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HADAMARD PRODUCT ON QUATERNION HERMITIAN MATRICES

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Abstract. Recently, Kum, Lim, Jeong define an Hadamard product in the setting of Jordan spin algebra, under the scheme of the Peirce decomposition. It is shown that this novel product satisfies an analog of the Schur product theorem as well as the inequalities of Hadamard, Oppenheim, Fiedler, etc. In fact, they also try to extend their results to Euclidean Jordan algebras, but they encounter troubles on Hermitian Quaternion Matrices. In this paper, we propose an Hadamard product on Hermitian Quaternion Matrices, and we show this new product satisfies the same inequalities as well.

1. Introduction

Let $\mathbb{A} = (\mathbb{V}, \langle \cdot, \cdot \rangle, \circ)$ be an n-dimensional Euclidean Jordan algebra (see Section 2). It is known taht every Euclidean Jordan algebra can be written as a direct sum of so-called *simple* ones, which are not themselves direct sums in a nontrivial way. In finite dimensions, the simple Euclidean Jordan algebras come from the following five basic structures.

Theorem 1.1 ([1, Chapter V.3.7]). *Every simple Euclidean Jordan algebra is isomorphic to one of the following.*

(i) *The Jordan spin algebra $\mathbb{L}^n$.*
(ii) *The algebra $\mathbb{S}^n$ of $n \times n$ real symmetric matrices.*
(iii) *The algebra $\mathbb{H}^n$ of all $n \times n$ complex Hermitian matrices.*
(iv) *The algebra $\mathbb{Q}^n$ of all $n \times n$ quaternion Hermitian matrices.*
(v) *The algebra $\mathbb{O}^3$ of all 3×3 octonion Hermitian matrices.*

In particular, $\mathbb{S}^n$ is a primary example of simple Euclidean Jordan algebras. There have been numerous efforts to extend the Hadamard product to general setting of Euclidean Jordan algebras. To name a few, for Euclidean Jordan algebra $\mathbb{V}$ of rank n, Gowda-Tao [2, 3] and Sznajder et al. [7] define two possible generalizations of the Hadamard product, one of which is from $\mathbb{S}^n \times \mathbb{V} \to \mathbb{V}$ and the other is from $\mathbb{V} \times \mathbb{V} \to \mathbb{S}^n$, and prove extensions of the Schur product theorem in $\mathbb{V}$. Nonetheless, even though the products suggested by Gowda-Tao and Sznajder et al. successfully generalize the classical Hadamard product in $\mathbb{S}^n$ to Euclidean Jordan algebras, those products seem "incomplete" in the sense that they are not well defined on Euclidean Jordan algebra $\mathbb{V}$ as a bilinear mapping from $\mathbb{V} \times \mathbb{V}$ to $\mathbb{V}$. Thus, it is still questionable

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whether or not it is possible to define a Hadamard product from $\mathbb{V} \times \mathbb{V}$ to $\mathbb{V}$.

To proceed, we briefly recall the definition of Hadamard product for any two matrices. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices, their Hadamard product or entry-wise product is defined as $A \cdot B = [a_{ij}b_{ij}]$. The Hadamard product has been an active research topic for a long time. For instance, in matrix theory, especially proving inequalities involving Hadamard products, statistics, and physics. For an interested reader, we refer to [4, 9]. In order to overcome the aforementioned problem, Kum, Lim, Jeong [5] propose a novel Hadamard product on the Jordan Spin Algebra $\mathbb{L}^n$, which is really a bilinear mapping from $\mathbb{L}^n \times \mathbb{L}^n$ to $\mathbb{L}^n$. Their main idea is described as below. We say an element $u \in \mathbb{L}^n$ is *semipositive* if

$$u \in \mathcal{K}^n = \{(u_1, \bar{u}) \in \mathbb{L}^n \,|\, u_1 \geq \|\bar{u}\|\}.$$

In light of this concept, the celebrated Schur Theorem, which indicates every Hadamard product of two semipositive elements is still semipositive, holds true as well under their novel Hadamard product. Then, they try to find a bilinear Hadamard product on the other Euclidean Jordan Algebras, but they encounter troubles on quaternion Hermitian matrices $\mathbb{Q}^n$. More specifically, suppose that

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}.$$

In [5], they try to define

$$A \cdot B = \begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}.$$

However, this entry-wise product, which is the way people do in $\mathbb{S}^n$, is not Hermitian again. This is the main difficulty behind. In this paper, we propose our Hadamard Product on $\mathbb{Q}^n$ with rank $n = 2$, and show that Schur Theorem and the corresponding inequalities hold true. Although it is only a certain quaternion case, it may open a new thinking to conquer the hurdle.

2. Preliminaries

This section recalls some results on Euclidean Jordan algebras that will be used in subsequent analysis and definition of semismoothness . More detailed expositions of Euclidean Jordan algebras can be found in Koecher's lecture notes [6] and the monograph by Faraut and Korányi [1].

Let $\mathbb{V}$ be an n-dimensional vector space over the real field $\mathbb{R}$, endowed with a bilinear mapping $(x, y) \mapsto x \circ y$ from $\mathbb{V} \times \mathbb{V}$ into $\mathbb{V}$. The pair $(\mathbb{V}, \circ)$ is called a *Jordan algebra* if

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$.

Note that a Jordan algebra is not necessarily associative, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$ may not hold for all $x, y, z \in \mathbb{V}$. We call an element $e \in \mathbb{V}$ the *identity* element if

$x \circ e = e \circ x = x$ for all $x \in \mathbb{V}$. A Jordan algebra $(\mathbb{V}, \circ)$ with an identity element e is called a *Euclidean Jordan algebra* if there is an inner product, $\langle \cdot, \cdot \rangle_{\mathbb{V}}$, such that

(iii) $\langle x \circ y, z \rangle_{\mathbb{V}} = \langle y, x \circ z \rangle_{\mathbb{V}}$ for all $x, y, z \in \mathbb{V}$.

Given a Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \circ, \langle \cdot, \cdot \rangle_{\mathbb{V}})$, we denote the set of squares as

$$\mathcal{K} := \left\{ x^2 \mid x \in \mathbb{V} \right\}.$$

By [1, Theorem III.2.1], $\mathcal{K}$ is a symmetric cone. This means that $\mathcal{K}$ is a self-dual closed convex cone with nonempty interior and for any two elements $x, y \in \mathrm{int}(\mathcal{K})$, there exists an invertible linear transformation $\mathcal{T} : \mathbb{V} \to \mathbb{V}$ such that $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ and $\mathcal{T}(x) = y$.

With this cone $\mathcal{K}$, we can define a partial order on $\mathbb{V}$.

Definition 2.1. $x \succeq y$ if $x - y \in \mathcal{K}$ and $x \succ y$ if $x - y \in \mathrm{int}(\mathcal{K})$.

For any given $x \in \mathbb{A}$, let $\zeta(x)$ be the degree of the minimal polynomial of x, i.e.,

$$\zeta(x) := \min \Big\{ k : \{e, x, x^2, \cdots, x^k\} \text{ are linearly dependent} \Big\}.$$

Then the *rank* of $\mathbb{A}$ is defined as $\max\{\zeta(x) : x \in \mathbb{V}\}$. In this paper, we use r to denote the rank of the underlying Euclidean Jordan algebra. Recall that an element $c \in \mathbb{V}$ is *idempotent* if $c^2 = c$. Two idempotents c_i and c_j are said to be *orthogonal* if $c_i \circ c_j = 0$. One says that $\{c_1, c_2, \ldots, c_k\}$ is a *complete system of orthogonal idempotents* if

$$c_j^2 = c_j, \quad c_j \circ c_i = 0 \text{ if } j \neq i \text{ for all } j, i = 1, 2, \cdots, k, \quad \text{and } \textstyle\sum_{j=1}^k c_j = e.$$

An idempotent is *primitive* if it is nonzero and cannot be written as the sum of two other nonzero idempotents. We call a complete system of orthogonal primitive idempotents a *Jordan frame*. Now we state the second version of the spectral decomposition theorem.

Theorem 2.2 ([1, Theorem III.1.2])**.** *Suppose that $\mathbb{A}$ is a Euclidean Jordan algebra with the rank r. Then for any $x \in \mathbb{V}$, there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1(x), \ldots, \lambda_r(x)$, arranged in the decreasing order $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$, such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

The numbers $\lambda_j(x)$ (counting multiplicities), which are uniquely determined by x, are called the eigenvalues and $\mathrm{tr}(x) = \sum_{j=1}^r \lambda_j(x)$ the trace of x.

To move on, we point out a few facts. Suppose $x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r$. Then,

$$x^2 := x \circ x = \lambda_1(x)^2 c_1 + \lambda_2(x)^2 c_2 + \cdots + \lambda_r(x)^2 c_r.$$

Moreover, if $x \succ 0$, then $\lambda_i(x) > 0$ for all i, and x has the inverse expression as

$$x^{-1} = \lambda_1(x)^{-1} c_1 + \lambda_2(x)^{-1} c_2 + \cdots + \lambda_r(x)^{-1} c_r.$$

Theorem 2.3. *Suppose that $\mathbb{A}$ is a Euclidean Jordan algebra with the rank r. For any $x \in \mathbb{V}$, if $x \succ 0$, then $x + x^{-1} \succeq 2e$.*

Proof. The proof is straightforward by verifying

$$x + x^{-1} = (\lambda_1(x) + \lambda_1(x)^{-1})c_1 + \cdots + \left(\lambda_r(x) + \lambda_r(x)^{-1}\right) c_r \succeq 2c_1 + \cdots + 2c_r = 2e.$$

□

3. Hadamard product in quaternion Hermitian matrices

The algebra $\mathbb{Q}^n$ of $n \times n$ quaternion Hermitian matrices.

The linear space of quaternions over $\mathbb{R}$, denoted by $\mathbb{Q}$, is 4-dimensional vector space [8] with a basis $\{1, i, j, k\}$. This space becomes an associated algebra via the following multiplication table.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

For any $\zeta = x_0 1 + x_1 i + x_2 j + x_3 k \in \mathbb{Q}$, we define its *real part* by $\mathrm{I\!R}(\zeta) := x_0$, its *conjugate* by $\bar{\zeta} := x_0 1 - x_1 i - x_2 j - x_3 k$, and its norm by $|x| = \sqrt{x\bar{x}}$. **Note in general $\zeta\xi \neq \xi\zeta$, i.e., $\mathbb{Q}$ is Not commutative. We do not have $\bar{\zeta\xi} = \bar{\zeta}\,\bar{\xi}$, either. But luckily we do have $\bar{\zeta\xi} = \bar{\xi}\,\bar{\zeta}$. Moreover $\|\zeta\xi\|^2 = \|\zeta\|^2\|\xi\|^2$.** These are crucial tools that we can employ for proceeding the analysis.

A square matrix A with quaternion entries is called *Hermitian* if A coincides with its conjugate transpose. Let $\mathbb{Q}^n$ be the set of all $n \times n$ quaternion Hermitian matrices. For any $X, Y \in \mathbb{Q}^n$, we define

$$X \circ Y := \frac{1}{2}(XY + YX) \quad \text{and} \quad \langle X, Y \rangle := \mathrm{I\!R}(\mathrm{trace}(XY)).$$

Then, $\mathbb{Q}^n$ is a Euclidean Jordan algebra of rank n with e being the $n \times n$ identity matrix I_n. Analogous to complex number, each quaternion $\zeta = a1 + bi + cj + dk \in \mathbb{Q}$ can be represented as a 4×4 real matrix $\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$ which is also equivalent to

$$a \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, embedding $\mathbb{Q}^n$ into $\mathbb{S}^{4n}$ yields that $\mathbb{Q}^n$ can be viewed as a Jordan sub-algebra of $\mathbb{S}^{4n}$. In particular, the embedding map under the case for $\mathbb{Q}^2$ is

$$\mathbb{Q}^2 \ni \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix} \longmapsto \begin{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_1 \end{bmatrix} & \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \\ \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} & \begin{bmatrix} \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} \end{bmatrix} \in \mathbb{S}^8$$

where $x = a1 + bi + cj + dk$.

Moreover, the general embedding map under this case is given by

$$\mathbb{Q}^n \ni \begin{bmatrix} \alpha_1 & x & \cdots & y \\ \bar{x} & \alpha_2 & \cdots & z \\ \vdots & \vdots & \ddots & \vdots \\ \bar{y} & \bar{z} & \cdots & \alpha_n \end{bmatrix} \longmapsto$$

$$\begin{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_1 \end{bmatrix} & \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} & \cdots & \begin{bmatrix} e & f & g & h \\ -f & e & -h & g \\ -g & h & e & -f \\ -h & -g & f & e \end{bmatrix} \\ \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} & \begin{bmatrix} \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} & \cdots & \begin{bmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} e & -f & -g & -h \\ f & e & h & -g \\ g & -h & e & f \\ h & g & -f & e \end{bmatrix} & \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix} & \cdots & \begin{bmatrix} \alpha_n & 0 & 0 & 0 \\ 0 & \alpha_n & 0 & 0 \\ 0 & 0 & \alpha_n & 0 \\ 0 & 0 & 0 & \alpha_n \end{bmatrix} \end{bmatrix} \in \mathbb{S}^{4n}$$

where $x = a1 + bi + cj + dk$, $y = e1 + fi + gj + hk$ and $z = p1 + qi + rj + sk$.

Note that the symmetric cone $\mathcal{K} = \{A^2 | A \in \mathbb{Q}^n\}$ is the set of all squares in $\mathbb{Q}^n$. The following Proposition is easily to be verified.

Proposition 3.1. *Suppose a quaternion Hermitian matrix* $A = \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix} \in \mathbb{Q}^2$. *The following hold.*

(i) $\mathrm{tr}(A) = \alpha_1 + \alpha_2$, *and* $\det(A) = \alpha_1\alpha_2 - \|\zeta\|^2$.

(ii) $A \succeq 0$ *i.e.* $A \in \mathcal{K}$ *if and only if one of the following holds*

(a) $\alpha_1, \alpha_2 \geq 0$ *and* $\alpha_1\alpha_2 - \|\zeta\|^2 \geq 0$.

(b) $\mathrm{tr}(A) \geq 0$ *and* $\det(A) \geq 0$.

(iii) $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \alpha_2 & -\zeta \\ -\bar{\zeta} & \alpha_1 \end{bmatrix}$.

Now, we propose our version of Hadamard product associated with $\mathbb{Q}^2$.

Definition 3.2. Suppose two quaternion Hermitian matrices $A = \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix}$ and $B = \begin{bmatrix} \beta_1 & \xi \\ \bar{\xi} & \beta_2 \end{bmatrix} \in \mathbb{Q}^2$. We define the Hadamard product of A and B as

$$A \cdot B := \begin{bmatrix} \alpha_1\beta_1 & \zeta\xi \\ \bar{\xi}\bar{\zeta} & \alpha_2\beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\beta_1 & \zeta\xi \\ \bar{\zeta\xi} & \alpha_2\beta_2 \end{bmatrix} \in \mathbb{Q}^2.$$

It is an easy observation that this product is not commutative, since $\mathbb{Q}$ is not commutative in general. But it is associative and distributive. To sum up, we see that

- $A \cdot B = B \cdot A$ does not hold.
- $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
- $A \cdot (B + C) = A \cdot B + A \cdot C$.

We next show that our Hadamard product defined on $\mathbb{Q}^2$ satisfies the Schur product theorem.

Theorem 3.3. *Suppose two quaternion Hermitian matrices* $A = \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix} \succeq 0$ *and* $B = \begin{bmatrix} \beta_1 & \xi \\ \bar{\xi} & \beta_2 \end{bmatrix} \succeq 0 \in \mathbb{Q}^2$. *Then,* $A \cdot B \succeq 0$.

Proof. Obviously, we have $\mathrm{tr}(A \cdot B) = \alpha_1\beta_1 + \alpha_2\beta_2 \geq 0$ and $\det(A \cdot B) = \alpha_1\alpha_2\beta_1\beta_2 - \|\zeta\|^2\|\xi\|^2 \geq 0$. By Proposition 3.1 (ii), the desired result holds. □

4. Inequalities involving Hadamard products

4.1. Oppenheim type inequalities.

Theorem 4.1. *Suppose two quaternion Hermitian matrices* $A = \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix} \succeq 0$ *and* $B = \begin{bmatrix} \beta_1 & \xi \\ \bar{\xi} & \beta_2 \end{bmatrix} \succeq 0 \in \mathbb{Q}^2$. *Then, we have* $\det A \det B \leq \alpha_1\alpha_2 \det B \leq \det(A \cdot B)$, *and* $\det A \det B \leq \beta_1\beta_2 \det A \leq \det(A \cdot B)$.

Proof. First, it is easy to see that $A \cdot B = \begin{bmatrix} \alpha_1\beta_1 & \zeta\xi \\ \bar{\zeta\xi} & \alpha_2\beta_2 \end{bmatrix}$. This leads to

$$\det A \det B = (\alpha_1\alpha_2 - \|\zeta\|^2) \det B \leq \alpha_1\alpha_2 \det B.$$

Similarly, we have $\det A \det B \leq \beta_1\beta_2 \det A$. Moreover, it cab be verified that

$$\begin{aligned} \alpha_1\alpha_2 \det B = \alpha_1\alpha_2 (\beta_1\beta_2 - \|\xi\|^2) &= \alpha_1\alpha_2\beta_1\beta_2 - \alpha_1\alpha_2\|\xi\|^2 \\ &\leq \alpha_1\alpha_2\beta_1\beta_2 - \|\zeta\xi\|^2 = \det(A \cdot B), \end{aligned}$$

where we use $\alpha_1\alpha_2 \geq \|\zeta\|^2$ since $A \succeq 0$. Likewise, we have $\beta_1\beta_2 \det A \leq \det(A \cdot B)$. $\square$

Corollary 4.2. *Suppose that $X \succ 0 \in \mathbb{Q}^2$ is a quaternion Hermitian matrix. Then, we have $1 \leq \det(X \cdot X^{-1})$.*

Proof. Let $A = X$ and $B = X^{-1}$. By Theorem 4.1, the inequality follows immediately. $\square$

In fact, for general quaternion Hermitian matrices, we have the following special equality.

Theorem 4.3. *Suppose that $A = \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix}$ and $B = \begin{bmatrix} \beta_1 & \xi \\ \bar{\xi} & \beta_2 \end{bmatrix}$ are two quaternion Hermitian matrices in $\mathbb{Q}^2$. Then, we have*

$$\alpha_1\alpha_2 \det B + \beta_1\beta_2 \det A = \det A \det B + \det(A \cdot B).$$

Proof. It is easy to see that $A \cdot B = \begin{bmatrix} \alpha_1\beta_1 & \zeta\xi \\ \bar{\zeta}\bar{\xi} & \alpha_2\beta_2 \end{bmatrix}$. Accordingly, direct computation gives

$$\alpha_1\alpha_2 \det B + \beta_1\beta_2 \det A = \alpha_1\alpha_2\beta_1\beta_2 - \alpha_1\alpha_2\|\xi\|^2 + \alpha_1\alpha_2\beta_1\beta_2 - \beta_1\beta_2\|\zeta\|^2,$$

$$\det A \det B + \det(A \cdot B) = (\alpha_1\alpha_2 - \|\zeta\|^2)(\beta_1\beta_2 - \|\xi\|^2) + \alpha_1\alpha_2\beta_1\beta_2 - \|\zeta\xi\|^2.$$

Then, the proof is complete. $\square$

4.2. **Fiedler type inequalities.**

Theorem 4.4. *Suppose that $X = \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix} \succ 0$ is a quaternion Hermitian matrices in $\mathbb{Q}^2$. Then, we have*

$$X \cdot X^{-1} \succeq I_2 \succeq (X \cdot X^{-1})^{-1}.$$

Proof. First, we compute that $X \cdot X^{-1} = \frac{1}{\det X}\begin{bmatrix} \alpha_1\alpha_2 & -\zeta^2 \\ -\bar{\zeta}^2 & \alpha_1\alpha_2 \end{bmatrix}$ since $X^{-1} = \frac{1}{\det X}\begin{bmatrix} \alpha_2 & -\zeta \\ -\bar{\zeta} & \alpha_1 \end{bmatrix}$. Thus,

$$X \cdot X^{-1} - I_2 = \frac{1}{\det X}\begin{bmatrix} \alpha_1\alpha_2 - \det X & -\zeta^2 \\ -\bar{\zeta}^2 & \alpha_1\alpha_2 - \det X \end{bmatrix} = \frac{1}{\det X}\begin{bmatrix} \|\zeta\|^2 & -\zeta^2 \\ -\bar{\zeta}^2 & \|\zeta\|^2 \end{bmatrix}.$$

hen, the first inequality follows from Proposition 3.1(ii).

Now, denote $A = X \cdot X^{-1} - I_2 = \lambda_1(A)c_1 + \lambda_2(A)c_2$. By previous inequality $A \succeq I_2$, we have $\lambda_1(A) \geq 1$ and $\lambda_2(A) \geq 1$. Then, its inverse $A^{-1} = \lambda_1(A)^{-1}c_1 + \lambda_2(A)^{-1}c_2$. In other words, the last inequality holds. $\square$

Theorem 4.5. *Suppose that* $A = \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix} \succ 0$, *and* $B = \begin{bmatrix} \beta_1 & \xi \\ \bar{\xi} & \beta_2 \end{bmatrix} \succ 0$ *are two quaternion Hermitian matrices in* $\mathbb{Q}^2$. *Then, we have*

$$A^{-1} \cdot B^{-1} \succeq (A \cdot B)^{-1}.$$

Proof. First, we compute that

$$A^{-1} \cdot B^{-1} = \frac{1}{\det A \det B} \begin{bmatrix} \alpha_2 & -\zeta \\ -\bar{\zeta} & \alpha_1 \end{bmatrix} \cdot \begin{bmatrix} \beta_2 & -\xi \\ -\bar{\xi} & \beta_1 \end{bmatrix} = \frac{1}{\det A \det B} \begin{bmatrix} \alpha_2\beta_2 & -\zeta\xi \\ -\bar{\zeta}\bar{\xi} & \alpha_1\beta_1 \end{bmatrix},$$

and

$$(A \cdot B)^{-1} = \frac{1}{\det(A \cdot B)} \begin{bmatrix} \alpha_2\beta_2 & -\zeta\xi \\ -\bar{\zeta}\bar{\xi} & \alpha_1\beta_1 \end{bmatrix}.$$

These yield

$$A^{-1} \cdot B^{-1} - (A \cdot B)^{-1} = \left[\frac{1}{\det A \det B} - \frac{1}{\det(A \cdot B)}\right] \begin{bmatrix} \alpha_2\beta_2 & -\zeta\xi \\ -\bar{\zeta}\bar{\xi} & \alpha_1\beta_1 \end{bmatrix}.$$

From the proof of Theorem 4.1, we have $\left[\frac{1}{\det A \det B} - \frac{1}{\det(A \cdot B)}\right] > 0$. Then, the inequality follows from Proposition 3.1 (ii). $\square$

Corollary 4.6. *Suppose that* $A \succ 0, B \succ 0$ *are two quaternion Hermitian matrices in* $\mathbb{Q}^2$. *Then, we have*

$$A \cdot B^{-1} + A^{-1} \cdot B \succeq 2I_2.$$

Proof. By Theorem 4.5, we obtain $A^{-1} \cdot B \succeq (A \cdot B^{-1})^{-1}$. Hence, there holds

$$A \cdot B^{-1} + A^{-1} \cdot B \succeq A \cdot B^{-1} + (A \cdot B^{-1})^{-1} \succeq 2I_2,$$

where the last inequality follows from Theorem 2.3. $\square$

Corollary 4.7. *Suppose that* $X_i \succ 0$ *are quaternion Hermitian matrices in* $\mathbb{Q}^2$ *for* $1 \le i \le k$. *Then, we have*

$$\left(\Sigma_{i=1}^k X_i\right) \cdot \left(\Sigma_{i=1}^k X_i^{-1}\right) \succeq k^2 I_2.$$

Proof. Note that

$$\left(\Sigma_{i=1}^k X_i\right) \cdot \left(\Sigma_{i=1}^k X_i^{-1}\right) = \Sigma_{i=1}^k X_i \cdot X_i^{-1} + \Sigma_{i<j}^k X_i \cdot X_j^{-1} + X_i^{-1} \cdot X_j \succeq kI_2 + \frac{k(k-1)}{2} 2I_2,$$

the inequality follows by Theorem 4.4 and Corollary 4.6. $\square$

4.3. **Other inequalities.**

Theorem 4.8. *Suppose that* $A = \begin{bmatrix} \alpha_1 & \zeta \\ \bar{\zeta} & \alpha_2 \end{bmatrix} \succeq 0$, *and* $B = \begin{bmatrix} \beta_1 & \xi \\ \bar{\xi} & \beta_2 \end{bmatrix} \succeq 0$ *are two quaternion Hermitian matrices in* $\mathbb{Q}^2$. *Then, we have*

$$A^2 \cdot B^2 \succeq (A \cdot B)^2 .$$

Proof. First, we compute that

$$A^2 = \begin{bmatrix} \alpha_1^2 + \|\zeta\|^2 & (\alpha_1+\alpha_2)\zeta \\ (\alpha_1+\alpha_2)\bar{\zeta} & \alpha_2^2 + \|\zeta\|^2 \end{bmatrix}, B^2 = \begin{bmatrix} \beta_1^2 + \|\xi\|^2 & (\beta_1+\beta_2)\xi \\ (\beta_1+\beta_2)\bar{\xi} & \beta_2^2 + \|\xi\|^2 \end{bmatrix},$$

$$A^2 \cdot B^2 = \begin{bmatrix} (\alpha_1^2 + \|\zeta\|^2)(\beta_1^2 + \|\xi\|^2) & (\alpha_1+\alpha_2)(\beta_1+\beta_2)\zeta\xi \\ (\alpha_1+\alpha_2)(\beta_1+\beta_2)\bar{\zeta}\bar{\xi} & (\alpha_2^2 + \|\zeta\|^2)(\beta_2^2 + \|\xi\|^2) \end{bmatrix},$$

and

$$(A \cdot B)^2 = \begin{bmatrix} \alpha_1^2\beta_1^2 + \|\zeta\xi\|^2 & (\alpha_1\beta_1+\alpha_2\beta_2)\zeta\xi \\ (\alpha_1\beta_1+\alpha_2\beta_2)\bar{\zeta}\bar{\xi} & \alpha_2^2\beta_2^2 + \|\zeta\xi\|^2 \end{bmatrix}.$$

Thus, we obtain

$$A^2 \cdot B^2 - (A \cdot B)^2 = \begin{bmatrix} \alpha_1^2\|\xi\|^2 + \beta_1^2\|\zeta\|^2 & (\alpha_1\beta_2+\alpha_2\beta_1)\zeta\xi \\ (\alpha_1\beta_2+\alpha_2\beta_1)\bar{\zeta}\bar{\xi} & \alpha_2^2\|\xi\|^2 + \beta_2^2\|\zeta\|^2 \end{bmatrix}.$$

Note that

$$\begin{aligned}(\alpha_1^2\|\xi\|^2 + \beta_1^2\|\zeta\|^2)(\alpha_2^2\|\xi\|^2 + \beta_2^2\|\zeta\|^2) - (\alpha_1\beta_2+\alpha_2\beta_1)^2\|\zeta\xi\|^2 \\ = \alpha_1^2\alpha_2^2\|\xi\|^4 + \beta_1^2\beta_2^2\|\zeta\|^4 - 2\alpha_1\alpha_2\beta_1\beta_2\|\zeta\xi\|^2 \geq 0.\end{aligned}$$

Then, by A-G inequality, the inequality follows fromProposition 3.1(ii). □

For general quaternion Hermitian matrices, the following inequalities hold.

Theorem 4.9. *Suppose that A is a quaternion Hermitian matrix in $\mathbb{Q}^2$. Then, we have*

(i) $A \cdot A \preceq A^2 \cdot I_2$,
(ii) $(A \cdot I_2)^2 \preceq \frac{1}{2}\left(A^2 \cdot I_2 + A \cdot A\right) \preceq A^2 \cdot I_2$.

Proof. By direct computation, we have

$$A \cdot A = \begin{bmatrix} \alpha_1^2 & \zeta^2 \\ \bar{\zeta}^2 & \alpha_2^2 \end{bmatrix}, A^2 \cdot I_2 = \begin{bmatrix} \alpha_1^2 + \|\zeta\|^2 & 0 \\ 0 & \alpha_2^2 + \|\zeta\|^2 \end{bmatrix}, (A \cdot I_2)^2 = \begin{bmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{bmatrix}.$$

Then, the inequalities hold by Proposition 3.1(ii). □

5. Conclusions

In this short paper, we propose a Hadamard product on $\mathbb{Q}^2$. In general the quaternion number system $\mathbb{Q}$ is not commutative. It causes the trouble when people try to extend entry-wise product(Hadamard product in $\mathbb{S}^n$) to $\mathbb{Q}^n$. But, we do have $\bar{\zeta}\xi = \bar{\xi}\bar{\zeta}$ in $\mathbb{Q}$. This is the key property that we could define our Hadamard product in $\mathbb{Q}^2$. In general, let

$$A = \begin{bmatrix} \alpha_1 & x & \cdots & y \\ \bar{x} & \alpha_2 & \cdots & z \\ \vdots & \vdots & \ddots & \vdots \\ \bar{y} & \bar{z} & \cdots & \alpha_n \end{bmatrix}, B = \begin{bmatrix} \beta_1 & u & \cdots & v \\ \bar{u} & \beta_2 & \cdots & w \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v} & \bar{w} & \cdots & \beta_n \end{bmatrix} \in \mathbb{Q}^n.$$

Define

$$A \cdot B := \begin{bmatrix} \alpha_1\beta_1 & xu & \cdots & yv \\ \bar{u}\bar{x} & \alpha_2\beta_2 & \cdots & zw \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v}\bar{y} & \bar{w}\bar{z} & \cdots & \alpha_n\beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1\beta_1 & xu & \cdots & yv \\ \bar{xu} & \alpha_2\beta_2 & \cdots & zw \\ \vdots & \vdots & \ddots & \vdots \\ \bar{yv} & \bar{zw} & \cdots & \alpha_n\beta_n \end{bmatrix} \in \mathbb{Q}^n. \tag{1}$$

It is natural to make the following conjecture.

Conjecture 5.1. The product in (1) satisfy all inequalities in section 4.

In addition, as we point out in section 3, our Hadamard product is not commutative. We leave it for future work to find out a Hadamard product in $\mathbb{Q}^n$ that is commutative, too.

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