\cup \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{yz} \geqslant |x|,\ |y| \leqslant \tfrac{2}{\pi}|x|,\ y,z \geqslant 0\}\end{aligned} \tag{4.20}$$

Their respective geometric representations are illustrated in Figs. 13(a) and (b).



(a) The $\mathcal{K}_{\sqrt{xz}}^{\cos}$ cone (b) The $\mathcal{K}_{\sqrt{yz}}^{\cos}$ cone

Fig. 13: The variant cosine cones $\mathcal{K}_{\sqrt{xz}}^{\cos}$ and $\mathcal{K}_{\sqrt{yz}}^{\cos}$

5 Conclusions

This paper aims to refine the methodology for deriving the dual and polar cones of given closed convex cones. By leveraging the homogeneity of the associated explicit real-valued functions, we establish a more efficient analytical framework. Drawing inspiration from several case studies, we extend the construction framework for closed convex cones previously developed in our earlier work.

By providing a more systematic and refined construction for dual and polar cones, this research significantly enhances the analytical toolkit available for conic programming and nonsmooth optimization. The proposed framework not only simplifies the derivation of dual problems but also offers deeper insights into the sensitivity analysis and duality gaps of complex optimization models.

6 Appendix

In this section, we present the detailed derivations of the support functions referenced in Examples 3.1 and 3.3. Due to their computational complexity, these calculations are collected here for the reader's convenience and future reference.

6.1 The support functions in Example 3.1

(1) Let $B(1) = \{(b,c) \in \mathbb{R}^2_+ \mid (\sqrt{b} - \sqrt{c})^2 \leqslant 1\}$. For any $(y,z) \in \mathbb{R}^2$, we divide the plane into four major parts (a)-(d) as in Fig. 14.

For any $(b,c) \in B(1)$, $-1 \leqslant \sqrt{b} - \sqrt{c} \leqslant 1$. We have

$$\sqrt{b} \leqslant \sqrt{c} + 1 \text{ and } \sqrt{c} \leqslant \sqrt{b} + 1$$

(a) (a1) If $y + z > 0$, consider the line $b = c$ where $b, c \geqslant 0$ lies in $B(1)$, we have

$$\langle (y,z),(b,c) \rangle = yb + zc = (y+z)b.$$

In this case, $h_{B(1)}(y,z)$ will approach infinity when b approach infinity.

(a2) If $y + z = 0$ and $y > 0$, along the curve $\sqrt{b} = \sqrt{c} + 1$, we have

$$\langle (y,z),(b,c) \rangle = yb + zc = y(b-c) = y((\sqrt{c}+1)^2 - c) = y(2\sqrt{c}+1)$$

$h_{B(1)}(y,z)$ will approach infinity when c approach infinity.

(a3) If $y + z = 0$ and $z > 0$, along the curve $\sqrt{c} = \sqrt{b} + 1$, we have

$$\langle (y,z),(b,c) \rangle = yb + zc = z(c-b) = z((\sqrt{b}+1)^2 - b) = z(2\sqrt{b}+1)$$

$h_{B(1)}(y,z)$ will approach infinity when b approach infinity.

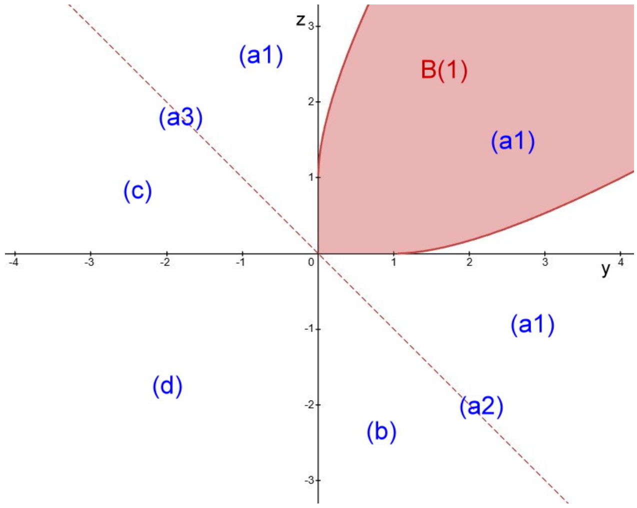


Fig. 14: The set $B(1)$

(b) If $y+z<0$ and $y>0$, we have

$$\begin{aligned}\langle (y,z),(b,c)\rangle &= yb+zc\\ &\leqslant y(\sqrt{c}+1)^2+zc\\ &= (y+z)(\sqrt{c}+\tfrac{y}{y+z})^2+\tfrac{yz}{y+z}\end{aligned}$$

$h_{B(1)}(y,z)=\dfrac{yz}{y+z}$ when $\sqrt{c}=-\dfrac{y}{y+z}$.

(c) If $y+z<0$ and $z>0$, we have

$$\begin{aligned}\langle (y,z),(b,c)\rangle &= yb+zc\\ &\leqslant yb+z(\sqrt{b}+1)^2\\ &= (y+z)(\sqrt{b}+\tfrac{z}{y+z})^2+\tfrac{yz}{y+z}\end{aligned}$$

$h_{B(1)}(y,z)=\dfrac{yz}{y+z}$ when $\sqrt{b}=-\dfrac{z}{y+z}$.

(d) If $y+z<0$ and $y\leqslant 0$, $z\leqslant 0$, we have

$$\langle (y,z),(b,c)\rangle = yb+zc\leqslant 0.$$

In this case, $h_{B(1)}(y,z)=0$.

Based on the preceding analysis in (a) through (d), the support function of the set $B(1)$ is formulated as follows:

$$h_{B(1)}(y,z)=\begin{cases}\frac{yz}{y+z} & \text{, if } y+z<0,\ y>0 \text{ or } z>0;\\ 0 & \text{, if } y\leqslant 0 \text{ and } z\leqslant 0.\end{cases}$$

(2) Let $B(0)=\{(b,c)\in \mathbb{R}^2_+ \mid (\sqrt{b}-\sqrt{c})^2\leqslant 0\}$, it is equivalent to the half line $b=c$, $b,c\geqslant 0$. We divide the plane into 2 major parts (a),(b) as in Fig. 15.

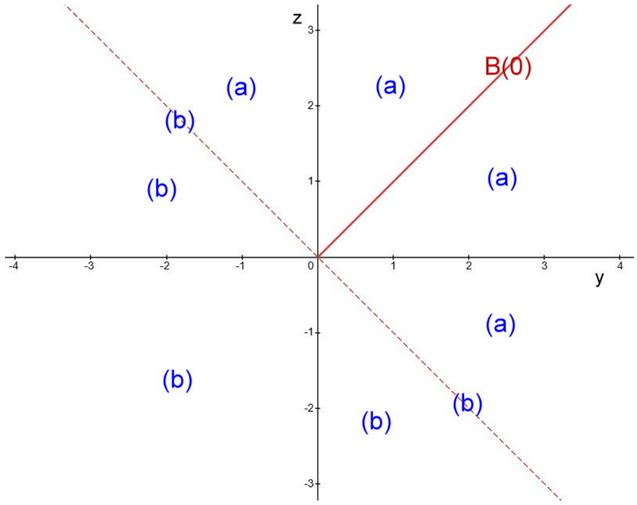


Fig. 15: The set $B(0)$

(a) If $y + z > 0$,

$$\langle (y, z), (b, c) \rangle = yb + zc = (y + z)b.$$

In this case, $h_{B(0)}(y, z)$ will approach infinity when b approach infinity.

(b) If $y + z \leqslant 0$, then

$$\langle (y, z), (b, c) \rangle = yb + zc = (y + z)b \leqslant 0.$$

In this case, $h_{B(0)}(y, z) = 0$.

Based on the preceding analysis in (a),(b), the support function of the set $B(0)$ is formulated as follows:

$$h_{B(0)}(y, z) = 0, \ y + z \leqslant 0.$$

6.2 The support functions in Example 3.3

We will use the approach from the geometric point of view mentioned in [8, Appendix 5.1].

(1) Let $B(1) = \{(b, c) \in \mathbb{R}^2_+ \mid \sqrt{b^2 + c^2} - (b + c) \leqslant 1\}$. The graph of $B(1)$ will be illustrated in Fig. 16.

For any (y, z) belonging to the first, second, or third quadrants, not including the negative y-axis and negative z-axis, it is readily apparent that the support function diverges to infinity. Consequently, the subsequent discussion focuses exclusively on the fourth quadrant.

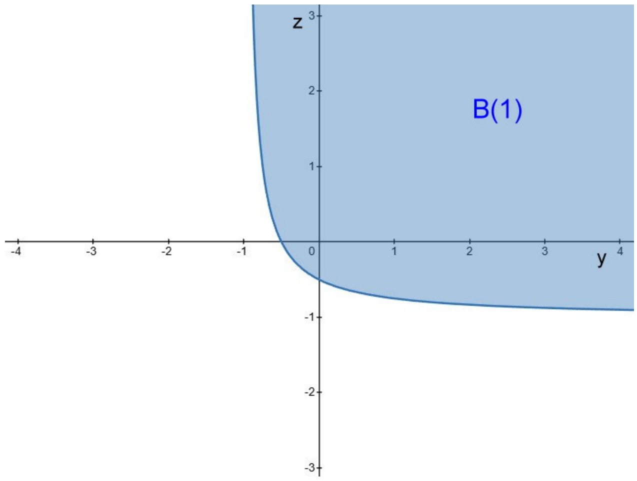


Fig. 16: The set $B(1)$ of FB cone

Suppose (b, c) is the vector that satisfies $h_{B(1)}(y, z) = \langle (y, z), (b, c) \rangle$ for a fixed (y, z) in the fourth quadrant, we have

$$\sqrt{b^2 + c^2} - (b + c) = 1. \tag{6.1}$$

Note that $b > -1$ and $c > -1$. The tangent line at (b, c) with the slope $-\frac{c+1}{b+1}$, is perpendicular to the vector (y, z), where $y \neq 0,\ z \neq 0$. It leads to

$$-\frac{c+1}{b+1} \cdot \frac{z}{y} = -1 \tag{6.2}$$

From (6.1), we can have the relation $b = \frac{1}{2(c+1)} - 1$, $c = \frac{1}{2(b+1)} - 1$. By taking b, c into (6.2) respectively, we have

$$b = \sqrt{\frac{z}{2y}} - 1,\ \ c = \sqrt{\frac{y}{2z}} - 1.$$

Then,

$$h_{B(1)}(y, z) = \begin{cases} y\sqrt{\dfrac{z}{2y}} + z\sqrt{\dfrac{y}{2z}} - (y + z) & ,\ y < 0,\ z < 0; \\ -y & ,\ y \leqslant 0,\ z = 0; \\ -z & ,\ y = 0,\ z \leqslant 0. \end{cases}$$

Note that if $z = 0$, $\langle (y, 0), (b, c) \rangle = yb \leqslant -y$ because of $b > -1$, while if $y = 0$, $\langle (0, z), (b, c) \rangle = zc \leqslant -z$ because of $c > -1$.

The aforementioned support function can be reformulated into the following simplified expression.

$$h_{B(1)}(y, z) = -\sqrt{2yz} - (y + z),\ \ y \leqslant 0,\ \ z \leqslant 0.$$

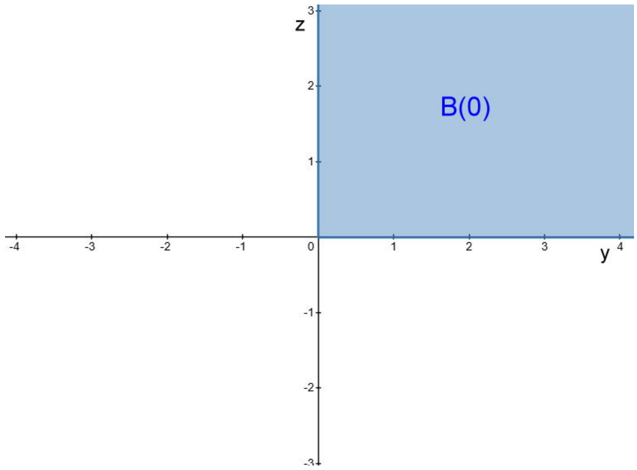


Fig. 17: The set $B(0)$ of FB cone

(2) Let $B(0) = \{(b, c) \in \mathbb{R}^2_+ \mid \sqrt{b^2 + c^2} - (b + c) \leqslant 0\}$, the graph of $B(0)$ is illustrated in Fig. 17.

In this case, it is evident that

$$h_{B(0)}(y, z) = 0, \ y \leqslant 0, \ z \leqslant 0$$

(3) Let $B(-1) = \{(b, c) \in \mathbb{R}^2_+ \mid \sqrt{b^2 + c^2} - (b + c) \leqslant -1\}$, please refer to Fig. 18 for its graph.

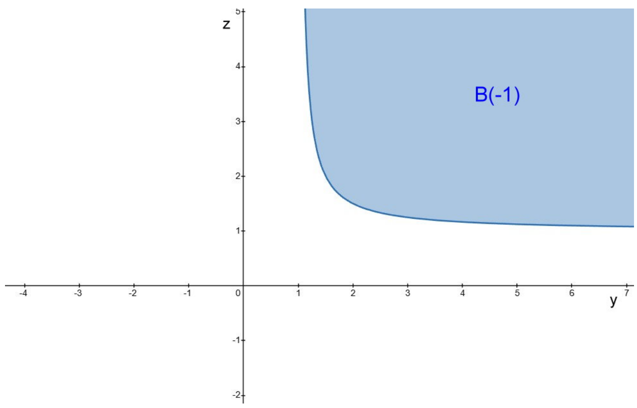


Fig. 18: The set $B(-1)$ of FB cone

The yz plane will be divided into four parts just like the four quadrants. Note that for any (b, c) in $B(-1)$, there exist the following inequalities.

$$b \geqslant \frac{1}{2(c-1)} + 1,\ c \geqslant \frac{1}{2(b-1)} + 1,\ b \geqslant 1,\ c \geqslant 1.$$

(a) In the first quadrant, it's not hard to see that the support function of $B(-1)$ will approach infinity, we omit it here.

(b) In the second quadrant, $y \leqslant 0$ and $z > 0$. Consider the boundary curve $b = \frac{1}{2(c-1)} + 1$ of $B(-1)$.

$$\langle (y, z), (b, c) \rangle = yb + zc = y(\frac{1}{2(c-1)} + 1) + zc$$

The above equation will approach infinity if c approaches infinity.

(c) In the third quadrant, $y > 0$ and $z \leqslant 0$. Consider the boundary curve $c = \frac{1}{2(b-1)} + 1$ of $B(-1)$.

$$\langle (y, z), (b, c) \rangle = yb + zc = yb + z(\frac{1}{2(b-1)} + 1)$$

The above equation will approach infinity if b approaches infinity.

(d) In the fourth quadrant where $y < 0$ and $z < 0$, suppose (b, c) is the vector that satisfies $h_{B(-1)}(y, z) = \langle (y, z), (b, c) \rangle$ for a fixed (y, z) in the fourth quadrant. We have

$$\sqrt{b^2 + c^2} - (b + c) = -1 \tag{6.3}$$

Note that $b > 1$ and $c > 1$. The tangent line at (b, c) with the slope $-\frac{c-1}{b-1}$, is perpendicular to the vector (y, z), where $y \neq 0,\ z \neq 0$. It leads to

$$-\frac{c-1}{b-1} \cdot \frac{z}{y} = -1 \tag{6.4}$$

From (6.3), we can have the relation $b = \frac{1}{2(c-1)} + 1$, $c = \frac{1}{2(b-1)} + 1$. By taking b, c into (6.4) respectively, we have

$$b = \sqrt{\frac{z}{2y}} + 1,\ c = \sqrt{\frac{y}{2z}} + 1.$$

Then,

$$h_{B(-1)}(y, z) = yb + zc = y\sqrt{\frac{z}{2y}} + z\sqrt{\frac{y}{2z}} + (y + z),\ y < 0, z < 0.$$

In addition, we can merge the special cases that $y = 0$ or $z = 0$ in above equation as follows.

$$h_{B(-1)}(y, z) = \begin{cases} y\sqrt{\frac{z}{2y}} + z\sqrt{\frac{y}{2z}} + (y + z) & , \ y < 0,\ z < 0; \\ y & , \ y \leqslant 0,\ z = 0; \\ z & , \ y = 0,\ z \leqslant 0. \end{cases}$$

Note that if $z = 0$, $\langle (y,0),(b,c)\rangle = yb \leqslant y$ because of $b > 1$. if $y = 0$, $\langle (0,z),(b,c)\rangle = zc \leqslant z$ because of $c > 1$.

The aforementioned support function can be reformulated into the following simplified expression.

$$h_{B(-1)}(y,z) = -\sqrt{2yz} + (y+z),\ y \leqslant 0,\ z \leqslant 0.$$

Data availability No data was used for the research described in the article.

Declaration of competing interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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