Commun Nonlinear Sci Numer Simulat 152 (2026) 109471

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research Paper

Viscosity inexact projection methods with applications to electricity production models

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ARTICLE INFO

2020 MSC:
47H05
47J20
47J25
49J40
65K15

Keywords:
Multivalued variational inequalities
Pseudomonotone
Hausdorff Lipschitz continuous
Viscosity method
Inexact projection
Electricity production models

ABSTRACT

In this paper, we propose a new inexact projection approach for multivalued variational inquality problems without using inertial technique in a real Hilbert space. First, we introduce a new inxact projection and show some its properties. By combining the inexact projection, viscosity technique and Mann-type iteration method via self-adaptive step size, we introduce a new VIP-Viscosity Inexact Projection Algorithm and prove its strong convergence under Lipschitz continuous and pseudomonotone assumptions of the cost multivalued mapping. Next, we apply the algorithm VIP to propose a new algorithm for the electricity production models. The effectiveness of the algorithm VIP is then validated by numerical experiments via computational examples of the electricity production models.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $\mathbb{C}$ be a nonempty, closed and convex subset of the $\mathcal{H}$. Denote the weak convergence by "$\rightharpoonup$" and the strong convergence by "$\to$". We consider the multivalued variational inequality problem, first introduced by Kinderlehrer and Stampacchia in [1], shortly MVI$(\mathbb{C}, \mathcal{A})$, that is given in the following form:

$$\text{Find } x^* \in \mathbb{C}, w^* \in \mathcal{A}(x^*) \text{ such that } \langle w^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathbb{C},$$

where $\mathcal{A} : \mathbb{C} \to 2^{\mathcal{H}}$ is usually called *multivalued cost mapping*. It is well-known to see that this problem is difficult from the cost multivalued mapping $\mathcal{A}$. The problem MVI$(\mathbb{C}, \mathcal{A})$ has a widely applicable across various disciplines including game theory, control theory, mathematical economics such as well-known Nash-Cournot economic model, and other applied sciences. Its solution methods have witnessed an interesting attention among many researchers in arrays of disciplines. This is due to its active links with many fields of study, such as poroelasticity for petroleum engineering [2–6], financial analysis in economics [7], the reconstruction of images

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https://doi.org/10.1016/j.cnsns.2025.109471
Received 16 August 2025; Received in revised form 27 September 2025; Accepted 27 October 2025
Available online 31 October 2025

in imaging processing [8–10], telecommunication networks or public roads and noncooperative games [11,46], among many others [7,12–16,47]. It is remarkably known that if $\mathcal{A} = \nabla h$, where $h : \mathbb{C} \to \mathcal{R}$ is a convex and differentiable function with its gradient ∇h, then the problem MVI($\mathbb{C}, \mathcal{A}$) is equivalent to the convex problem:

$$\min\{h(x) : x \in \mathbb{C}\}.$$

It is clear that a point $\bar{x} \in \mathbb{C}$ is a solution of the problem MVI($\mathbb{C}, \mathcal{A}$) if and only if it belongs to the fixed point set of the multivalued solution mapping: $\mathcal{A}_\lambda : \mathbb{C} \to 2^{\mathbb{C}}$ defined in the form

$$\mathcal{A}_\lambda(x) = \left\{\Pi_{\mathbb{C}}[x - \lambda w_x] : w_x \in \mathcal{A}(x)\right\}, \quad \forall x \in \mathbb{C},$$

where $\lambda \in (0, \infty)$ and $\Pi_{\mathbb{C}}$ is the metric projection from $\mathcal{H}$ onto $\mathbb{C}$. To apply the problem MVI($\mathbb{C}, \mathcal{A}$), one of the earliest iterative algorithms is the projection gradient algorithm (PGA) in [17], defined by the update rule:

$$x^0 \in \mathbb{C}, x^{k+1} \in \mathcal{A}_\lambda(x^k).$$

Under the $\beta-$strongly monotone and Hausdorff $L-$Lipschitz continuous assumptions of the cost mapping $\mathcal{A}$, and $0 < \lambda < \frac{2\beta}{L}$, then the solution mapping $\mathcal{A}_\lambda$ is contractive. By the Banach Contraction Principle, the sequence $\{x^k\}$ converges strongly to the unique solution of the problem MVI($\mathbb{C}, \mathcal{A}$). Basing on this idea, the algorithm (PGA) has attracted significant attention, and several interesting projection algorithms have been derived to solve the problem MVI($\mathbb{C}, \mathcal{A}$). Let us list some popular algorithms such as Cegielski and Zalas in [18,19], Ceng et al. in [20,21], Gibali et al. in [22,23], Anh et al. in [24–29], Maingé in [30,31], Shehu and Iyiola in [32], Iusem and Svaiter in [33], Izuchukwu et al. in [8], Fan et al. [34,35] and many research results in some interesting books [7,36]. Most of these algorithms, at each iteration, it requires to compute the metric projection point of an iteration point onto the convex set $\mathbb{C}$. However, computing of this projection point has some the following restrictions:

- It is easy to calculate in practice with special structures of $\mathbb{C}$. Examples as, the nonnegative orthant, more generally, a box, a half space or the intersection of two half spaces;
- On computational softwares, the exact projection point requires solving a convex quadratic optimization problem. It is too expensive in some general cases of the set $\mathbb{C}$. In fact, this can significantly raise the cost per iteration if the number of unknowns is large.

There exist some solution methods that proposed in an effort to reduce the computational cost without using the metric projection $\Pi_{\mathbb{C}}$. Let us briefly list some popular algorithms for solving the problem MVI($\mathbb{C}, \mathcal{A}$), where the $\mathcal{A}$ only is single-valued, such as Outer Approximation Schemes proposed by Burachik and Lopes [37], Block-Iterative Outer Approximation Methods introduced by Combettes [38], Outer Approximation Methods of Gibali et al. [39] and some recent methods in some interesting books [7,36].

Basing on this fact, there exists a natural question that: *Can we propose an inexact projection-type approach with one evaluation of the multivalued cost mapping $\mathcal{A}$, one computation of the inexact projection point without using $\Pi_{\mathbb{C}}$, by using viscosity technique to solve the problem MVI($\mathbb{C}, \mathcal{A}$)?*

Our contribution in this paper is to answer the above question in the affirmative and offers a brief overview of our results and their distinction from prior results. We propose a new inexact projection algorithm to solve the problem MVI($\mathbb{C}, \mathcal{A}$) with the following details:

- It differs from existing algorithms, even in special cases that one inexact projection onto the constraint set $\mathbb{C}$ at each iteration is instead of the metric projection $\Pi_{\mathbb{C}}$;
- our iteration algorithm only uses two evaluations of the multivalued cost mapping $\mathcal{A}$ at each iteration;
- we use viscosity technique via a contractive mapping to show that the cluster point of iteration sequences is a unique solution of another strongly monotone variational inequality problem;
- finally, we apply the proposed algorithm for the electricity production models with numerical experiments.

The outline of the paper is as follows. Section 2 introduces a foundation by basic definitions, comparing the metric projection with the inxact projection, reviewing essential concepts and relevant prior lemmas. One important concept is the "inexact projection onto the constraint $\mathbb{C}$". Basing on the inexact projection, Section 3 presents a new viscosity inexact projection algorithm and shows its strong convergence. In Section 4, the numerical experiences are performed to evaluate our proposed algorithm via the electricity production models.

2. Preliminaries

We recall several concepts which will be used in this paper. These definitions, lemmas and properties are well-known and can be found, e.g., in two interesting books [7,36].

Now, we work with a real Hilbert space $\mathcal{H}$. Let $\mathbb{C}$ be a nonempty, convex and closed subset of $\mathcal{H}$. The metric projection of $a \in \mathcal{H}$ onto $\mathbb{C}$ is denoted by $\Pi_{\mathbb{C}}(a)$. It is the unique solution of the quadratical convex programming:

$$\Pi_{\mathbb{C}}(a) = \operatorname{argmin}\left\{\|a - y\|^2 : a \in \mathbb{C}\right\}. \tag{2.1}$$

It is clear that the mapping $\Pi_{\mathbb{C}}$ is from $\mathcal{H}$ onto $\mathbb{C}$, and $\|\Pi_{\mathbb{C}}(a) - \Pi_{\mathbb{C}}(b)\| \leq \|a-b\|, \quad \forall a, b \in \mathcal{H}$. This property is usually called *nonexpansive*. Moreover, the mapping $\Pi_{\mathbb{C}}$ is the 1−strongly quasi-nonexpansive, i.e.,

$$\|\Pi_{\mathbb{C}}(a) - \bar{a}\|^2 \leq \|a-\bar{a}\|^2 - \|a - \Pi_{\mathbb{C}}(a)\|^2, \quad \forall a \in \mathcal{H}, \bar{a} \in \mathbb{C}. \tag{2.2}$$

Let us recall some popular definitions of Lipschitz continuous and monotone mappings in $\mathcal{H}$ used in the problem MVI$(\mathbb{C}, \mathcal{A})$. The cost mapping $\mathcal{A} : C \to 2^{\mathcal{H}}$ is called to be *monotone* on $\mathbb{C}$, if

$$\langle w_u - w_v, u - v\rangle \geq 0, \quad \forall u, v \in \mathbb{C}, w_u \in \mathcal{A}(u), w_v \in \mathcal{A}(v);$$

pseudomonotone on $\mathbb{C}$, if

$$\langle w_v, u - v\rangle \geq 0 \Rightarrow \langle w_u, u - v\rangle \geq 0, \quad \forall u, v \in \mathbb{C}, w_u \in \mathcal{A}(u), w_v \in \mathcal{A}(v);$$

Hausdorff $\mathcal{L}$−Lipschitz continuous, where constant $\mathcal{L} > 0$, on $\mathbb{C}$, if

$$\rho(\mathcal{A}(u), \mathcal{A}(v)) \leq \mathcal{L}\|u - v\|, \quad \forall u, v \in \mathbb{C};$$

η−partially pseudomonotone with constant $\eta > 0$, on $C \subset \mathbb{C}$, if

$$\langle w_z, v - z\rangle \geq 0 \Rightarrow \langle w_v, v - z\rangle \geq \eta\|v - z\|^2, \quad \forall v \in \mathbb{C}, z \in C, w_z \in \mathcal{A}(z), w_v \in \mathcal{A}(v);$$

partially pseudomonotone on $C \subset \mathbb{C}$, if we have

$$\langle w_z, v - z\rangle \geq 0 \Rightarrow \langle w_v, v - z\rangle \geq 0, \quad \forall v \in \mathbb{C}, z \in C, w_z \in \mathcal{A}(z), w_v \in \mathcal{A}(v).$$

Consider in the problem MVI$(\mathbb{C}, \mathcal{A})$. Note that $\mathbb{C}$ is a nonempty, closed and convex subset of $\mathcal{H}$. In this paper, we use an *inexact projection* of $u \in \mathcal{H}$ onto $\mathbb{C}$ relatives to any point $z \in \mathcal{H}$ with computational error $\tau \geq 0$, shortly $\Pi_{\mathbb{C}}^{\tau,z}(u)$. It is defined as follows:

$$\Pi_{\mathbb{C}}^{\tau,z}(u) = \left\{w \in \mathbb{C} : \langle u - w, v - w\rangle \leq \tau\|w - z\|^2, \quad \forall v \in \mathbb{C}\right\}. \tag{2.3}$$

Remark 1. By using (2.3), it follows that

$$\Pi_{\mathbb{C}}(u) \in \Pi_{\mathbb{C}}^{\tau,z}(u), \quad \forall u \in \mathcal{H}, z \in \mathcal{H}, \tau \geq 0.$$

Indeed, by the definition (2.1) of the metric projection $\Pi_{\mathbb{C}}$ of a point $u \in \mathcal{H}$ onto $\mathbb{C}$, we get that $\Pi_{\mathbb{C}}(u) \in \mathbb{C}$ and $\langle u - \Pi_{\mathbb{C}}(u), y - \Pi_{\mathbb{C}}(u)\rangle \leq 0$ for all $y \in \mathbb{C}$. From this and $\tau \geq 0$, it yields

$$\langle u - \Pi_{\mathbb{C}}(u), y - \Pi_{\mathbb{C}}(u)\rangle \leq \tau\|\Pi_{\mathbb{C}}(u) - z\|^2.$$

Thus, $\Pi_{\mathbb{C}}(u) \in \Pi_{\mathbb{C}}^{\tau,z}(u)$.

Otherwise, the $\Pi_{\mathbb{C}}^{\tau,z}$ is a multivalued mapping from $\mathcal{H}$ to $\mathbb{C}$. In the case $\tau = 0$, we have $\Pi_{\mathbb{C}}^{0,z} = \{\Pi_{\mathbb{C}}\}$ for all $z \in \mathcal{H}$. Thus, for each $\tau \geq 0$ and $z \in \mathcal{H}$, the inexact projection $\Pi_{\mathbb{C}}^{\tau,z}$ is an extended formulation of the metric projection $\Pi_{\mathbb{C}}$. However, there exists a problem arose on the computer: *What is the benefit of using the inexact projection $\Pi_{\mathbb{C}}^{\tau,z}$ without the projection $\Pi_{\mathbb{C}}$?* First, computing the projection of a point x onto $\mathbb{C}$ is to solve the quadratical convex programming:

$$\min\{\|x - y\|^2 : y \in \mathbb{C}\}.$$

By using the well-known Gradient Descent Method, computing an inexact projection point $y_x \in \Pi_{\mathbb{C}}^{\tau,z}(x)$ of a point x onto the constraint $\mathbb{C}$ with respect to the iteration point $z \in \mathcal{H}$ and the parameter $\tau > 0$ is very effective on computer as follows:

Procedure 2.1. *(for finding $y_x \in \Pi_{\mathbb{C}}^{\tau,z}(x)$)*

St. 1: Taking $k = 0, y^0 \in \mathbb{C}, z \in \mathcal{H}$ and $\tau \in (0, \infty)$.
St. 2: Solve the linear programming with the convex constraint:

$$v^k = \operatorname{argmin}\{\langle y^k - x, y - y^k\rangle : y \in \mathbb{C}\}.$$

If $\langle y^k - x, y^k - v^k\rangle \leq \tau\|y^k - z\|^2$, then Stop, i.e., $y_x = y^k$. Otherwise, compute $y^{k+1} = y^k + \delta_k(v^k - y^k)$ with the stepsize $\delta_k = \min\left\{1, \frac{\langle y^k - x, y^k - v^k\rangle}{\|v^k - y^k\|^2}\right\}$.
St. 3: Repeat $k := k + 1$ and come back to St. 2.

On MATLAB Software, computing y_x of a point x on computer via the inexact projection $\Pi_{\mathbb{C}}^{\tau,u}$ proves more efficient than obtaining the metric projection $\Pi_{\mathbb{C}}(x)$ with an approximation error $\tau > 0$.

Remark 2. Let $\tau \geq 0$ and $\xi > 0$. A point $x^* \in \mathbb{C}$ is a solution of the problem MVI$(\mathbb{C}, \mathcal{A})$ if and only if $x^* \in \Pi_{\mathbb{C}}^{\tau,x^*}(x^* - \xi u^*)$, where $u^* \in \mathcal{A}(x^*)$.

Indeed, for each $u^* \in \mathcal{A}(x^*)$, by the definition of $\Pi_{\mathbb{C}}^{\tau,x}$ and $x^* \in \Pi_{\mathbb{C}}^{\tau,x^*}(x^* - \xi u^*)$ is equivalent to

$$\langle x^* - \xi u^* - x^*, y - x^*\rangle \leq \tau\|x^* - x^*\|^2, \quad \forall y \in \mathbb{C}.$$

Consequently

$$\langle u^*, y - x^*\rangle \geq 0, \quad \forall y \in \mathbb{C}.$$

Thus, the point x^* also is a solution of the problem MVI$(\mathbb{C}, \mathcal{A})$.

□

Lemma 2.1. *[30, Remark 4.4] Let $\{a_k\}$ be a positive sequence. For each any positive integer h, there exists a positive integer $p > h$ such that $a_p \le a_{p+1}$. For each any positive integer k_0 such that $a_{k_0} \le a_{k_0+1}$, set*

$$\xi(k) = \max\{i \in \mathcal{N} : k_0 \le i \le k, a_i \le a_{i+1}\}.$$

Then, $0 \le a_k \le a_{\xi(k)+1}$ for all $k \ge k_0$. Moreover, the sequence $\{\xi(k)\}_{k\ge k_0}$ is nondecreasing and $\lim_{k\to\infty} \xi(k) = +\infty$.

Lemma 2.2. *[40, Lemma 2.5] Assume that $\{\zeta_k\}$ is a positive sequence such that*

$$\zeta_{k+1} \le (1-\theta_k)\zeta_k + \theta_k\tau_k, \quad \forall k \ge 1.$$

Let $\{\theta_k\}$ and $\{\tau_k\}$ are two real sequences satisfying the following conditions:

(i) $\{\theta_k\} \subset (0,1)$ *and* $\sum_{k=1}^{\infty} \theta_k = \infty$;

(ii) $\limsup_{k\to\infty} \tau_k \le 0$ *or* $\sum_{k=1}^{\infty} |\theta_k\tau_k| < +\infty$.

Then, we get $\lim_{k\to\infty} \zeta_k = 0$.

Lemma 2.3. *[41, lemma 2.1] Consider the problem MVI$(\mathbb{C},\mathcal{A})$ with $\mathbb{C}$ being a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $\mathcal{A} : \mathbb{C} \to \mathcal{H}$ being pseudomonotone and continuous. Then, $x^* \in \mathbb{C}$ and $u^* \in \mathcal{A}(x^*)$ is a solution of MVI$(\mathbb{C},\mathcal{A})$ if and only if*

$$\langle u_x, x - x^*\rangle \ge 0, \quad \forall x \in \mathbb{C}, u_x \in \mathcal{A}(x).$$

3. Convergent results

In the viscosity technique, we use the contraction mapping $f : \mathcal{H} \to \mathcal{H}$ with constant $\delta \in [0,1)$ and the positive viscosity sequence $\{\alpha_k\}$ is chosen at each iteration. To solve the problem MVI$(\mathbb{C},\mathcal{A})$, we assume conditions of the cost mapping $\mathcal{A}$ as follows:

(S_1) the cost mapping $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is pseudomonotone, Hausdorff $\mathcal{L}-$Lipschitz continuous and sequentially weakly continuous on $\mathbb{C}$;
(S_2) the solution set of the problem MVI$(\mathbb{C},\mathcal{A})$ is nonempty.

Algorithm 3.1. *(VIP - Viscosity Inexact Projection Algorithm)*
Initialization: *Choose an arbitrary point $x^0 \in \mathcal{H}$ and $L \ge \mathcal{L}$.*
Iteration step: $k = 0, 1, \dots$

Step 1. Take an arbitrary point $u^k \in \mathcal{A}(x^k)$. Find an inexact projection point by Procedure 2.1:

$$y^k \in \Pi_{\mathbb{C}}^{\epsilon_k, x^k}\left[x^k - \lambda_k u^k\right].$$

If $y^k = x^k$, then Stop. Otherwise, go to Step 2.

Step 2. Evaluate a point $v^k \in \mathcal{A}(y^k) \cap \{w \in \mathcal{H} : \|w - u^k\| \le L\|y^k - x^k\|\}$ and

$$x^{k+1} = \beta_k f(x^k) + (1-\beta_k)\left[y^k - \lambda_k(v^k - u^k)\right].$$

Step 3. Increase k by 1 and return to Step 1.

Parameters satisfy the following restrictions:

$$\begin{cases} b \in (0,1), 0 < \lambda_k < a \le \frac{\sqrt{1-b}}{L}, \\ 1 - 2\epsilon_k - \lambda_k^2 L^2 \ge b, \epsilon_k > 0, \\ \beta_k \in (0,1), \sum_{k=0}^{\infty} \beta_k = \infty, \lim_{k\to\infty} \beta_k = 0. \end{cases} \tag{3.1}$$

The following shows the existence of the stopping condition in Step 1 and the iteration point v^k in Step 2.

Remark 3. (i) In the case $y^k = x^k$, from *Step 1* it follows

$$x^k \in \Pi_{\mathbb{C}}^{\epsilon_k, x^k}\left(x^k - \lambda_k u^k\right),$$

where $u^k \in \mathcal{A}(x^k)$. By Remark 2, the x^k is a solution the problem MVI$(\mathbb{C},\mathcal{A})$ under Condition (3.1).
(ii) For each x^k and y^k, by the definition of the Hausdorff $\mathcal{L}-$Lipschitz continuous mapping, we have

$$\begin{aligned} \rho(\mathcal{A}(x^k), \mathcal{A}(y^k)) &= \max\left\{d(\mathcal{A}(x^k), \mathcal{A}(y^k)), d(\mathcal{A}(y^k), \mathcal{A}(x^k))\right\} \\ &\le \mathcal{L}\|y^k - x^k\|, \end{aligned} \tag{3.2}$$

where

$$d(\mathcal{A}(x^k), \mathcal{A}(y^k)) = \sup\left\{d(x, \mathcal{A}(y^k)) : x \in \mathcal{A}(x^k)\right\},$$

and

$$d(x^k, \mathcal{A}(y^k)) = \inf\left\{\|x^k - y\| : y \in \mathcal{A}(y^k)\right\}.$$

Choosing a point $v^k \in \mathcal{A}(y^k)$ such that $\|u^k - v^k\| \le \|u^k - v\|$ for all $v \in \mathcal{A}(y^k)$. Combining $\mathcal{L} \le L$, this and (3.2), it yields $v^k \in \mathcal{A}(y^k)$ and

$$\|u^k - v^k\| \le \rho(\mathcal{A}(x^k), \mathcal{A}(y^k)) \le \mathcal{L}\|x^k - y^k\| \le L\|x^k - y^k\|.$$

Thus, there exists a point v^k at Step 2 such that $v^k \in \mathcal{A}(y^k) \cap \{w \in \mathcal{H} : \|w - u^k\| \le L\|y^k - x^k\|\}$

The next lemma shows the relation between the iteration point $y^k - \lambda_k(v^k - u^k)$ in *Step 2* and any solution of the problem MVI$(\mathbb{C}, \mathcal{A})$.

Lemma 3.1. *Set $z^k := y^k - \lambda_k(v^k - u^k)$. Let p be a solution of the problem MVI$(\mathbb{C}, \mathcal{A})$. Then, the following inequality holds:*

$$\|z^k - p\|^2 \le \|x^k - p\|^2 - (1 - 2\epsilon_k - \lambda_k^2 L^2)\|y^k - x^k\|^2.$$

Proof. Since $z^k = y^k - \lambda_k(v^k - u^k)$, we have

$$\begin{aligned}
\|z^k - p\|^2 =& \|y^k - \lambda_k(v^k - u^k) - p\|^2\\
=& \|y^k - p\|^2 + \lambda_k^2\|v^k - u^k\|^2 - 2\lambda_k\langle y^k - p, v^k - u^k\rangle\\
=& \|x^k - p\|^2 + \|x^k - y^k\|^2 + 2\langle y^k - x^k, x^k - p\rangle + \lambda_k^2\|v^k - u^k\|^2\\
& - 2\lambda_k\langle y^k - p, v^k - u^k\rangle\\
=& \|x^k - p\|^2 + \|x^k - y^k\|^2 - 2\langle y^k - x^k, y^k - x^k\rangle + 2\langle y^k - x^k, y^k - p\rangle\\
& + \lambda_k^2\|v^k - u^k\|^2 - 2\lambda_k\left\langle y^k - p, v^k - u^k\right\rangle\\
=& \|x^k - p\|^2 - \|x^k - y^k\|^2 + 2\langle y^k - x^k, y^k - p\rangle + \lambda_k^2\|v^k - u^k\|^2\\
& - 2\lambda_k\left\langle y^k - p, v^k - u^k\right\rangle.
\end{aligned} \tag{3.3}$$

By using the definition of the inexact projection $y^k \in \Pi_{\mathbb{C}}^{\epsilon_k, x^k}(x^k - \lambda_k u^k)$, it yields

$$\langle y^k - x^k + \lambda_k u^k, y^k - p\rangle \le \epsilon_k\|y^k - x^k\|^2.$$

This is equivalent to

$$\langle y^k - x^k, y^k - p\rangle \le -\lambda_k\langle u^k, y^k - p\rangle + \epsilon_k\|y^k - x^k\|^2. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned}
\|z^k - p\|^2 =& \|x^k - p\|^2 - \|x^k - y^k\|^2 + 2\langle y^k - x^k, y^k - p\rangle + \lambda_k^2\|v^k - u^k\|^2\\
& - 2\lambda_k\langle y^k - p, v^k - u^k\rangle\\
\le& \|x^k - p\|^2 - \|x^k - y^k\|^2 - 2\lambda_k\langle u^k, y^k - p\rangle + 2\epsilon_k\|y^k - x^k\|^2\\
& + \lambda_k^2\|v^k - u^k\|^2 - 2\lambda_k\langle y^k - p, v^k - u^k\rangle.\\
=& \|x^k - p\|^2 - (1 - 2\epsilon_k)\|x^k - y^k\|^2 + \lambda_k^2\|v^k - u^k\|^2 - 2\lambda_k\langle y^k - p, v^k\rangle\\
\le& \|x^k - p\|^2 - (1 - 2\epsilon_k - \lambda_k^2 L^2)\|y^k - x^k\|^2,
\end{aligned}$$

where the last inequality is deduced from the assumptions that p is a solution of the problem MVI$(\mathbb{C}, \mathcal{A})$, $y^k \in \mathbb{C}$ of the inexact projection and the pseudomonotonicity of $\mathcal{A}$ in (S_1), i.e.,

$$\langle w_p, y^k - p\rangle \ge 0 \Rightarrow \langle v^k, y^k - p\rangle \ge 0,$$

where $w_p \in \mathcal{A}(p)$. This completes the proof. □

Lemma 3.2. *The sequences $\{x^k\}, \{f(x^k)\}$ and $\{y^k\}$ are bounded.*

Proof. By Lemma 3.1 and the condition $1 - 2\epsilon_k - \lambda_k^2 L^2 > 0$ of (3.1), it yields

$$\|z^k - p\| \le \|x^k - p\|, \quad \forall k \ge 0. \tag{3.5}$$

Since f is $\delta-$contractive and $\beta_k \in (0, 1)$, for each $k \ge 1$, we have

$$\begin{aligned}
\|x^{k+1} - p\| &= \|\beta_k(f(x^k) - p) + (1 - \beta_k)(z^k - p)\|\\
&\le \beta_k\|f(x^k) - p\| + (1 - \beta_k)\|z^k - p\|\\
&\le \beta_k\|f(x^k) - f(p)\| + \beta_k\|f(p) - p\| + (1 - \beta_k)\|z^k - p\|\\
&\le \beta_k\delta\|x^k - p\| + \beta_k\|f(p) - p\| + (1 - \beta_k)\|x^k - p\|
\end{aligned}$$

$$= [1-(1-\delta)\beta_k]\|x^k - p\| + (1-\delta)\beta_k \frac{\|f(p)-p\|}{1-\delta}$$
$$\le \max\left\{\|x^k - p\|, \frac{\|f(p)-p\|}{1-\delta}\right\},$$

where z^k is defined in Lemma 3.1. By induction, we have

$$\|x^k - p\| \le \max\left\{\|x^0 - p\|, \frac{\|f(p)-p\|}{1-\delta}\right\},$$

which implies that $\{x^k\}$ is bounded. By (3.5), the $\{z^k\}$ is also bounded. Using Lemma 3.1 and the condition $1-2\epsilon_k-\lambda_k^2L^2 \ge b > 0$ of (3.1), it follows

$$b\|y^k - x^k\|^2 \le \|x^k - p\|^2 - \|z^k - p\|^2, \quad \forall k \ge 0.$$

Consequently, $\{y^k\}$ is bounded. Also, since f is contractive, it deduces that $\{f(x^k)\}$ is bounded. This finishes the proof. □

Lemma 3.3. *Let the subsequence $\{x^{k_j}\}$ of $\{x^k\}$ satisfy that the $\{x^{k_j}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{j\to\infty}\|x^{k_j} - y^{k_j}\| = 0$. Under Condition (3.1), we get $z \in Sol(\mathbb{C}, \mathcal{A})$.*

Proof. From the assumptions $x^{k_j} \rightharpoonup z$ and $\lim_{j\to\infty}\|x^{k_j} - y^{k_j}\| = 0$, it follows $y^{k_j} \rightharpoonup z$. By using $\{y^{k_j}\} \subset \mathbb{C}$ and $\mathbb{C}$ is convex and closed, it yields $z \in \mathbb{C}$. Since *Step 2* and the definition of the inexact projection $y^{k_j} \in \Pi_{\mathbb{C}}^{\epsilon_{k_j}, x^{k_j}}\left(x^{k_j} - \lambda_{k_j}u^{k_j}\right)$, we have

$$\langle x^{k_j} - \lambda_{k_j}u^{k_j} - y^{k_j}, x - y^{k_j}\rangle \le \epsilon_{k_j}\|y^{k_j} - x^{k_j}\|^2, \quad \forall x \in \mathbb{C}.$$

Using $\lambda_k > 0$, we get

$$\frac{1}{\lambda_{k_j}}\langle x^{k_j} - y^{k_j}, x - y^{k_j}\rangle \le \langle u^{k_j}, x - y^{k_j}\rangle + \frac{\epsilon_{k_j}}{\lambda_{k_j}}\|y^{k_j} - x^{k_j}\|^2, \quad \forall x \in \mathbb{C}.$$

This is equivalent to

$$\frac{1}{\lambda_{k_j}}\langle x^{k_j} - y^{k_j}, x - y^{k_j}\rangle + \langle u^{k_j}, y^{k_j} - x^{k_j}\rangle - \frac{\epsilon_{k_j}}{\lambda_{k_j}}\|y^{k_j} - x^{k_j}\|^2 \le \langle u^{k_j}, x - x^{k_j}\rangle.$$

Since the boundedness of the sequences $\{y^k\}$ and $\{x^k\}$ in Lemma 3.2, $0 < \lambda_k \le a < \infty$ in Condition (3.1) and passing the liminf as $j \to \infty$, we obtain

$$\liminf_{j\to\infty}\langle u^{k_j}, x - x^{k_j}\rangle \ge 0, \quad \forall x \in \mathbb{C}. \tag{3.6}$$

Otherwise, from $v^{k_j} \in \mathcal{A}(y^{k_j})$, it follows that

$$\langle v^{k_j}, x - y^{k_j}\rangle = \langle v^{k_j} - u^{k_j}, x - x^{k_j}\rangle + \langle u^{k_j}, x - x^{k_j}\rangle + \langle v^{k_j}, x^{k_j} - y^{k_j}\rangle. \tag{3.7}$$

Since $\mathcal{A}$ is Hausdorff L−Lipschitz continuous and the assumption $\lim_{j\to\infty}\|x^{k_j} - y^{k_j}\| = 0$, we deduce

$$0 \le \lim_{j\to\infty}\|v^{k_j} - u^{k_j}\| \le \lim_{j\to\infty} L\|x^{k_j} - y^{k_j}\| = 0.$$

Combining this, (3.6) and (3.7), it yields

$$\liminf_{j\to\infty}\langle v^{k_j}, x - y^{k_j}\rangle \ge 0.$$

This implies that we can take a sequence $\{\xi_j\} \subset (0,1)$ satisfying $\lim_{j\to\infty}\xi_j = 0$, for all $j \ge 1$ there exists the smallest positive integer $m_j \ge k_j$ such that

$$\langle v^i, x - y^i\rangle + \xi_j \ge 0, \quad \forall i \ge m_j. \tag{3.8}$$

Note that $\|v^k\| \ne 0$. So we can set $g^{m_j} = \frac{v^{m_j}}{\|v^{m_j}\|^2}$. Then, we have $\langle v^{m_j}, g^{m_j}\rangle = 1, \quad \forall j \ge 1$. From (3.8), it follows

$$\langle v^{m_j}, x + \xi_j g^{m_j} - y^{m_j}\rangle \ge 0.$$

Combining this and the pseudomonotonicity of $\mathcal{A}$, it yields

$$\langle w_x^j, x + \xi_j g^{m_j} - y^{m_j}\rangle \ge 0, \quad \forall w_x^j \in \mathcal{A}(x + \xi_j g^{m_j}),$$

and hence

$$\langle w_x, x - y^{m_j}\rangle \ge \langle w_x - w_x^j, x + \xi_j g^{m_j} - y^{m_j}\rangle - \xi_j\langle w_x, g^{m_j}\rangle, \quad \forall j \ge 0, w_x \in \mathcal{A}(x). \tag{3.9}$$

Next we prove that $\lim_{j\to\infty}\xi_j\|g^{m_j}\| = 0$. As the above proof, $y^{k_j} \rightharpoonup z$ and $z \in \mathbb{C}$. Since $\mathcal{A}$ is sequentially weakly continuous on $\mathbb{C}$, so $\{v^{k_j}\}$ converges weakly to $w_z \in \mathcal{A}(z)$. The proof is complete with $w_z = 0$, i.e., $z \in Sol(\mathbb{C}, \mathcal{A})$. Now we consider the case $w_z \ne 0$. By using the sequentially weak lower semicontinuity of the norm $\|\cdot\|$, we get

$$0 < \|w_z\| \le \liminf_{j\to\infty}\|v^{k_j}\|.$$

From $\{y^{m_j}\} \subset \{y^{k_j}\}$, $\{\xi_j\} \subset (0,\infty)$ and $\lim_{j\to\infty} \xi_j = 0$, it follows

$$0 \le \limsup_{j\to\infty} \|\xi_j g^{m_j}\| = \limsup_{j\to\infty} \left(\frac{\xi_j}{\|v^{k_j}\|}\right) \le \frac{\limsup_{j\to\infty} \xi_j}{\liminf_{j\to\infty} \|v^{k_j}\|} \le \frac{\lim_{j\to\infty} \xi_j}{\|\mathcal{A}z\|} = 0.$$

It means that $\lim_{j\to\infty} \xi_j \|g^{m_j}\| = 0$. Taking the limit as $j \to \infty$, then the right hand side of (3.9) tends to zero under the fact that $\mathcal{A}$ is Lipschitz continuous, $\{x^{m_j}\}$, $\{g^{m_j}\}$ are bounded and $\lim_{j\to\infty} \xi_j g^{m_j} = 0$. So, from (3.9) it follows, for each $w_x \in \mathcal{A}(x)$,

$$\liminf_{j\to\infty} \langle w_x, x - y^{m_j}\rangle \ge 0.$$

Since the sequence $\{y^{m_j}\}$ converges weakly to $z \in \mathbb{C}$, we deduce

$$\langle w_x, x - z\rangle = \lim_{j\to\infty} \langle w_x, x - y^{m_j}\rangle = \liminf_{j\to\infty} \langle w_x, x - y^{m_j}\rangle \ge 0, \quad \forall x \in \mathbb{C}.$$

By Lemma 2.3, we have $z \in Sol(\mathbb{C}, \mathcal{A})$. The proof is completed. □

Theorem 3.1. *Let the cost mapping $\mathcal{A}$ satisfy Assumptions $(S_1)-(S_2)$. Under Conditions (3.1), the sequence $\{x^k\}$ generated by Algorithm 3.1 converges strongly to an element $p \in Sol(\mathbb{C}, \mathcal{A})$. Moreover, the point p is the unique solution of the following variational inequality problem:*

$$\text{Find } x^* \in Sol(\mathbb{C}, \mathcal{A}) \text{ such that } \langle (f - I)(x^*), y - x^*\rangle \ge 0, \quad \forall y \in Sol(\mathbb{C}, \mathcal{A}). \tag{3.10}$$

Proof. Note that under Assumptions $(S_1)-(S_2)$, the set $Sol(\mathbb{C}, \mathcal{A})$ is nonempty, closed and convex. Since f is $\delta-$contractive, so the problem (3.10) exists a unique solution. Firstly, we show that there exists a positive number $M > 0$ such that

$$(1 - 2\epsilon_k - \lambda_k^2 L^2)\|x^k - y^k\|^2 \le \|x^k - p\|^2 - \|x^{k+1} - p\|^2 + \beta_k M, \quad \forall k \ge 0.$$

Indeed, by the viscosity technique in *Step 2* and using the inequality

$$\|a + b\|^2 \le \|a\|^2 + 2\langle b, a + b\rangle, \quad \forall a, b \in \mathcal{H}, \tag{3.11}$$

we get

$$\begin{aligned}
\|x^{k+1} - p\|^2 &= \|z^k - p + \beta_k(f(x^k) - z^k)\|^2 \\
&\le \|z^k - p\|^2 + 2\beta_k\langle f(x^k) - z^k, x^{k+1} - p\rangle \\
&\le \|z^k - p\|^2 + 2\beta_k\|f(x^k) - z^k\|\|x^{k+1} - p\| \\
&\le \|z^k - p\|^2 + \beta_k M,
\end{aligned} \tag{3.12}$$

where $M = \sup\{2\|f(x^k) - z^k\|\|x^{k+1} - p\| : k \ge 1\}$. By using lemma 3.2, the sequences $\{x^k\}$ and $\{z^k\}$ are bounded and hence $M < \infty$. Lemma 3.1 shows that

$$\|z^k - p\|^2 \le \|x^k - p\|^2 - (1 - 2\epsilon_k - \lambda_k^2 L^2)\|x^k - y^k\|^2, \quad \forall k \ge 0. \tag{3.13}$$

By using (3.12) and (3.13), it follows that

$$\|x^{k+1} - p\|^2 \le \|x^k - p\|^2 - (1 - 2\epsilon_k - \lambda_k^2 L^2)\|x^k - y^k\|^2 + \beta_k M,$$

which means that

$$(1 - 2\epsilon_k - \lambda_k^2 L^2)\|x^k - y^k\|^2 \le \|x^k - p\|^2 - \|x^{k+1} - p\|^2 + \beta_k M. \tag{3.14}$$

Combining (3.13) and the contractiveness of f, we deduce that

$$\begin{aligned}
\|x^{k+1} - p\|^2 =& \|\beta_k(f(x^k) - f(p)) + (1 - \beta_k)(z^k - p) + \beta_k(f(p) - p)\|^2 \\
\le& \|\beta_k(f(x^k) - f(p)) + (1 - \beta_k)(z^k - p)\|^2 + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle \\
\le& \beta_k\|f(x^k) - f(p)\|^2 + (1 - \beta_k)\|z^k - p\|^2 + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle \\
\le& \beta_k\delta^2\|x^k - p\|^2 + (1 - \beta_k)\|z^k - p\|^2 + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle \\
\le& \beta_k\delta^2\|x^k - p\|^2 + (1 - \beta_k)\left[\|x^k - p\|^2 - (1 - 2\epsilon_k - \lambda_k^2 L^2)\|x^k - y^k\|^2\right] \\
& + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle \\
=& [1 - \beta_k(1 - \delta^2)]\|x^k - p\|^2 - (1 - \beta_k)(1 - 2\epsilon_k - \lambda_k^2 L^2)\|x^k - y^k\|^2 \\
& + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle.
\end{aligned}$$

This implies

$$\begin{aligned}
\|x^{k+1} - p\|^2 \le& [1 - \beta_k(1 - \delta^2)]\|x^k - p\|^2 - (1 - \beta_k)(1 - 2\epsilon_k - \lambda_k^2 L^2)\|x^k - y^k\|^2 \\
& + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle.
\end{aligned} \tag{3.15}$$

Set $a_k := \|x^k - p\|^2$. Let us divide two the following cases.

Case 1. There exists a positive integer k_0 such that $a_{k+1} \le a_k$ for all $k \ge k_0$. Then, there exists the limit $\lim_{k\to\infty} \|x^k - p\|^2 = A \in [0, \infty)$. Taking the limit into (3.14) as $k \to \infty$ and using the condition $1 - 2\epsilon_k - \lambda_k^2 L^2 \ge b > 0$ in (3.1), it yields

$$\lim_{k\to\infty} \|x^k - y^k\| = 0. \tag{3.16}$$

Since $\mathcal{A}$ is Hausdorff L–Lipschitz continuous, Step 2 and (3.16), we have

$$0 \le \lim_{k\to\infty} \|z^k - y^k\| = \lim_{k\to\infty} \lambda_k \|v^k - u^k\| \le \lim_{k\to\infty} \lambda_k L \|y^k - x^k\| = 0. \tag{3.17}$$

Using (3.16) and (3.17), we obtain

$$0 \le \lim_{k\to\infty} \|x^k - z^k\| \le \lim_{k\to\infty} (\|x^k - y^k\| + \|y^k - z^k\|) = 0. \tag{3.18}$$

From Lemma 3.2 and $\lim_{k\to\infty} \beta_k = 0$, it follows

$$\begin{aligned} 0 &\le \lim_{k\to\infty} \|x^{k+1} - x^k\| \\ &\le \lim_{k\to\infty} (\|x^{k+1} - z^k\| + \|z^k - x^k\|) \\ &= \lim_{k\to\infty} (\beta_k \|f(x^k) - z^k\| + \|z^k - x^k\|) \\ &= 0. \end{aligned}$$

Consequently

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{3.19}$$

By Lemma 3.2 that $\{x^k\}$ is a bounded sequence, so there exists $\{x^{k_j}\} \subset \{x^k\}$ such that $\{x^{k_j}\}$ converges weakly to $z \in \mathcal{H}$ and

$$\limsup_{k\to\infty} \langle f(p) - p, x^k - p\rangle = \lim_{j\to\infty} \langle f(p) - p, x^{k_j} - p\rangle = \langle f(p) - p, z - p\rangle. \tag{3.20}$$

Since $x^{k_j} \rightharpoonup z$, (3.16) and Lemma 3.1, we get $z \in \mathcal{S}(\mathbb{C}, \mathcal{A})$. It is clear that the projection mapping $\Pi_{\mathcal{S}(\mathbb{C},\mathcal{A})} f : \mathcal{H} \to \mathcal{S}(\mathbb{C}, \mathcal{A})$ is contractive with constant $\delta \in (0, 1)$. Therefore, it has a unique fixed point. We assume that $p = \Pi_{\mathcal{S}(\mathbb{C},\mathcal{A})} f(p)$. From the definition of the metric projection $\Pi_{\mathcal{S}(\mathbb{C},\mathcal{A})}$ and $z \in \mathcal{S}(\mathbb{C}, \mathcal{A})$, it implies

$$\langle f(p) - p, z - p\rangle \le 0.$$

Using this and (3.20), we get

$$\limsup_{k\to\infty} \langle f(p) - p, x^k - p\rangle = \langle f(p) - p, z - p\rangle \le 0. \tag{3.21}$$

From (3.19) and (3.21), it follows that

$$\begin{aligned} \limsup_{k\to\infty} \langle f(p) - p, x^{k+1} - p\rangle &\le \limsup_{k\to\infty} \langle f(p) - p, x^{k+1} - x^k\rangle + \limsup_{k\to\infty} \langle f(p) - p, x^k - p\rangle \\ &\le \limsup_{k\to\infty} (\|f(p) - p\| \|x^{k+1} - x^k\|) + \limsup_{k\to\infty} \langle f(p) - p, x^k - p\rangle \\ &\le 0. \end{aligned} \tag{3.22}$$

Applying Lemma 2.2 for (3.15) with

$$\zeta_k := \|x^k - p\|^2, \theta_k := \beta_k(1 - \delta^2), \tau_k := 2\beta_k \langle f(p) - p, x^{k+1} - p\rangle,$$

we have the limit $\lim_{k\to\infty} a_k = 0$. Note that $\lim_{k\to\infty} \tau_k \le 0$ by (3.22). Thus, three sequences $\{x^k\}$ and $\{y^k\}$ converge strongly to the unique solution p of the problem (3.10).

Case 2. It does not exist a positive integer k_0 such that

$$a_{k+1} \le a_k, \quad \forall k \ge k_0.$$

By using the interesting subsequence introduced first by Maingé in Lemma 2.1, there exists a nondecreasing sequence $\{n_j\} \subset \{1, 2, \dots\}$ such that $\lim_{j\to\infty} n_j = \infty$ and the following relations hold:

$$a_{n_j} \le a_{n_j+1}, a_j \le a_{n_j+1}, \quad \forall j \ge 1. \tag{3.23}$$

By (3.14), we have

$$\begin{aligned} b\|x^{n_j} - y^{n_j}\|^2 &\le (1 - 2\epsilon_k - \lambda_k^2 L^2) \|x^{n_j} - y^{n_j}\|^2 \\ &\le \|x^{n_j} - p\|^2 - \|x^{n_j+1} - p\|^2 + \beta_{n_j} \boldsymbol{M} \\ &\le \beta_{n_j} \boldsymbol{M}. \end{aligned}$$

Taking the limit as $j \to \infty$ and using $\lim_{k\to\infty} \beta_k = 0$, we get

$$\lim_{j\to\infty} \|y^{n_j} - x^{n_j}\| = 0.$$

By a similar work as in **Case 1**, we also have

$$\lim_{j\to\infty} \|x^{n_j} - z^{n_j}\| = 0, \lim_{j\to\infty} \|x^{n_j+1} - x^{n_j}\| = 0,$$

and

$$\limsup_{j\to\infty} \langle f(p) - p, x^{n_j+1} - p\rangle \le 0.$$

Substituting $k := n_j$ into (3.15) and using (3.23), it yields

$$\begin{aligned} a_{n_j+1} =& \|x^{n_j+1} - p\|^2 \\ \le& [1 - \beta_{n_j}(1-\delta^2)]a_{n_j} + 2\beta_{n_j}\langle f(p) - p, x^{n_j+1} - p\rangle \\ \le& [1 - \beta_{n_j}(1-\delta^2)]a_{n_j+1} + 2\beta_{n_j}\langle f(p) - p, x^{n_j+1} - p\rangle. \end{aligned}$$

Which implies that

$$a_{n_j} \le a_{n_j+1} = \|x^{n_j+1} - p\|^2 \le \frac{2\left\langle f(p) - p, x^{n_j+1} - p\right\rangle}{1-\delta^2}.$$

Consequently,

$$\limsup_{j\to\infty} a_{n_j+1} = 0,$$

and hence $\limsup_{j\to\infty} a_j = 0$. Thus, the sequences $\{x^k\}$ and $\{y^k\}$ converge strongly to the unique solution p of the problem (3.10). This completes the proof.

□

4. Application to electricity production models

In this section, we consider electricity production models which can be regarded as an extension formulation of the Nash-Cournot economic equilibrium model in electricity markets. The latter model has been investigated in some research results in [36, oligopolistic electricity models], used later in [42]. In these electricity production models, it is assumed that

- Let N companies, each company j $(j = 1, \dots, N)$ may possess x_j generating units;
- Denote $a_{l,j}$ by the quantity of material l $(l = 1, \dots, m)$ for producing one unit electricity by the generating unit j $(j = 1, \dots, N)$. Let A be the matrix whose entries are $a_{l,j}$ and it is usually called *quantity matrix*. Then, the entry l of the vector Ax is the quantity of material l for producing x. Using materials for production may cause pollution to environment for which companies have to pay environmental fee. Suppose that $g(Ax)$ is the total environmental fee for producing x;
- The price $p_j(s) = \alpha - \beta_j s$ is a decreasing affine function, where $s = \sum_{j=1}^N x_j$;
- Let $c_j(x_j)$ be the cost for generating x_j by unit j;
- Assume that the strategy set of the model is $x = (x_1, \dots, x_N)^\top \in \mathbb{C}$.

The task now is to find a production x^* such that it is a Nash equilibrium point with minimum environmental fee. By [7,42], this problem can be formulated as the variational inequality problem: Find $x^* \in \mathbb{C}$ such that

$$\begin{cases} \langle \bar{B}x^* - \bar{a} + \nabla\varphi(x^*), x - x^*\rangle \ge 0, \quad \forall x \in \mathbb{C}, \\ x^* = \arg\min\{g(Ax) : x \in \mathbb{C}\}, \end{cases} \tag{4.1}$$

where the $g : \mathcal{R}^m \to \mathcal{R}$ and c_j $(j = 1, \dots, N)$ are convex and differentialbe,

$$\varphi(x) := x^\top Bx + \sum_{j=1}^N c_j(x_j), \mathcal{H} := \mathcal{R}^N, \bar{a} = (\alpha, \dots, \alpha)^\top \in \mathcal{H}, \tag{4.2}$$

and

$$B = \begin{pmatrix} \beta_1 & 0 & 0 & \dots & 0 \\ 0 & \beta_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \beta_N \end{pmatrix}_{N\times N}, \bar{B} = \begin{pmatrix} 0 & \beta_1 & \beta_1 & \dots & \beta_1 \\ \beta_2 & 0 & \beta_2 & \dots & \beta_2 \\ \dots & \dots & \dots & \dots & \dots \\ \beta_N & \beta_N & \beta_N & \dots & 0 \end{pmatrix}_{N\times N}.$$

Now we define two new mappings $\mathcal{A}_i (i = 1, 2)$ from $\mathcal{H}$ to $\mathcal{H}$ as follows:

$$\mathcal{A}_1(x) = \bar{B}x - \bar{a} + \nabla\varphi(x), \tag{4.3}$$

$$\mathcal{A}_2(x) = A^\top \nabla g(Ax). \tag{4.4}$$

Then, it is clear that the problem (4.1) is written in the form of finding a common solution x^* of the problem MVI$(\mathbb{C}, \mathcal{A}_1)$ and the problem MVI$(\mathbb{C}, \mathcal{A}_2)$.

Proposition 4.1. *Assume that the $\nabla c_j (j = 1, \dots, N)$ is $\mathcal{L}_j$−Lipschitz continuous. Then, the cost mapping $\mathcal{A}_1$ defined in (4.3), is pseudomonotone and Lipschitz continuous with constant*

$$L_1 = \|\bar{B}\| + 2\|B\| + \max\{\mathcal{L}_j : j = 1, \dots, N\}.$$

Proof. By using [43, Proposition 3.2], the cost mapping $\mathcal{A}_1$ is pseudomonotone on $\mathcal{H}$.

For each $x, y \in \mathcal{H}$, we have

$$\begin{aligned}
&\|\mathcal{A}_1(x) - \mathcal{A}_1(y)\| \\
&= \left\|\bar{B}x - \bar{a} + \nabla\varphi(x) - \left(\bar{B}y - \bar{a} + \nabla\varphi(y)\right)\right\| \\
&\le \|\bar{B}(x-y)\| + \|\nabla\varphi(x) - \nabla\varphi(y)\| \\
&\le \|\bar{B}\|\|x-y\| + \|\nabla\varphi(x) - \nabla\varphi(y)\|.
\end{aligned} \tag{4.5}$$

On the other hand, from (4.2) it follows that

$$\nabla\varphi(x) = 2Bx + \left(\nabla c_1(x_1), \dots, \nabla c_N(x_N)\right)^\top,$$

and hence

$$\begin{aligned}
&\|\nabla\varphi(x) - \nabla\varphi(y)\| \\
&= \left\|2B(x-y) + \left(\nabla c_1(x_1) - \nabla c_1(y_1), \dots, \nabla c_N(x_N) - \nabla c_N(y_N)\right)^\top\right\| \\
&\le 2\|B\|\|x-y\| + \left\|\left(\nabla c_1(x_1) - \nabla c_1(y_1), \dots, \nabla c_N(x_N) - \nabla c_N(y_N)\right)^\top\right\| \\
&= 2\|B\|\|x-y\| + \sqrt{\|\nabla c_1(x_1) - \nabla c_1(y_1)\|^2 + \dots + \|\nabla c_N(x_N) - \nabla c_N(y_N)\|^2} \\
&\le (2\|B\| + \bar{\mathcal{L}})\|x-y\|,
\end{aligned}$$

where $\bar{\mathcal{L}} := \max\{\mathcal{L}_j : j = 1, \dots, N\}$. Combining this and (4.5), it yields

$$\|\mathcal{A}_1(x) - \mathcal{A}_1(y)\| \le \left(\|\bar{B}\| + 2\|B\| + \bar{\mathcal{L}}\right)\|x-y\|, \quad \forall x, y \in \mathcal{H}.$$

Thus, the cost mapping $\mathcal{A}_1$ is L_1−Lipschitz continuous, where

$$L_1 = \|\bar{B}\| + 2\|B\| + \bar{\mathcal{L}}.$$

The proof is complete. □

Proposition 4.2. *Assume that the ∇g is $\mathcal{L}$−Lipschitz continuous. Then, the cost mapping $\mathcal{A}_2$ defined in (4.4), is monotone and $L_2 := \|A\|^2\mathcal{L}$−Lipschitz continuous.*

Proof. Since g is convex and differentiable, so the gradient mapping ∇g is monotone. Then, for every $x, y \in \mathcal{H}$, we have

$$\begin{aligned}
\langle \mathcal{A}_2(x) - \mathcal{A}_2(x), x - y\rangle &= \left\langle A^\top\nabla g(Ax) - A^\top\nabla g(Ay), x - y\right\rangle \\
&= \langle \nabla g(Ax) - \nabla g(Ay), Ax - Ay\rangle \\
&\ge 0.
\end{aligned}$$

Since (4.4) and ∇g is $\mathcal{L}$−Lipschitz continuous, it yields

$$\begin{aligned}
\|\mathcal{A}_2(x) - \mathcal{A}_2(y)\| &= \|A^\top\nabla g(Ax) - A^\top\nabla g(Ay)\| \\
&\le \|A\|\|\nabla g(Ax) - \nabla g(Ay)\| \\
&\le \|A\|\mathcal{L}\|Ax - Ay\| \\
&\le \|A\|^2\mathcal{L}\|x-y\|, \quad \forall x, y \in \mathcal{H}.
\end{aligned}$$

This implies the proof. □

Basing on the idea of Von Neumann Alternating Method for finding common solutions to variational inequality problems proposed by Censor et al. [44]., we apply the algorithm VIP for finding a common solution of the problem MVI$(\mathbb{C}, \mathcal{A}_1)$ and the problem MVI$(\mathbb{C}, \mathcal{A}_2)$ as follows.

Algorithm 4.1. *(MVIP - Modified Viscosity Inexact Projection Algorithm)*
Initialization: *Choose an arbitrary point $x^0 \in \mathcal{H}, \gamma \in (0,1)$.*
Iteration step: $k = 0, 1, \dots$.

Step 1. Find two inexact projection points:

$$y_i^k \in \Pi_{\mathbb{C}}^{\epsilon_{i,k}, x^k}\left[x^k - \lambda_{i,k}\mathcal{A}_i(x^k)\right], \quad \forall i = 1, 2.$$

If $y_i^k = x^k$ for all $i \in \{1, 2\}$, then Stop. Otherwise, go to Step 2.

Step 2. Evaluate

$$z_i^k = \beta_{i,k} f(x^k) + (1-\beta_{i,k})\big[y_i^k - \lambda_{i,k}(\mathcal{A}_i(y_i^k) - \mathcal{A}_i(x^k))\big], \quad \forall i = 1,2,$$
$$x^{k+1} = \gamma z_1^k + (1-\gamma) z_2^k.$$

Step 3. Increase k by 1 and return to Step 1.

In light of the condition (3.1), parameters are chosen in the following restrictions, for each $i = 1,2$:

$$\begin{cases} b_i \in (0,1), 0 < \lambda_{i,k} < a \le \min\left\{ \frac{\sqrt{1-b_i}}{L_i} : i = 1,2 \right\}, \\ 1 - 2\epsilon_{i,k} - \lambda_{i,k}^2 L_i^2 \ge b_i, \epsilon_{i,k} > 0, \\ \beta_{i,k} \in (0,1), \sum_{k=0}^{\infty} \beta_{i,k} = \infty, \lim_{k\to\infty} \beta_{i,k} = 0, \end{cases} \tag{4.6}$$

where L_i is Lipschitz constant of the mapping $\mathcal{A}_i$.

Theorem 4.1. *Let the cost mappings $\mathcal{A}_i (i = 1,2)$ be defined in (4.3)-(4.4). Under Conditions (4.6), the sequence $\{x^k\}$ generated by the algorithm MVIP converges strongly to a common solution of the problem MVI($\mathbb{C}, \mathcal{A}_1$) and the problem MVI($\mathbb{C}, \mathcal{A}_2$).*

Proof. It is the same ways as the proof of Theorem 3.1 and [44, Theorem 3.10].

To test the algorithm MVIP, the strategy set is given in the form:

$$\mathbb{C} := \{x \in \mathcal{H} : \|x\| \le R, \langle \ddot{a}, x\rangle \le r\},$$

where $R > 0, \ddot{a} \in \mathcal{H}$ and $r \in \mathcal{R}$. Now, we test the proposed algorithm MVIP with the cost function given by a convex quadratic programming:

$$c_j(x_j) = \frac{1}{2}\hat{a}_j x_j^2 + \hat{b}_j x_j + \hat{c}_j, \quad \hat{a}_j > 0, \hat{b}_j \in \mathcal{R}, \hat{c}_j \in \mathcal{R}, j = 1, \dots, N. \tag{4.7}$$

The total environmental fee $g(v)$ is given in the form:

$$g(v) = \frac{1}{2} v^\top E v + e^\top v, \quad \forall v \in \mathcal{H}, \tag{4.8}$$

where the $E = (N \times N)$ is a positive semidefinite matrix and $e \in \mathcal{R}^N$. Then, we get

$$\begin{aligned} \nabla\varphi(x) &= 2Bx + \left(\nabla c_1(x_1), \dots, \nabla c_N(x_N)\right)^\top \\ &= 2Bx + \left(\hat{a}_1 x_1 + \hat{b}_1, \dots, \hat{a}_N x_N + \hat{b}_N\right)^\top. \end{aligned}$$

Using (4.3), it yields

$$\begin{aligned} \mathcal{A}_1(x) &= \bar{B}x - \bar{a} + \nabla\varphi(x) \\ &= (\bar{B} + 2B)x - \bar{a} + (\hat{a}_1 x_1 + \hat{b}_1, \dots, \hat{a}_N x_N + \hat{b}_N)^\top. \end{aligned}$$

From (4.4), it follows

$$\begin{aligned} \mathcal{A}_2(x) &= A^\top \nabla g(Ax) \\ &= A^\top [E(Ax) + e]. \end{aligned}$$

Note that at each iteration, the main iteration step of the algorithm MVIP is to compute an inexact projection point via x^k as follows:

$$y_i^k \in \Pi_{\mathbb{C}}^{\epsilon_{i,k}, x^k}(x^k - \lambda_{i,k}\mathcal{A}_i(x^k)).$$

By using (2.3), it yields

$$\langle u_i^k - y_i^k, y - y_i^k\rangle \le \epsilon_{i,k}\|y_i^k - x^k\|^2, \quad \forall y \in \mathbb{C},$$

where $u_i^k := x^k - \xi_{i,k}\mathcal{A}_i(x^k)$. This is equivalent to

$$\begin{aligned} \max\left\{\langle u_i^k - y_i^k, y - y_i^k\rangle : y \in \mathbb{C}\right\} &= -\min\left\{\langle y_i^k - u_i^k, y - y_i^k\rangle : y \in \mathbb{C}\right\} \\ &\le \epsilon_{i,k}\|y_i^k - x^k\|^2. \end{aligned} \tag{4.9}$$

Then, by using the well-known Gradient Descent Method, it is formally defined to compute an inexact projection point y_i^k of a point u_i^k onto the set $\mathbb{C}$ by Procedure 2.1. Moreover, solving the problem (4.9) with error $\epsilon_{i,k}\|y_i^k - x^k\|^2$ is simple and more effective on software Matlab 2023 than computing exact the projection $\Pi_{\mathbb{C}}(x^k - \lambda_{i,k}\mathcal{A}_i(x^k))$ via auxilary function such as Quadratic Program.

All programming is implemented in Matlab R2023a running on a PC with Intel(R) Core(TM) $i9-9900KS\ CPU$ @ $4.00GHz$ 32.0 $GB\ Ram$. The inexact projection $\Pi_{\mathbb{C}}$ is computed via the Matlab optimization toolbox by fmincon or quadratic functions.

Test 1. Let us test in $\mathcal{R}^N$ where $N = 5$. We apply the proposed algorithm MVIP for solving the electricity production model (4.1) in the space $\mathcal{R}^N$. Data of the (4.1) are described as follows:

Table 1
Iterations and CPU times with randomly different starting points x^0.

Problem	Starting point x^0	Iterations	CPU times
1	$(1,2,3,4,5)^\top$	2756	77.4063
2	$(0,1,2,3,4)^\top$	2995	100.0781
3	$(-1,0,1,2,3)^\top$	1890	81.0938
4	$(-2,-1,0,1,2)^\top$	2007	82.0781
5	$(-3,-2,-1,0,1)^\top$	2769	89.7031
6	$(-4,-3,-2,-1,0)^\top$	2746	82.4844
7	$(-5,-4,-3,-2,-1)^\top$	2780	81.9219
8	$(1,0,0,0,10)^\top$	2773	86.6094

- the matrices B and $\bar{B}$:

$$\beta_j = 12 + 3j, \quad \forall j = 1, \dots, N;$$

- the vector $\bar{a}$: $\alpha = 20$;
- the cost function $c_j (j = 1, \dots, 5)$ is defined by (4.7):

$$\hat{a}_j = 3j - 1, \hat{b}_j = 4j^2 - 5, \hat{c}_j = 5 - 4j;$$

- the quantity matrix A:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 7 & 9 & 11 \\ -1 & 0 & 5 & 3 & -7 \\ 0 & 3 & 4 & 8 & 9 \\ -2 & 9 & 2 & -6 & 3 \end{pmatrix}_{N\times N};$$

- the environmental fee $g : \mathcal{R}^m \to \mathcal{R}$ is defined by (4.8):

$$E = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 9 \end{pmatrix}_{N\times N}, e = \begin{pmatrix} 1 \\ 2 \\ 6 \\ 9 \\ -5 \end{pmatrix}_{N\times 1};$$

- the strategy set $\mathbb{C}$:

$$R = 10, \ddot{a} = (2, -3, 6, -9, 8)^\top, r = 4.$$

By using Proposition (4.1) and Proposition (4.2), computing on Matlab Sotfware shows that

$$\begin{aligned} L_1 &= \|\bar{B}\| + 2\|B\| + \max\{\mathcal{L}_j : j = 1, \dots, N\} = 154.1276, \\ \mathcal{L} &= \|E\| = 9.4142, \\ L_2 &= \|A\|\mathcal{L} = 4.7977e + 03. \end{aligned}$$

From $eig(E) = \{0.1716, 2.0000, 5.8284, 6.5858, 9.4142\}$, it follows that the matrix E is semipositive. (Fig. 2)

Parameters in the algorithm MVIP are chosen as follows: $\gamma = 0.5395$, $f(x) = \frac{1}{2}x$, $b = 0.001$, $a = \min\left\{\frac{\sqrt{1-b}}{2L_i} : i = 1,2\right\}$,

$$\lambda_{i,k} = a - \frac{1}{20k + 1000}, \epsilon_{i,k} = \frac{1 - L_i^2\lambda_{i,k}^2 - b}{4}, \beta_{i,k} = \frac{1}{25k + 150};$$

By Remark 2 and Step 1 of the algorithm MVIP, if $y_i^k = x_i^k$ for all $i = 1, 2$, then x_i^k is a solution of the problem MVI($\mathbb{C}, \mathcal{A}_i$). This implies that the stopping criterion is $\Gamma_k := \max\{\|y_i^k - x_i^k\| : i = 1, 2\} \le \epsilon$, where $\epsilon > 0$. The numerical results are showed in Fig. 2, Table 1 and Fig. 3. The approximation solution is $x^{2756} = (0.0026, 0.0028, -0.0007, 0.0051, 0.0019)^\top$.

Since the preliminary numerical results reported in Figs. 2–3 and Table 1 of the algorithm MVIP for solving electricity production models (4.1), we can see that the convergence speed of the algorithm is very sensitive to the different starting points x^0. The CPU time (second) and the number of iterations of the proposed algorithm are quite effective when solving the problem (4.1). (Table 2)

Test 2. Let us compare the algorithm MVIP with three other algorithms via the electricity production model (4.1): The Projection Algorithm (PA) for solving split feasibility problems proposed by Yen and Muu in [42, Algorithm 3.1] and the Von Neumann Alternating Algorithm (VNAA) introduced by Censor, Gibali and Reich in [44, Algorithm 3.4] and the Efficient Self-Adaptive Algorithm (ESAA) of Tuyen and Ha in [45, Algorithm 1].

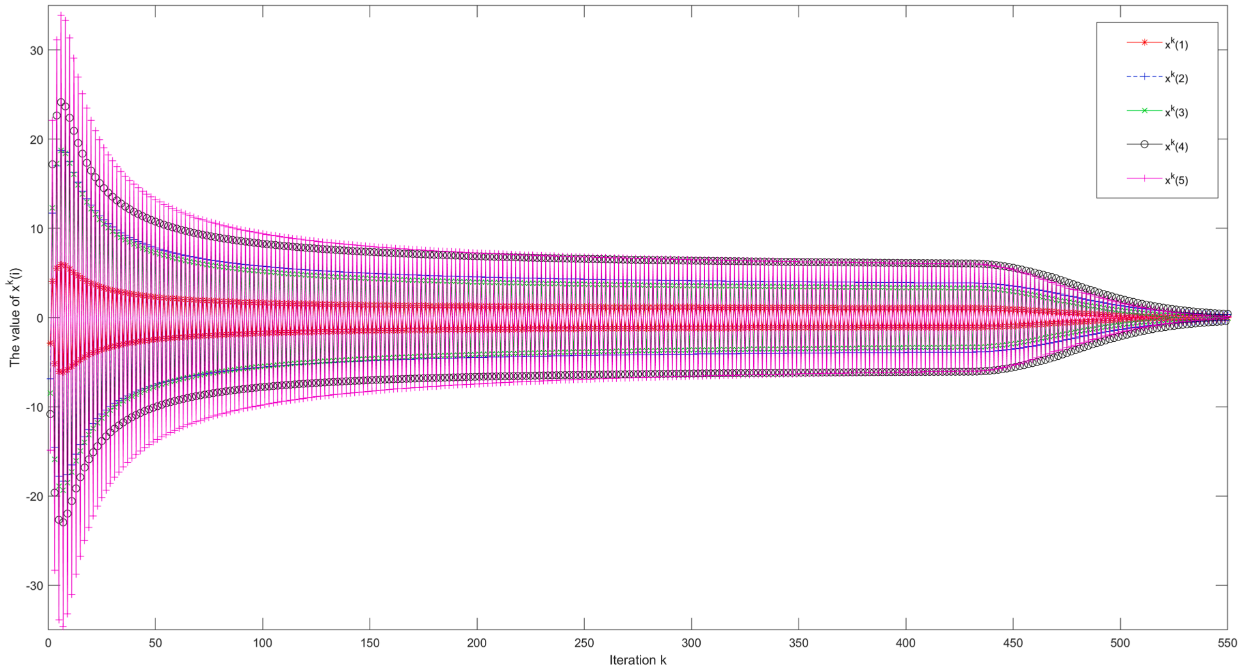

Fig. 1. The algorithm MVIP: Convergence of the sequence $\{x^k\}$ with the tolerance $\epsilon = 10^{-3}$ and the starting point $x^0 = (1, 2, \ldots, 5)^\top$.

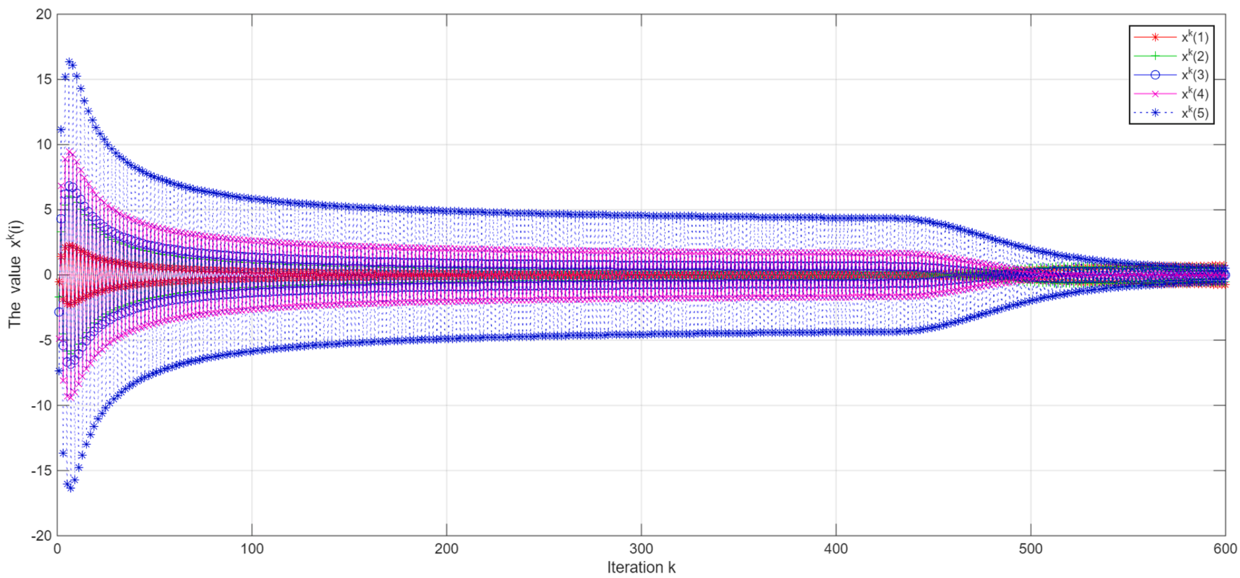

Fig. 2. The algorithm MVIP: Convergence of the sequence $\{x^k\}$ with the tolerance $\epsilon = 10^{-3}$, the starting point $x^0 = (-1, 0, 1, 2, 3)^\top$, $\mathbb{C} = \{x \in \mathcal{R}^5 : \|x\| \le 10, \sum_{i=1}^5 \frac{x_i^2}{i^2} \le 1\}$.

Now we test with the following data. Choose randomly $x^0 = (1, 2, \ldots, N)^\top$, the tolerence is $\Gamma_k \le \epsilon = 10^{-5}$. Let A_0 be an $n \times n$ matrix, B_0 be an $N \times N$ skew-symmetric matrix, D_0 be an $N \times N$ diagonal matrix and the matrix $E = A_0^\top A_0 + B_0 + D_0$ (as generated in [24, Section 5]). To choose randomly on MATLAB software, we use the commands as follows:

$$A = 2 * rand(N, N), A_0 = 5 * rand(N, N), B_0 = skewdec(N, 1), D_0 = diag(1 : N),$$

and $\ddot{a} = 3 * rand(N, 1), e = 4 * rand(N, 1)$. Then, the E is a positive semidefinite matrix. All other remaining parameters are the same as those given is Test 1.

Parameters and data processing in each above algorithm are evaluated as the follows.

- MVIP: As in Test 1.

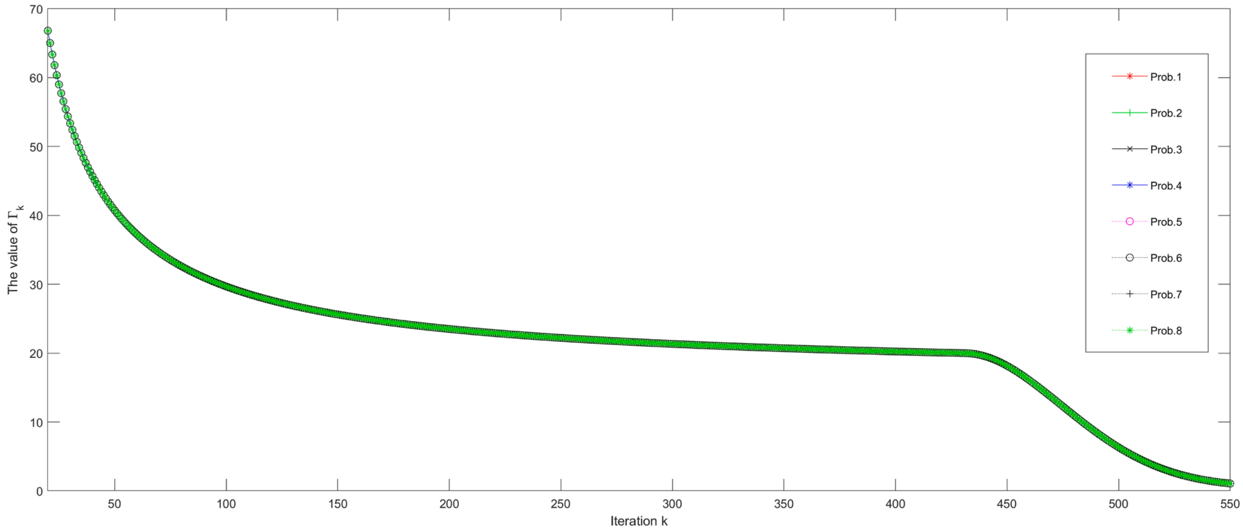

Fig. 3. The algorithm MVIP: Convergence of the sequence $\{\Gamma_k\}$ with the tolerance $\epsilon = 10^{-3}$ and the different starting points x^0 of the problems $1, \dots, 8$.

Table 2
Iterations and CPU times with randomly different parameters, where $x^0 = (1,2,3,4,5)^\top$ and $b = 0.001, a = \min\left\{\frac{\sqrt{1-b}}{2L_i} : i = 1,2\right\}$.

Test	$\beta_{i,k}$	$\epsilon_{i,k}$	$\lambda_{i,k}$	Iter.	CPU times
1	$\frac{1}{25k+150}$	$\frac{1-L_i^2\lambda_{i,k}^2-b}{4}$	$a - \frac{1}{20k+1000}$	2756	6.9375
2	$\frac{1}{20k+100}$	$\frac{1-L_i^2\lambda_{i,k}^2-b}{5}$	$a - \frac{1}{10k+100}$	3104	8.7856
3	$\frac{1}{100k+15}$	$\frac{1-L_i^2\lambda_{i,k}^2-b}{10}$	$a - \frac{1}{100k+500}$	1805	5.1753
4	$\frac{1}{100k+15}$	$\frac{1-L_i^2\lambda_{i,k}^2-b}{10}$	$0.7a - \frac{1}{100k+500}$	1905	5.0081
5	$\frac{1}{120k+200}$	$\frac{1-L_i^2\lambda_{i,k}^2-b}{2}$	$0.9a - \frac{1}{10k+150}$	3820	7.8651
6	$\frac{1}{5k+50}$	$\frac{1-L_i^2\lambda_{i,k}^2-b}{6}$	$0.5a - \frac{1}{10k+10}$	4002	8.2571
7	$\frac{1}{125k+1000}$	$\frac{1-L_i^2\lambda_{i,k}^2-b}{7}$	$0.8a - \frac{1}{110k+150}$	1852	5.3825

- PA:

$$a = 0.01, b = 0.84, a_k = 0.001 + \frac{b}{10K + 100}, \xi = 1, \rho_k = 3, \delta = 0.12,$$

$$\delta_k = 2 + \frac{1}{3k+7}, -a_k = \frac{k+1}{2k+100}, \beta_k = \frac{1}{10k+9}.$$

- VNAA:

$$\alpha_1 := \min\{\beta_i, \hat{a}_i : i = 1, \dots, N\}, \alpha_2 = \|A^\top(EA)\|, \lambda = \min\{\alpha_1, \alpha_2\}.$$

- ESAA: $H_1 = H_2 = \mathcal{R}^N$, the matrix $\mathcal{L}$ is the unit, $B_k = \mathcal{A}_k (k = 1,2)$, the mappings $T_i (i = 1, \dots, N)$ and $S_j (j = 1, \dots, M)$ are the identity mapping,

$$\beta_k = \frac{1}{k+1}, \epsilon_k = \beta_k^2, \zeta_k = 0.5, \alpha_k = 0.5, \theta = 1, \mu = 0.9999, a_k = 10, \lambda_1 = 9.$$

From the test results via the electricity production models of our Modified Viscosity Inexact Projection Algorithm MVIP and the algorithms PA, VNAA and ESAA reported in the tables and figures, we observe that

- The convergence rate of MVIP is quite sensitive to choice of the starting point x^0;
- Test with different $N-$dimensions, the number iterations (No. Iter.) of the algorithms PA, VNAA and ASAA are less than of our proposed algorithm MVIP. Note that, at each iteration k, the MVIP only requires computing two evaluations of the cost mapping $\mathcal{A}_i (i = 1,2)$ and two computations of the inexact projection point without using $\Pi_{\mathbb{C}}$. This leads to fact that its CPU time is the smallest as shown in Table 3.

Table 3
Comparison of the algorithms with different dimensions of $\mathbb{R}^N$.

Alg.	$N = 5$ No. Iter.	$N = 5$ CPU Times	$N = 10$ No. Iter.	$N = 10$ CPU Times	$N = 20$ No. Iter.	$N = 20$ CPU Times	$N = 100$ No. Iter.	$N = 100$ CPU Times
MVIP	2702	6.9375	3024	9.2031	5639	17.9903	23855	85.0482
PA	270	12.0864	371	19.4493	492	48.3319	9501	97.5938
VNAA	43	7.0382	59	11.9405	103	23.0492	796	185.4495
ESAA	403	19.0842	683	57.0052	1092	179.9905	12,058	957.5538

5. Conclusion

In this paper, we present a new viscosity inexact projection algorithm for solving the multivalued variational inquality problem MVI$(\mathbb{C}, \mathcal{A})$ and applications to electricity production models via self-adaptive step size, viscosity techniques, and inexact projection in a real Hilbert space. Very interesting is that strong convergence of iteration sequences is proved and numerical experiments onto the electricity production model have showed the effective performances of the proposed algorithm.

CRediT authorship contribution statement

N.D. Hien: Writing – original draft, Software, Methodology; **J.-S Chen:** Writing – original draft, Software, Methodology, Supervision; **N.V. Hong:** Formal analysis, Data curation, Methodology, Software; **P.N. Anh:** Writing – review & editing, Methodology, Supervision.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We would like to thank the Editor and four anonymous Referees very much for their really helpful, constructive comments and suggestions that helped us very much to improve quality of the paper. The 2nd author's work is supported by National Science and Technology Council, Taiwan and Higher Education Sprout Project - Center for Optimal Intelligent Data Analytics and Prediction. The fourth author's work is supported by Posts and Telecommunications Institute of Technology, Vietnam.

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