ON A NEW EQUATION: GREATEST INTEGER VALUE EQUATION

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Abstract. This paper initiates the study of a novel class of equations, referred to as the greatest integer value equation, given by $Ax - \lfloor x \rfloor = b$, where $A \in \Re^{n\times n}$, $b \in \Re^n$, and $\lfloor x \rfloor$ denotes the componentwise greatest integer value of $x \in \Re^n$. This formulation represents a significant extension of the classical absolute value equation to an integer-type structure, and to the best of our knowledge constitutes the first such investigation in the literature.

To solve this new equation, we propose an innovative approach by reformulating it as a combinatorial problem: counting integer lattice points within a parallelotope. This perspective allows us to establish a sufficient condition for the existence of solutions and to derive an explicit upper bound on the number of such solutions in $\Re^2$.

1. Introduction

During the past decades, there has been much attention paid to the *absolute value equation (AVE)*, which is described by

$$Ax + B|x| = b, \tag{1.1}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times n}$, $b \in \Re^n$, and $|x|$ denotes the componentwise absolute value of $x \in \Re^n$. Equation (1.1) was first introduced by Rohn in [20] as a generalization of the equation $Ax - |x| = b$, the latter being the subject of numerous research works for almost two decades now; see [3, 7, 10, 11, 14, 15, 16, 17, 21, 24]. Inspired by the work of Rohn [20], Mangasarian introduced the general non-square system in [13]. It is worth noting that interest in this equation is primarily motivated by its equivalence with the *linear complementarity problem (LCP)*, which encompasses several optimization problems [5, 6, 13, 17, 18]. In addition, the AVEs are also

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intimately related with mixed integer programming [18] and interval linear equations [19].

With its equivalence with the LCP, it is known that solving the AVE (1.1) is likewise an NP-hard problem [13]. Meanwhile, some issues like conditions for existence, non-existence and uniqueness of solutions of the AVE (1.1) are reported in [11, 17, 21, 23]. On the numerical side, there are already many algorithms aimed at solving the AVE (1.1). Roughly, these algorithms can be classified into four categories: (i) Newton methods; (ii) Picard iteration methods; (iii) Matrix splitting iteration method; (iv) Concave minimization approach. Please refer to [1] for more details.

Due to curiosity and the success of existing approaches for the AVE, we propose a replacement of the absolute value function by the greatest integer value function. In particular, we focus on the *greatest integer value equation (GIVE)* given by

$$Ax - \lfloor x \rfloor = b, \tag{1.2}$$

where $A \in \Re^{n\times n}$, $b \in \Re^n$, and $\lfloor x \rfloor$ denotes the componentwise greatest integer value of $x \in \Re^n$.

One may think that only the absolute value function in the AVE is replaced by the greatest integer value function, nothing would be surprising. However, we notice that the main difference between the GIVE (1.2) and the AVE (1.1) is that the greatest integer value function is not continuous at all. This causes a big trouble and hurdle, since we cannot directly apply any fixed point theory on the GIVE. For example, it is seen that applying Banach contraction Principle is adopted for solution analysis of the AVE, but it does not work on the GIVE. Nonetheless, the inconvenience of discontinuity of the greatest integer value function may become an advantage from some other aspect. More specifically, due to the discreteness of the greatest integer value function, we can establish a one-to-one correspondence between the set of integer lattice points in a paralleltope and the set of solutions to a GIVE. The former one is indeed a combinatorial problem [2]. With this observation, we offer another approach to solving the problem, which is a novel idea different from traditional methods. In this paper a parallelotope $\mathbb{P}$ is of the form $\mathbb{P} = d + P[0,1)^n$ where P is an $n \times n$ nonsigular real matrix and $d \in \Re^n$.

To close this section, we point out that there are a few real world problems in signal engineering industry [4, 8, 9, 12, 22, 25], where $\lfloor x \rfloor$ is the main variable or the step function play a key role. Some possible reformulations related to their KKT conditions may lead to the GIVE (1.2) or its variants.

We believe that the study of the GIVE (1.2) will pave a new way to these problems. In other words, a naive start-up of this topic may provide more possibilities and directions.

2. The GIVEs associated with $\Re$

In order to demonstrate some concrete concepts on the GIVEs (1.2), let us begin with a few examples in the setting of real numbers $\Re$.

Example 2.1. Consider the equation $2x - \lfloor x \rfloor = 1.2$, where $x \in \Re$. It is tedious to work out that there are exactly two solutions $x = 0.6$ and $x = 1.1$. An easier way to see this is to examine the graph of the function $f(x) = 2x - \lfloor x \rfloor$, see Figure 1.

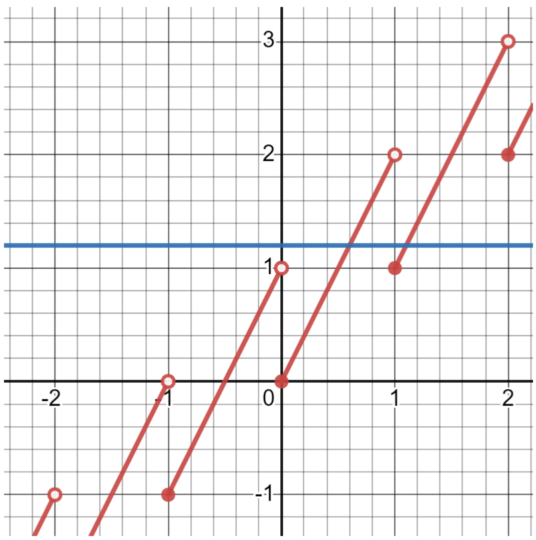


FIGURE 1. Graphs of $y = 2x - \lfloor x \rfloor$(red) and $y = 1.2$(blue). There are two intersection points

Example 2.2. Consider the equation $x - 2\lfloor x \rfloor = 1.2$, where $x \in \Re$. It is also tedious to figure out that there is only one solution $x = -0.8$. Again, an easier way to see this is to examine the graph of the function $f(x) = x - 2\lfloor x \rfloor$, see Figure 2.

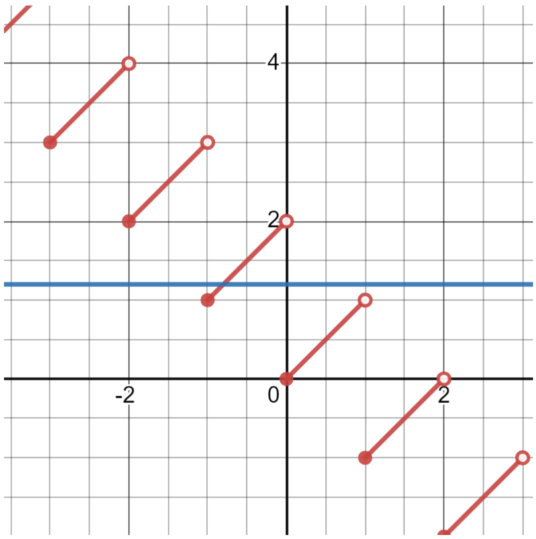


FIGURE 2. Graphs of $y = x - 2\lfloor x \rfloor$(red) and $y = 1.2$(blue). There is only one intersection point.

Example 2.3. Consider the equation $x - \lfloor x \rfloor = 1.2$, where $x \in \Re$. We see that there is a big difference between this equation and those in Example 2.1 and Example 2.2. Actually, the function $f(x) = x - \lfloor x \rfloor = \{x\}$ is the fractional part of $x \in \Re$, whose range is $[0, 1)$. Consequently, the equation $x - \lfloor x \rfloor = 1.2$ has no solution. On the other hand, the equation $x - \lfloor x \rfloor = 0.2$ has infinite solutions. Figure 3 depicts the graph of the function $f(x) = x - \lfloor x \rfloor = \{x\}$.

In general, let a and b be two given real numbers. Set $f(x) := ax - \lfloor x \rfloor$. There are two natural questions to ask. One is on the total number of solutions to the corresponding GIVE $f(x) = b$. The other is on the algorithm towards finding the solutions. We try to answer the first one as below.

First of all, if $a = 0$ or $a = 1$, the GIVE becomes

$$-\lfloor x \rfloor = b \text{ or } x - \lfloor x \rfloor = \{x\} = b.$$

It is easy to check the solutions. Hence, we focus on $a \neq 0, 1$. For convenience, set $I_n := [n, n+1)$. Then, we observe that the image of f on the interval I_n:

$$f(I_n) = \begin{cases} [an,\ a(n+1)) - n = [n(a-1),\ n(a-1)+a) & \text{if} \quad a > 0, \\ (a(n+1),\ an] - n = (n(a-1)+a,\ n(a-1)] & \text{if} \quad a < 0. \end{cases}$$

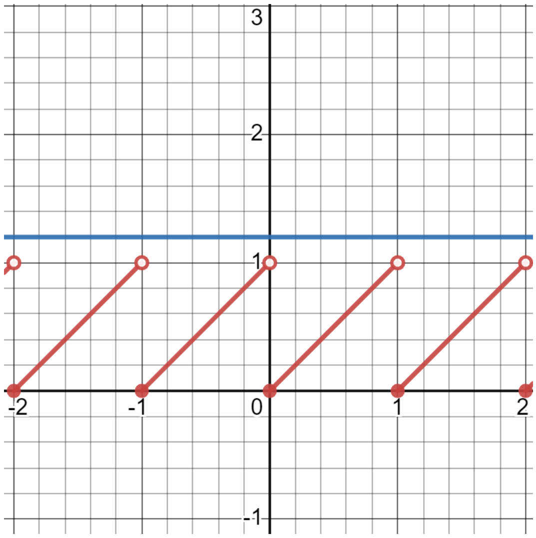


FIGURE 3. Graphs of $y = x - \lfloor x \rfloor$(red) and $y = 1.2$(blue). There is no intersection.

Then we know that $b \in f(I_n)$ for some n if and only if the GIVE $ax - \lfloor x \rfloor = b$ has a solution $\bar{x} \in I_n$ with $\lfloor \bar{x} \rfloor = n$. Hence, this solution $\bar{x} = \frac{b+n}{a}$ belongs to I_n. Therefore, we can conclude that the number of solution to $f(x) = b$ is the number of $f(I_n)'$s where the point b sits. In fact, we can expand $b \in f(I_n)$ as $n(a-1) \leq b < n(a-1) + a$ when $a > 0$ and $n(a-1) + a < b \leq n(a-1)$ when $a < 0$.

The above discussions and observations offer an idea to count the number of solutions to the corresponding GIVE, $f(x) = b$ in $\Re$.

Lemma 2.4. *Suppose I is an half open and half closed interval with length l. Then the number of integers belong to the interval I are $\lfloor l \rfloor$ or $\lfloor l \rfloor + 1$.*

Proof. Suppose $I = [a, b)$. We divide it into four cases.

- $a = \lfloor a \rfloor$ and $b = \lfloor b \rfloor$. Let $J = [\lfloor a \rfloor, \lfloor b \rfloor - 1]$. J contains $\lfloor b \rfloor - \lfloor a \rfloor = \lfloor b - a \rfloor = \lfloor l \rfloor$ integer points.
- $a = \lfloor a \rfloor$ and $b > \lfloor b \rfloor$. Let $J = [\lfloor a \rfloor, \lfloor b \rfloor]$. J contains $\lfloor b \rfloor - \lfloor a \rfloor + 1 = \lfloor b - a \rfloor = \lfloor l \rfloor + 1$ integer points.
- $a > \lfloor a \rfloor$ and $b = \lfloor b \rfloor$. Let $J = [\lfloor a \rfloor + 1, \lfloor b \rfloor - 1]$. In this case, we have $\lfloor b - a \rfloor = \lfloor b \rfloor - \lfloor a \rfloor - 1$. J contains $\lfloor b \rfloor - \lfloor a \rfloor - 1 = \lfloor b - a \rfloor = \lfloor l \rfloor$ integer points.

- $a > \lfloor a \rfloor$ and $b > \lfloor b \rfloor$. Let $J = [\lfloor a \rfloor + 1, \lfloor b \rfloor]$. J contains $\lfloor b \rfloor - \lfloor a \rfloor = \lfloor b - a \rfloor = \lfloor l \rfloor$ integer points.

Note that in each case the closed interval J contains all integers in I. Hence the theorem is true for $I = [a, b)$. For $I = (a, b]$ the proof follows the similar argument. □

Theorem 2.5. *Given two real numbers a and b. Let $a \neq 0, 1$, and $f(x) = ax - \lfloor x \rfloor$. Suppose $b \in f(I_{\bar{n}}) = [\bar{n}(a-1), \bar{n}(a-1)+a)$ when $a > 0$ for some $\bar{n}$ or $b \in f(I_{\bar{n}}) = (\bar{n}(a-1)+a, \bar{n}(a-1)]$ when $a < 0$ for some $\bar{n}$, then $x = \frac{b+\bar{n}}{a}$ is a solution to $f(x) = b$.*

On the other hand, suppose $\bar{x}$ is a solution to $f(x) = b$ with $\bar{n} = \lfloor \bar{x} \rfloor$, then $b \in f(I_{\bar{n}}) = [\bar{n}(a-1), \bar{n}(a-1)+a)$ when $a > 0$ or $b \in f(I_n) = (\bar{n}(a-1)+a, \bar{n}(a-1)]$ when $a < 0$. In either case we have $\bar{x} = \frac{b+\bar{n}}{a}$.

Moreover the number of solutions to $f(x) = b$ is $\lfloor |\frac{a}{a-1}| \rfloor$ or $\lfloor |\frac{a}{a-1}| \rfloor + 1$.

Proof. Suppose $b \in f(I_{\bar{n}})$. Let $\bar{x} = \frac{b+\bar{n}}{a}$. We want to claim $\lfloor \bar{x} \rfloor = \bar{n}$. Note that $b = a\bar{x} - \bar{n}$. That means $\bar{x}$ is a solution to $f(x) = b$. We divide our proof into following two cases.

- For $a > 0$, the inequality $\bar{n}(a-1) \leq b < \bar{n}(a-1) + a$ becomes
$$\bar{n}a \leq b + \bar{n} < (\bar{n}+1)a.$$
- For $a < 0$, the inequality $\bar{n}(a-1) + a < b \leq \bar{n}(a-1)$ becomes
$$\bar{n}a \geq b + \bar{n} > (\bar{n}+1)a.$$

In either case we have $\bar{n} \leq \bar{x} = \frac{b+\bar{n}}{a} < \bar{n} + 1$ which implies $\lfloor \bar{x} \rfloor = \bar{n}$.

Suppose $\bar{x}$ is a solution to $f(x) = b$. Since $\bar{n} = \lfloor \bar{x} \rfloor$, we have $\bar{x} = \bar{n} + \epsilon$ for some $0 \leq \epsilon < 1$. Plug in $f(x) = b$. We have
$$b = a(\bar{n} + \epsilon) - \bar{n} = (a-1)\bar{n} + a\epsilon.$$

Since $0 \leq a\epsilon < a$ when $a > 0$ or $0 \geq a\epsilon > a$ when $a < 0$, it is easy to see $b \in f(I_{\bar{n}})$, and $\bar{x} = \frac{b+\bar{n}}{a}$.

For the last part, we divide our proof into three cases:

- For $a > 1$, the inequality $n(a-1) \leq b < n(a-1) + a$ implies
$$\frac{b-a}{a-1} < n \leq \frac{b}{a-1}.$$
- For $0 < a < 1$, the inequality $n(a-1) \leq b < n(a-1) + a$ implies
$$\frac{b-a}{a-1} > n \geq \frac{b}{a-1}.$$

- For $a < 0$, the inequality $n(a-1) + a < b \leq n(a-1)$ implies
$$\frac{b-a}{a-1} < n \leq \frac{b}{a-1}.$$

In either case, the length of interval which contains integer n is $|\frac{a}{a-1}|$. Then the theorem follows by Lemma 2.4.

$\square$

Example 2.6. Here are examples to illustrate Theorem 2.5.

(i): Let $a = 2$, which says $|\frac{a}{a-1}| = 2$. It can be verified that the function $f(x) = 2x - \lfloor x \rfloor$ is a two-to-one mapping from Figure 1. It means that for any given b, we always have two solution to $f(x) = b$.

(ii): Let $a = 3$, which says $\lfloor|\frac{a}{a-1}|\rfloor = 1$. Consequently, we see from the graph of $3x - \lfloor x \rfloor$ that the number of solutions to $3x - \lfloor x \rfloor = b$ is one or two, which depends on b. Please see Figure 4.

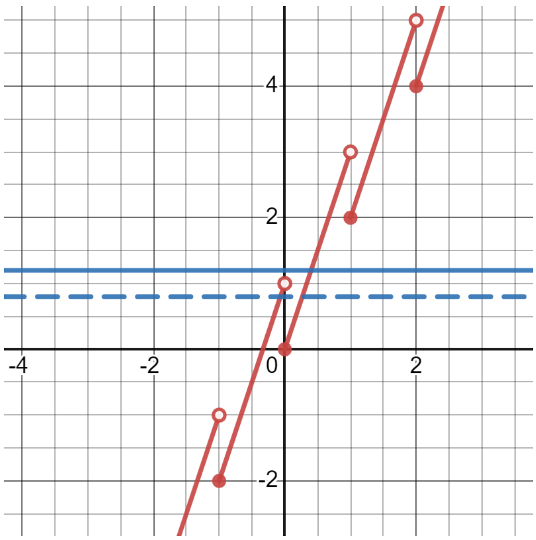


FIGURE 4. Graphs of $3x - \lfloor x \rfloor$(red), $y = 1.2$(blue), and $y = 0.8$(dash blue). There is one intersection point with the blue line, and two intersection points with the dash blue one. Further note that there is only one intersection point when $b \in [1 + 2k, 2 + 2k)$ where $k \in \mathbb{Z}$, and two intersection points otherwise.

3. The GIVEs associated with $\Re^n$

Now, we target the GIVEs (1.2) associated with $\Re^n$. In this setting, we assume that $A \in \Re^{n\times n}$ is a nonsingular $n \times n$ real matrix and $b \in \Re^n$. In order to make our arguments simpler, let us begin with demonstrating the homogeneous equation $Ax - \lfloor x \rfloor = 0$.

First, we know that $x = \lfloor x \rfloor + \{x\}$. Plugging it into the homogeneous equation yields

$$A\{x\} + (A - I)\lfloor x \rfloor = 0,$$

where I is the $n \times n$ identity real matrix. In other words, the homogeneous equation can be transformed into

$$(A - I)\lfloor x \rfloor = -A\{x\}.$$

Note that if there does exist a solution x, then we have that its integer part $\lfloor x \rfloor$ belongs to integer lattice $\mathbb{Z}^n$, and its fractional part $\{x\}$ belongs to the unit cube $[0, 1)^n$. Suppose $(A - I)$ is also a nonsingular real matrix. Then, the homogeneous equation can be further recast into

$$\lfloor x \rfloor = -(A - I)^{-1}A\{x\}.$$

Likewise, following the same idea in Theorem 2.5, we establish a similar result in higher dimension $\Re^n$.

Theorem 3.1. *Assume P and $P + I$ are nonsingular. Let $\mathbb{P} \subset \Re^n$ be a parallelotope of the form $d+P\,[0, 1)^n$. Let $A := P(P+I)^{-1}$ and $b := (A-I)d$. Suppose the translated parallelotope $\mathbb{P}$ contains a point c in the integer lattice $\mathbb{Z}^n$. Then, $\bar{x} = A^{-1}(b + c)$ is a solution to the GIVE, $Ax - \lfloor x \rfloor = b$.*

Proof. Note that $A := P(P+I)^{-1}$, by hypothesis, A is nonsingular. We also have $(P + I)A = PA + A = P$, hence $PA - P = P(A - I) = -A$. We have $A - I$ is also nonsingular, and $P = -(A - I)^{-1}A$.

From previous discussions, the equation $Ax - \lfloor x \rfloor = b$ can be reformulated as

$$(A - I)\lfloor x \rfloor = b - A\{x\},$$

and hence, the GIVE is transformed into

$$\lfloor x \rfloor = (A - I)^{-1}b - (A - I)^{-1}A\{x\} = d + P\{x\}.$$

By hypothesis, there is a $c \in \mathbb{Z}^n \cap \mathbb{P}$. That means there exists an x_0 in the cube $[0, 1)^n$ such that $c = d + Px_0$. Let $\bar{x} = c + x_0$. Then we have

$$\lfloor \bar{x} \rfloor = c, \text{ and } \{\bar{x}\} = x_0.$$

Then the equation $c = d + Px_0$ can be reformulated

$$\lfloor \bar{x} \rfloor = d + P\{\bar{x}\} = (A-I)^{-1}b - (A-I)^{-1}A\{\bar{x}\}.$$

That is $(A-I)\lfloor \bar{x} \rfloor = b - A\{\bar{x}\}$. Hence we have

$$A\bar{x} - \lfloor \bar{x} \rfloor = b,$$

and $\bar{x} = A^{-1}(b+c)$. □

Example 3.2. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$. Denote $\mathbb{P}$ as the parallelogram $-(A-I)^{-1}A\,[0,1)^2 = P[0,1)^2$, whose four vertices are $(0,0)$, $(1,-3)$, $(-1,-2)$ and $(-2,1)$. We observe that there are five integer lattice points inside the parallelogram $\mathbb{P}$. Pick up one of them, say $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

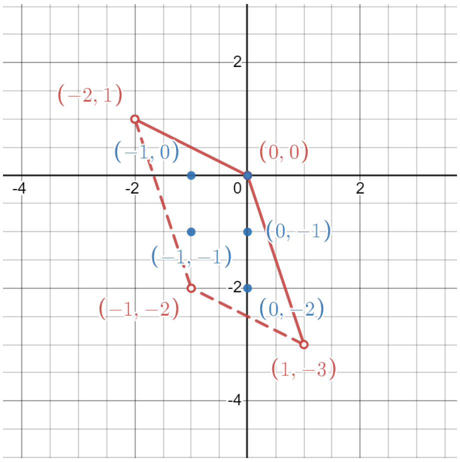
Translate it by $(A-I)^{-1}b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, we have $c = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is an integer lattice point inside the translated parallelogram $(A-I)^{-1}b + \mathbb{P}$. From Theorem 3.1, we see that

$$A^{-1}(b+c) = A^{-1}\left(\begin{bmatrix} 6 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1.2 \\ 3.4 \end{bmatrix}$$

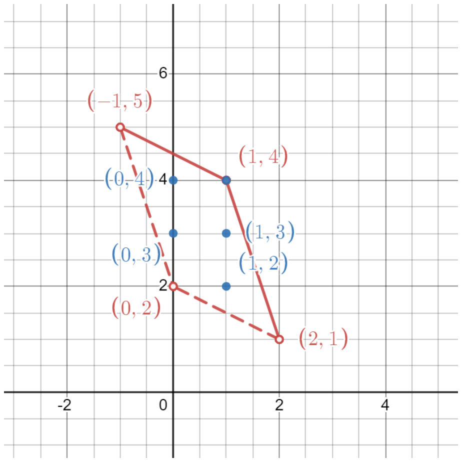
is a solution to the GIVE, $Ax - \lfloor x \rfloor = b$. Figure 5 shows what we discuss in the above.

Example 3.3. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$, which give $(A-I)^{-1}b = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$. Denote $\mathbb{P}$ as the parallelepiped $-(A-I)^{-1}A\,[0,1)^3$, whose eight vertices are $(0,0,0)$, $(1,-3,0)$, $(-1,-2,0)$, $(-2,1,0)$, $(0,0,-2)$, $(1,-3,-2)$, $(-1,-2,-2)$ and $(-2,1,-2)$. It can be verified that there are ten integer lattice points inside the parallelepiped $\mathbb{P}$. Pick up one of them, say $\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$.

Hence, we have $c = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ is an integer lattice point inside the



(A) Graph of the parallelogram $\mathbb{P}$ with five integer lattice points in $\mathbb{P}$.



(B) Graph of the translated parallelogram $(A-I)^{-1}b+\mathbb{P}$ with five integer lattice points in $(A-I)^{-1}b+\mathbb{P}$.

FIGURE 5. Graphs of the parallelograms.

translated parallelepiped $(A-I)^{-1}b+\mathbb{P}$. By Theorem 3.1, we see that

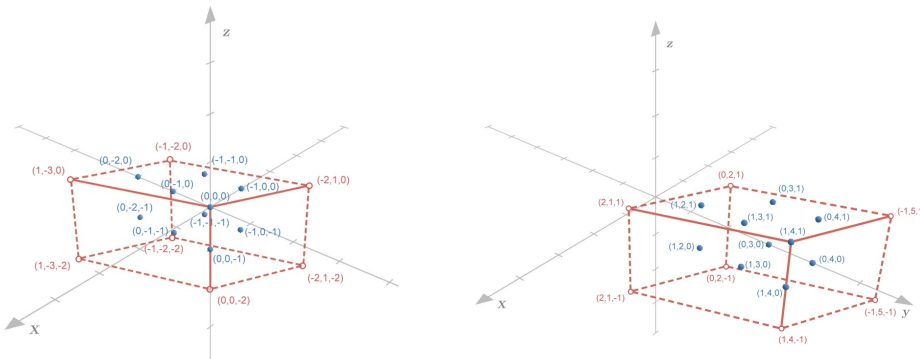
$$A^{-1}(b+c) = A^{-1}\left(\begin{bmatrix}6\\5\\1\end{bmatrix}+\begin{bmatrix}1\\3\\0\end{bmatrix}\right) = \begin{bmatrix}1.2\\3.4\\0.5\end{bmatrix}$$

is a solution to the GIVE, $Ax-\lfloor x\rfloor = b$. Please see Figure 6 for matching up the above discussions.

Theorem 3.4. *Assume P and $P+I$ are nonsingular. Let $\mathbb{P}\subset\Re^n$ be a parallelotope of the form $d+P\,[0,1)^n$. Let $A := P(P+I)^{-1}$ and $b := (A-I)d$. Then, there exists a one to one correspondence between the set of all integer lattice points contained in $\mathbb{P}$ and the set of all solutions to the GIVE, $Ax-\lfloor x\rfloor = b$.*

Proof. Since one direction is proved in Theorem 3.1, we need only prove the other direction.

Note that $A := P(P+I)^{-1}$, by hypothesis, A is nonsingular. We also have $(P+I)A = PA + A = P$, hence $PA - P = P(A-I) = -A$. We have $A-I$ is also nonsingular, and $P = -(A-I)^{-1}A$.



(A) Graph of the parallelepiped $\mathbb{P}$ with ten integer lattice points in $\mathbb{P}$.

(B) Graph of the translated parallelepiped $(A-I)^{-1}b+\mathbb{P}$ with ten integer lattice points in $(A-I)^{-1}b+\mathbb{P}$.

FIGURE 6. Graphs of the parallelepipeds .

Suppose we have a solution $\bar{x}$ to GIVE $Ax - \lfloor x \rfloor = b$. We want to claim that $c = \lfloor \bar{x} \rfloor$ belongs to $\mathbb{P}$. By hypothesis $c = \lfloor \bar{x} \rfloor = A\bar{x} - b$. Hence we $\bar{x} = A^{-1}(b+c)$. Let $x_0 = \bar{x} - c$. Since $c = \lfloor \bar{x} \rfloor$, we have x_0 is in the cube $[0,1)^n$. Hence $\{\bar{x}\} = x_0$.

Plug in $A\bar{x} - \lfloor \bar{x} \rfloor = b$. We have

$$(A-I)\lfloor \bar{x} \rfloor = b - A\{\bar{x}\} = b - Ax_0.$$

Thus $\lfloor x \rfloor = (A+I)^{-1}b - (A-I)^{-1}Ax_0 = d + Px_0$. That means $\lfloor x \rfloor \in \mathbb{Z} \cap \mathbb{P}$. □

The following is an intuitive scheme to find out all integer lattice points inside a parallelotope. In fact, it is an exhaustive method.

Suppose we are given a parallelotope $\mathbb{P} \subset \Re^n$, which has 2^n vertices

$$\{v_i \in \Re^n |\ v_i = (v_{i1}, \cdots, v_{ij}, \cdots, v_{in})_{1 \le i \le 2^n}\}.$$

We aim to count all integer lattice points $c = (k_1, \cdots, k_j, \cdots, k_n) \in \mathbb{Z}^n$, which fall inside the parallelotope $\mathbb{P}$. The exhaustive method is as below.

- Step 1: Find all $b_j := \max\limits_{1 \le i \le 2^n} v_{ij}$ and $a_j := \min\limits_{1 \le i \le 2^n} v_{ij}$ for each $1 \le j \le n$.
- Step 2: Find all candidates $c = (k_1, \cdots, k_j, \cdots, k_n) \in \mathbb{Z}^n$ satisfying
$$a_j \le k_j \le b_j.$$
- Step 3: We check if c falls in $\mathbb{P}$ by using the corresponding GIVE.

Actually, we construct a big cube $\mathcal{C} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ such that $\mathbb{P} \subset \mathcal{C}$. Those candidates c are the all integer lattice points in the cube $\mathcal{C}$. As one can expect it is much easier to check c in $\mathbb{P}$ provided $\mathbb{P}$ is more regular, for example $\mathbb{P}$ is a regular cube itself. If $\mathbb{P}$ is not that regular, we can apply the corresponding GIVE.

With this one to one correspondence, we can build up a sufficient condition for the existence of solutions to the GIVE, $Ax - \lfloor x \rfloor = b$, for any given b.

The following lemma is a consequence of Minkowski's Theorem. We give a proof here for reader's convenience.

Lemma 3.5. *Suppose $\mathbb{B} \subset \Re^n$ is a ball with radius $r \geq \frac{\sqrt{n}}{2}$. Then $\mathbb{B} \cap \mathbb{Z}^n \neq \emptyset$.*

Proof. Suppose the center of $\mathbb{B}$ is $(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$. Then the closest integer lattice points to the center are of the form $(k_1, k_2, \cdots, k_n)$ where $k_i \in \{0, 1\}$. The distance of these integer lattice points between the center is $\frac{\sqrt{n}}{2}$. Hence $\mathbb{B} \cap \mathbb{Z}^n \neq \emptyset$.

Suppose the center of $\mathbb{B}$ is $(\frac{1}{2} + \epsilon_1, \frac{1}{2} + \epsilon_2, \cdots, \frac{1}{2} + \epsilon_n)$ where $|\epsilon_i| \leq \frac{1}{2}$. Then the closest integer lattice points to the center are of the form $(k_1, k_2, \cdots, k_n)$ where $k_i = 1$ provided $\epsilon_i \geq 0$ and $k_i = 0$ provide $\epsilon_i \leq 0$. The distance of these integer lattice points between the center is not greater than $\frac{\sqrt{n}}{2}$. Hence $\mathbb{B} \cap \mathbb{Z}^n \neq \emptyset$.

Suppose in general the center is $(\frac{1}{2} + m_1 + \epsilon_1, \frac{1}{2} + m_2 + \epsilon_2, \cdots, \frac{1}{2} + m_n + \epsilon_n)$ where $m_i \in \mathbb{Z}$. Then the closest integer lattice points to the center are of the form $(k_1 + m_1, k_2 + m_2, \cdots, k_n + m_n)$ where $k_i = 1$ provided $\epsilon_i \geq 0$ and $k_i = 0$ provide $\epsilon_i \leq 0$. By the same argument we have $\mathbb{B} \cap \mathbb{Z}^n \neq \emptyset$. □

Corollary 3.6. *Let $\mathbb{P} \subset \Re^n$ be a parallelotope of the form $(A - I)^{-1}b - (A - I)^{-1}A\,[0, 1)^n$. Suppose that the radius of the largest ball enclosed in the paralleotope $\mathbb{P}$ is greater than $\frac{\sqrt{n}}{2}$, then there is at least one solution to the GIVE, $Ax - \lfloor x \rfloor = b$, for any given b.*

Proof. By Lemma 3.5 the enclosed ball must contain at least an integer lattice point. That is the parallelotope $\mathbb{P}$ contains one, too. Applying Theorem 3.4, the corollary follows. □

Here are some examples in $\Re^2$.

Example 3.7. As depicted in Figure 7, the red one is the graph of $2x + y - \lfloor x \rfloor = 2$. The blue one is the graph of $2x + 3y - \lfloor y \rfloor = 1$.

Example 3.8. As depicted in Figure 8, the red one is the graph of $2x + y - \lfloor x \rfloor = 2$. The blue one is the graph of $2x + 3y - \lfloor y \rfloor = 2.5$.

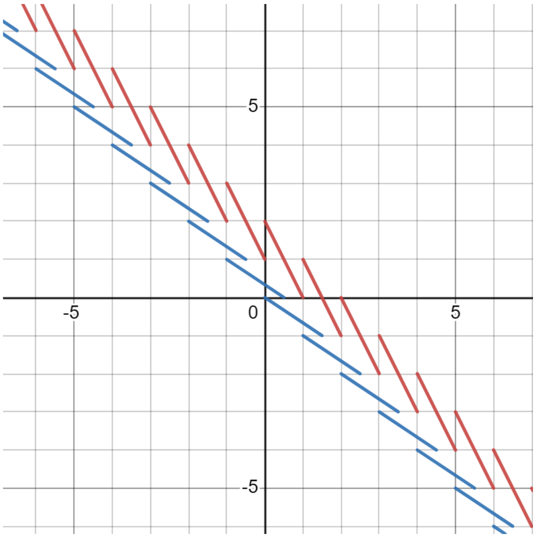


FIGURE 7. Graphs of $2x + y - \lfloor x \rfloor = 2$ and $2x + 3y - \lfloor y \rfloor = 1$.

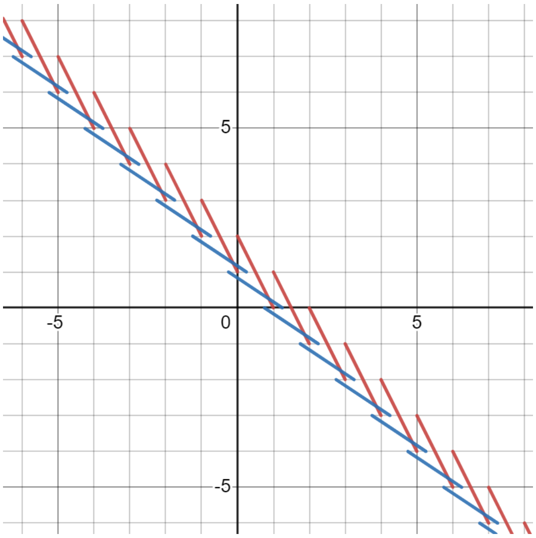


FIGURE 8. Graphs of $2x+y-\lfloor x \rfloor = 2$ and $2x+3y-\lfloor y \rfloor = 2.5$.

Example 3.7 and Example 3.8 show that GIVE in $\Re^2$ may also have no solutions or infinitely many solutions.

Let us consider the general system of equations

$$ax + by - \lfloor x \rfloor = g;$$

$$cx + dy - \lfloor y \rfloor = h.$$

Or in a compact form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \lfloor x \rfloor \\ \lfloor y \rfloor \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}. \tag{3.1}$$

Using $\lfloor x \rfloor = x - \{x\}$ we can rewrite the equation $ax + by - \lfloor x \rfloor = g$ as

$$(a-1)x + by = g - \{x\}.$$

Note that $0 \le \{x\} < 1$. We have

$$g - 1 < (a-1)x + by = g - \{x\} \le g.$$

Then we can point that the graph of $ax+by-\lfloor x \rfloor = g$ looks like the red ladders sitting inside the strip bounded by two green dash lines $(a-1)x+by = g$ and $(a-1)x + by = g - 1$, see Figure 9.

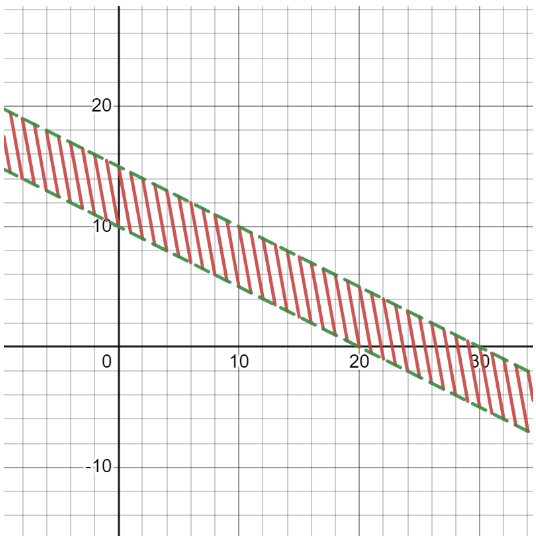


FIGURE 9. Graph of the ladders $ax + by - \lfloor x \rfloor = g$(red) and the strip bounded by two lines $(a-1)x + by = g$(dash green) and $(a-1)x + by = g - 1$(dash green).

In addition, as depicted in Figure 10, putting another strip bounded by $cx+(d-1)y=h$ and $cx+(d-1)y=h-1$ along with the strip bounded by $(a-1)x+by=g$ and $(a-1)x+by=g-1$. The intersection points are all in a parallelogram formed by the four lines, and these points are the solution to the system of equation 3.1.

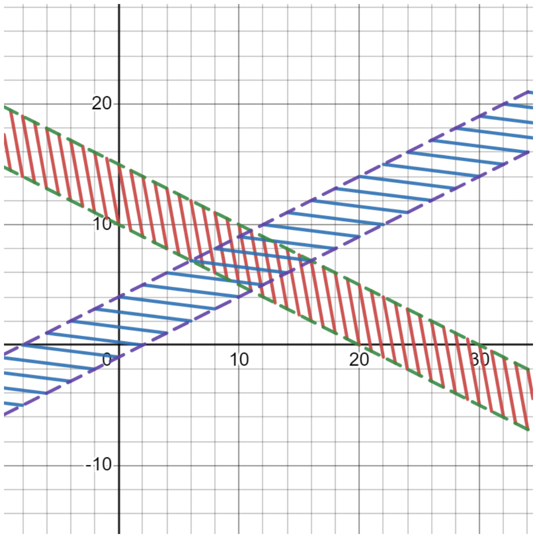


FIGURE 10. Graph of the ladders $ax+by-\lfloor x\rfloor=g$ and $cx+dy-\lfloor y\rfloor=h$ and the parallelogram bounded by four lines $(a-1)x+by=g, (a-1)x+by=g-1, cx+(d-1)y=h$ and $cx+(d-1)y=h-1$.

Rewrite the system of equation 3.1 as

$$\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} g \\ h \end{bmatrix}-\begin{bmatrix} \{x\} \\ \{y\} \end{bmatrix}. \tag{3.2}$$

For convenience, set $A:=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It is clear that these two strips are parallel if $D=\det(A-I)=(a-1)(d-1)-bc=0$. Moreover, the GIVE $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}-\begin{bmatrix} \lfloor x\rfloor \\ \lfloor y\rfloor \end{bmatrix}=\begin{bmatrix} g \\ h \end{bmatrix}$ have no solution or infinite solutions if $D=0$, as shown in Example 3.7 and Example 3.8.

In order to conclude the discussions and estimate the number of solutions, we set up

$$\begin{aligned} i &= \frac{1}{D}\left[(1+b-d)a+(1-a+c)b\right], \\ j &= \frac{1}{D}\left[(1-b-d)a+(a+c-1)b\right], \\ k &= \frac{1}{D}\left[(1+b-d)c+(1-a+c)d\right], \\ l &= \frac{1}{D}\left[(1-b-d)c+(a+c-1)d\right]. \end{aligned}$$

Lemma 3.9. *Suppose the parallelogram $\mathbb{P}$ is formed by four lines*

$$\begin{aligned} (a-1)x+by &= g, \\ (a-1)x+by &= g-1, \\ cx+(d-1)y &= h, \\ cx+(d-1)y &= h-1. \end{aligned}$$

Let L and M be the two diagonals of $\mathbb{P}$. Then the graph of $ax+by-\lfloor x\rfloor = g$ intersects L or M at most m times where $m = \max(\lfloor |i|\rfloor, \lfloor |j|\rfloor)+1$.

Proof. Let v_1, v_2, v_3 and v_4 be the four vertices of $\mathbb{P}$. By using Cramer's formula we can compute these vertices,

$$v_1 = \frac{1}{D}\left(\begin{vmatrix} g & b \\ h & d-1\end{vmatrix}, \begin{vmatrix} a-1 & g \\ c & h\end{vmatrix}\right); v_2 = \frac{1}{D}\left(\begin{vmatrix} g & b \\ h-1 & d-1\end{vmatrix}, \begin{vmatrix} a-1 & g \\ c & h-1\end{vmatrix}\right)$$

$$v_3 = \frac{1}{D}\left(\begin{vmatrix} g-1 & b \\ h-1 & d-1\end{vmatrix}, \begin{vmatrix} a-1 & g-1 \\ c & h-1\end{vmatrix}\right); v_4 = \frac{1}{D}\left(\begin{vmatrix} g-1 & b \\ h & d-1\end{vmatrix}, \begin{vmatrix} a-1 & g-1 \\ c & h\end{vmatrix}\right)$$

and its two diagonal vectors $\vec{d_1} = L = v_1v_3$ and $\vec{d_2} = M = v_2v_4$.

$$\vec{d_1} = \frac{1}{D}(1+b-d, 1-a+c), \vec{d_2} = \frac{1}{D}(1-b-d, a+c-1).$$

The gap **vector** between each step of the red ladder $ax+by-\lfloor x\rfloor = g$ is $\vec{n} = (\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2})$ in the normal direction of $ax+by = g$. Note that the gap vector is the direction one moves from the line $ax+by-0 = g$ to the other line $ax+by-1 = g$. See Figure 11.

The length of the projection vector of $\vec{d_1}$ along $\vec{n}$ is $p := |\vec{d_1}\cdot\vec{n}|/|\vec{n}|$. Let $q := p/|\vec{n}| = |i|$. We know q is the number of units of the length p along the direction of gap vector $\vec{n}$. Then we can see the diagonal L intersects the red ladder at most $\lfloor |i|\rfloor + 1$ times. Similarly, the diagonal M intersects the red ladder at most $\lfloor |j| \rfloor + 1$ times. Hence the theorem follows. See Figure 11. □

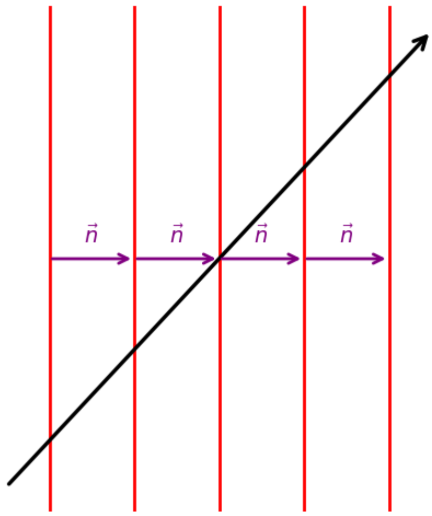


FIGURE 11. Gap vector $\vec{n}$ of red ladder and the black diagonal vector.

Theorem 3.10. *Suppose that* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a nonsingular* 2×2 *real matrix,* $(A - I)$ *is also nonsingular, and* $D = (a-1)(d-1) - bc \neq 0$. *Then, the corresponding GIVE*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \lfloor x \rfloor \\ \lfloor y \rfloor \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}$$

has at most mn *solutions, where* $m = \lfloor \max(|i|, |j|) \rfloor + 1$ *and* $n = \lfloor \max(|k|, |l|) \rfloor + 1$.

Proof. By Lemma 3.9 the red ladder $ax + by - \lfloor x \rfloor = g$ intersects the two diagonals of the parallelogram at most m times. Similarly, the the blue ladder $cx + dy - \lfloor x \rfloor = h$ intersects the two diagonals of the parallelogram at most n times.

Hence there are at most mn intersection points of red ladder and blue ladder inside the parallelogram. That is there are at most mn solutions of the corresponding GIVE. □

The following example shows that the number of solutions can exactly be mn.

Example 3.11. Let $A = \begin{bmatrix} 5 & 3 \\ 1 & -1 \end{bmatrix}$, which indicates $m = 2$ and $n = 1$. Picking up $\begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} 5 \\ 5.1 \end{bmatrix}$, we see that corresponding GIVE is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \lfloor x \rfloor \\ \lfloor y \rfloor \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}$, which has exactly 2 solutions. Figure 12 provides an illustration.

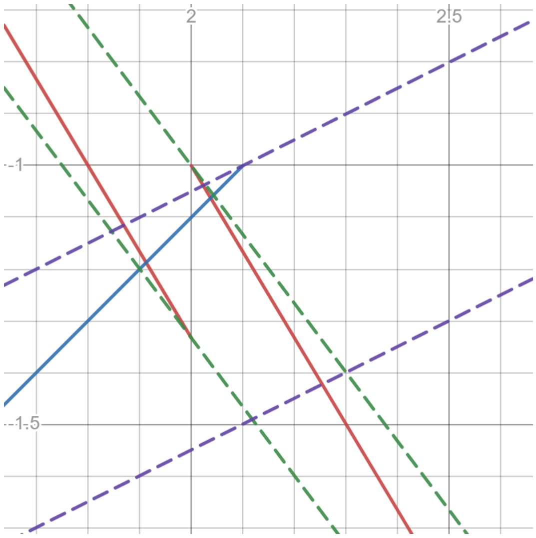


FIGURE 12. Graph of exact mn solutions

4. CONCLUSION

To sum up, we propose a novel equation, the greatest integer value equation (GIVE). To tackle with it, we establish a one to one correspondence between the set of all integer lattice points contained in a parallelotope $\mathbb{P}$ and the set of all solutions to the GIVE, $Ax - \lfloor x \rfloor = b$. We also provide a sufficient condition for existence of solutions to the GIVE and a upper bound for the number of solutions in $\Re^2$.

Here is a possible direct application of the GIVE to the integer programming.

$$\min f(x)$$
$$x \in \mathbb{P} \subset \Re^n, x \in \mathbb{Z}^n$$

The constraint condition $x \in \mathbb{P} \subset \Re^n, x \in \mathbb{Z}^n$ is much harder to check. By the one to one correspondence, we can reformulate it into GIVE. Of course the integer programming is NP-hard. That means GIVE is also NP-hard as well as AVE. However recall that we are inspired by the successful study on AVE, but in the end the GIVE shed us a light to an unexpected word. The works on AVE may help us to solve GIVE, or to integer programming or vice versa.

This is a start-up and initial study for the GIVE, $Ax - \lfloor x \rfloor = b$. We believe that there are more to be investigated in the future, and we list here some possible directions which we have not touched in this paper.

- Some other applications to the GIVE.
- More direct and efficient algorithms to the GIVE.
- Properties of solutions: for example, uniqueness of solution or lower bound for the number of solutions.

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