to appear in Journal of Nonlinear and Convex Analysis, 2017

On four discrete-type families of NCP-functions

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September 1, 2016
(revised on February 1, 2017)

Abstract. In this paper, we look into the detailed properties of four discrete-type families of NCP-functions, which are newly discovered in recent literature. With the discrete-oriented feature, we are motivated to know what differences there are compared to the traditional NCP-functions. The properties obtained in this paper not only explain the difference but also provide background bricks for designing solution methods based on such discrete-type families of NCP-functions.

Keywords. NCP-function; Complementarity; Semismooth.

1 Introduction

The nonlinear complementarity problem (NCP) [18, 27] is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \ F(x) \geq 0, \langle x, \ F(x) \rangle = 0$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \cdots, F_n)^T$ maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. The NCP has attracted much attention due to its various applications in operations research, economics, and engineering, see [13, 18, 27] and references therein. There have

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been many methods proposed for solving the NCP. Among which, one of the most popular and powerful approaches that has been studied intensively recently is to reformulate the NCP as a system of nonlinear equations [23] or as an unconstrained minimization problem [12, 14, 19]. Such a function that can constitute an equivalent unconstrained minimization problem for the NCP is called a merit function. In other words, a merit function is a function whose global minima are coincident with the solutions of the original NCP. For constructing a merit function, the class of functions, so-called NCP-functions plays an important role.

A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP-function if it satisfies

$$\phi(a,b) = 0 \iff a \geq 0,\ b \geq 0,\ ab = 0. \tag{1}$$

Many NCP-functions and merit functions have been explored and proposed in many literature, see [16] for a survey. Among them, the Fischer-Burmeister (FB) function and the Natural-Residual (NR) function are two effective NCP-functions. The FB function $\phi_{\rm FB} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\rm FB}(a,b) = \sqrt{a^2+b^2} - (a+b), \tag{2}$$

and the NR function $\phi_{\rm NR} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\rm NR}(a,b) = a - (a-b)_+ = \min\{a,b\}, \tag{3}$$

where $(t)_+$ means $\max\{0,t\}$ for any $t \in \mathbb{R}$.

Recently, the generalized Fischer-Burmeister function $\phi^p_{\rm FB}$ which includes the Fischer-Burmeister as a special case was considered in [2, 3, 4, 8, 33]. Indeed, the function $\phi^p_{\rm FB}$ is a natural extension of the $\phi_{\rm FB}$ function, in which the 2-norm in $\phi_{\rm FB}$ is replaced by general p-norm. In other words, $\phi^p_{\rm FB} : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\phi^p_{\rm FB}(a,b) = \|(a,b)\|_p - (a+b), \tag{4}$$

where $p > 1$ and $\|(a,b)\|_p = \sqrt[p]{|a|^p + |b|^p}$. The detailed geometric view of $\phi^p_{\rm FB}$ is depicted in [33]. Corresponding to $\phi^p_{\rm FB}$, there is a merit function $\psi^p_{\rm FB} : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$\psi^p_{\rm FB}(a,b) = \frac{1}{2}\left|\phi^p_{\rm FB}(a,b)\right|^2. \tag{5}$$

For any given $p > 1$, the function $\psi^p_{\rm FB}$ is a nonnegative NCP-function and smooth on $\mathbb{R}^2$. Note that $\phi^p_{\rm FB}$ is a natural "continuous" type of generalization of the FB function $\phi_{\rm FB}$.

To the contrast, what does "generalized natural-residual function" look like? In [7], Chen *et al.* give an answer to the long-standing open question. More specifically, the generalized natural-residual function, denoted by $\phi^p_{\rm NR}$, is defined by

$$\phi^p_{\rm NR}(a,b) = a^p - (a-b)^p_+ \tag{6}$$

with $p > 1$ being a positive odd integer. As remarked in [7], the main idea to create it relies on "discrete generalization", not the "continuous generalization". Note that when $p = 1$, $\phi^p_{_{\rm NR}}$ is reduced to the natural residual function $\phi_{_{\rm NR}}$.

Unlike the surface of $\phi^p_{_{\rm FB}}$, the surface of $\phi^p_{_{\rm NR}}$ is not symmetric which may cause some difficulties in further analysis in designing solution methods. To this end, Chang *et al.* [1] try to symmetrize the function $\phi^p_{_{\rm NR}}$. The first-type symmetrization of $\phi^p_{_{\rm NR}}$, denoted by $\phi^p_{_{\rm S-NR}}$ is proposed as

$$\phi^p_{_{\rm S-NR}}(a,b) = \begin{cases} a^p - (a-b)^p & \text{if} \quad a > b, \\ a^p = b^p & \text{if} \quad a = b, \\ b^p - (b-a)^p & \text{if} \quad a < b, \end{cases} \tag{7}$$

where $p > 1$ being a positive odd integer. It is shown in [1] that $\phi^p_{_{\rm S-NR}}$ is an NCP-function with symmetric surface, but it is not differentiable. Therefore, it is natural to ask whether there exists another symmetrization function that has not only symmetric surface but also is differentiable. Fortunately, Chang *et al.* [1] also figure out the second symmetrization of $\phi^p_{_{\rm NR}}$, denoted by $\psi^p_{_{\rm S-NR}}$, which is proposed as

$$\psi^p_{_{\rm S-NR}}(a,b) = \begin{cases} a^p b^p - (a-b)^p b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ a^p b^p - (b-a)^p a^p & \text{if} \quad a < b, \end{cases} \tag{8}$$

where $p > 1$ being a positive odd integer. As expected, the function $\psi^p_{_{\rm S-NR}}$ is not only differentiable but also possesses a symmetric surface. To sum up, there exist three discrete-type families of NCP-functions: $\phi^p_{_{\rm NR}}$, $\phi^p_{_{\rm S-NR}}$, and $\psi^p_{_{\rm S-NR}}$, which are based on the NR function $\phi_{_{\rm NR}}$.

Next, we elaborate more about the above three new NCP-functions.

(i) For p being an even integer, all of above are not NCP-functions. A counterexample is given as below.

$$\phi^2_{_{\rm NR}}(-1,-2) = (-1)^2 - (-1+2)^2_+ = 0.$$

$$\phi^2_{_{\rm S-NR}}(-1,-2) = (-1)^2 - (-1+2)^2 = 0.$$

$$\psi^2_{_{\rm S-NR}}(-1,-2) = (-1)^2(-2)^2 - (-1+2)^2(-2)^2 = 0.$$

(ii) The above three functions are neither convex nor concave functions. To see this, taking $p = 3$ and using the following arguments verify the assertion.

$$1 = \phi^3_{_{\rm NR}}(1,1) < \frac{1}{2}\phi^3_{_{\rm NR}}(0,1) + \frac{1}{2}\ \phi^3_{_{\rm NR}}(2,1) = \frac{0}{2} + \frac{7}{2} = \frac{7}{2}.$$

$$1 = \phi^3_{_{\rm NR}}(1,1) > \frac{1}{2}\phi^3_{_{\rm NR}}(1,-1) + \frac{1}{2}\ \phi^3_{_{\rm NR}}(1,3) = -\frac{7}{2} + \frac{1}{2} = -3.$$

$$1 = \phi^3_{_{\rm S-NR}}(1,1) < \frac{1}{2}\phi^3_{_{\rm S-NR}}(0,0) + \frac{1}{2}\phi^3_{_{\rm S-NR}}(2,2) = \frac{0}{2} + \frac{8}{2} = 4.$$

$$1 = \phi^3_{_{\rm S-NR}}(1,1) > \frac{1}{2}\phi^3_{_{\rm S-NR}}(2,0) + \frac{1}{2}\phi^3_{_{\rm S-NR}}(0,2) = \frac{0}{2} + \frac{0}{2} = 0.$$

$$1 = \psi^3_{_{\rm S-NR}}(1,1) < \frac{1}{2}\psi^3_{_{\rm S-NR}}(0,0) + \frac{1}{2}\psi^3_{_{\rm S-NR}}(2,2) = \frac{0}{2} + \frac{64}{2} = 32.$$

$$1 = \psi^3_{_{\rm S-NR}}(1,1) > \frac{1}{2}\psi^3_{_{\rm S-NR}}(2,0) + \frac{1}{2}\psi^3_{_{\rm S-NR}}(0,2) = \frac{0}{2} + \frac{0}{2} = 0.$$

The idea of "discrete generalization" looks simple, but it is novel and important. In fact, the authors also apply such idea to construct more NCP-functions. For example, the authors apply it to the Fischer-Burmeister function to obtain $\phi^p_{_{\rm D-FB}} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\phi^p_{_{\rm D-FB}}(a,b) = \left(\sqrt{a^2+b^2}\right)^p - (a+b)^p \tag{9}$$

where $p > 1$ being a positive odd integer. This function is proved as an NCP-function in [22]. In addition, it can also serve as a complementarity function for second-order cone complementarity problem (SOCCP) [22].

The aforementioned four discrete-type families of NCP-functions are newly discovered. Unlike the existing NCP-functions, we know that they are discrete-oriented in some sense. However, what other differences there are compared to the traditional continuous-type families of NCP-functions? This is the main motivation of this paper. Even though we have the feature of differentiability, we point out that the Newton method may not be applied directly because the Jacobian at a degenerate solution to NCP may be singular (see [19, 20]). Nonetheless, the feature of differentiability may enable that some other methods relying on differentiability (like quasi-Newton methods, neural network methods) can be employed directly for solving NCP. In this paper, we look into the detailed properties of these four discrete-type families of NCP-functions. The properties investigated in this paper not only explain the difference but also provide background bricks for designing solution methods based on such discrete-type families of NCP-functions.

The paper is organized as follows. In Section 2, we review some background definitions including locally Lipschitz, semismoothness, the known results about $\phi^p_{_{\rm FB}}$ and $\psi^p_{_{\rm FB}}$ and its related properties. In Section 3-6, we shall discuss the properties about $\phi^p_{_{\rm NR}}$, $\phi^p_{_{\rm S-NR}}$, $\psi^p_{_{\rm S-NR}}$, $\phi^p_{_{\rm D-FB}}$, respectively. Especially, we discuss the semismoothness of $\phi^p_{_{\rm S-NR}}$ in Section 4 as well.

2 Preliminaries

In this section, we recall some background concepts and materials which will play an important role in the subsequent analysis. We begin with the so-called semismooth functions. Semismooth function, as introduced by Mifflin [24] for functionals and further extended by Qi and Sun [30] for vector-valued functions, is of particular interest due to the central role it plays in the superlinear convergence analysis of certain generalized Newton methods, see [30]. First, we say that $F : \mathbb{R}^n \to \mathbb{R}^m$ is *strictly continuous* (also called locally Lipschitz continuous) at $x \in \mathbb{R}^n$ [31, Chap. 9] if there exist scalars $\kappa > 0$ and $\delta > 0$ such that

$$\|F(y) - F(z)\| \leq \kappa \|y - z\| \quad \forall y, z \in \mathbb{R}^n \ \text{ with } \ \|y - x\| \leq \delta \ \text{ and } \ \|z - x\| \leq \delta.$$

The mapping F is locally Lipschitz continuous if F is locally Lipschitz continuous at every $x \in \mathbb{R}^n$. If δ can be taken to be ∞, then F is Lipschitz continuous with Lipschitz constant κ. We say F is directionally differentiable at $x \in \mathbb{R}^n$ if

$$F'(x; h) := \lim_{t \to 0^+} \frac{F(x + th) - F(x)}{t} \quad \text{exists} \quad \forall h \in \mathbb{R}^n;$$

and F is directionally differentiable if F is directionally differentiable at every $x \in \mathbb{R}^n$. If F is locally Lipschitz continuous, then F is almost everywhere differentiable by Rademachers Theorem, see [31, Section 9J]. In this case, the generalized Jacobian $\partial F(x)$ of F at x (in the Clarke sense) can be defined as the convex hull of B-subdifferential $\partial_B F(x)$, where

$$\partial_B F(x) := \left\{ \lim_{x^j \to x} \nabla F(x^j) \,\middle|\, F \text{ is differentiable at } x^j \in \mathbb{R}^n \right\}.$$

Assume F is locally Lipschitz continuous. We say F is *semismooth at* $x \in \mathbb{R}^n$ if F is directionally differentiable at $x \in \mathbb{R}^n$ and, for any $V \in \partial F(x + h)$ and $h \to 0$, we have

$$F(x + h) - F(x) - Vh = o(\|h\|). \tag{10}$$

Moreover, F is called *ρ-order semismooth at* $x \in \mathbb{R}^n$ $(0 < \rho < \infty)$ if F is semismooth at $x \in \mathbb{R}^n$ and, for any $V \in \partial F(x + h)$ and $h \to 0$, we have

$$F(x + h) - F(x) - Vh = O(\|h\|^{1+\rho}). \tag{11}$$

The mapping F is semismooth (respectively, ρ-order semismooth) if F is semismooth (respectively, ρ-order semismooth) at every $x \in \mathbb{R}^n$. We say F is *strongly semismooth* if it is 1-order semismooth. Convex functions and piecewise continuously differentiable functions are examples of semismooth functions. The composition of two (respectively, ρ-order) semismooth functions is also a (respectively, ρ-order) semismooth function. The property of semismoothness plays an important role in nonsmooth Newton methods

[29, 30] as well as in some smoothing methods.

An important concept related to semismooth function is the SC^1 function, which is introduced as below.

Definition 2.1. *A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be an SC^1 function if F is continuously differentiable and its gradient is semismooth.*

We can view SC^1 functions are functions lying between C^1 and C^2 functions. By defining SC^1 functions, many results regarding the minimization of C^2 functions can be extended to the minimization of SC^1 functions, see [28] and references therein. In addition to SC^1 function, we also introduce LC^1 function here.

Definition 2.2. *A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called an LC^1 function if F is continuously differentiable and its gradient is locally Lipschitz continuous.*

In light of the above definitions, given any $F : \mathbb{R}^n \to \mathbb{R}^m$, we have the following relations.

$$
\begin{array}{ccccccccccc}
 & & & & \multicolumn{5}{c}{\text{strongly semismooth}} & & \\
 & & & & \Uparrow & & & & \Downarrow & & \\
C^2 & \Rightarrow & SC^1 & \Rightarrow & LC^1 & \Rightarrow & C^1 & \Rightarrow & \text{semismooth} & \Rightarrow & \text{locally Lipschitz} \\
 & & & & & & & & \Uparrow & & \\
 & & & & & & & & \text{convex} & &
\end{array}
$$

To close this section, we present some well-known properties of ϕ^p_{FB} and ψ^p_{FB}, defined as in (4) and (5), respectively, that are important for designing a descent algorithm that is indeed derivative-free method.

Property 2.1. ([8, Propostion 3.1]) *Let ϕ^p_{FB} be defined as in (4). Then, the following hold.*

(a) *ϕ^p_{FB} is a NCP-function, i.e., it satisfies (1).*

(b) *ϕ^p_{FB} is sub-additive, i.e., $\phi^p_{\text{FB}}(w + w') \le \phi^p_{\text{FB}}(w) + \phi^p_{\text{FB}}(w')$ for all $w, w' \in \mathbb{R}^2$.*

(c) *ϕ^p_{FB} is positive homogeneous, i.e., $\phi^p_{\text{FB}}(\alpha w) = \alpha \phi^p_{\text{FB}}(w)$ for all $w \in \mathbb{R}^2$ and $\alpha \ge 0$.*

(d) *ϕ^p_{FB} is convex, i.e., $\phi^p_{\text{FB}}(\alpha w + (1-\alpha) w') \le \alpha \phi^p_{\text{FB}}(w) + (1-\alpha)\phi^p_{\text{FB}}(w')$ for all $w, w' \in \mathbb{R}^2$ and $\alpha \in [0, 1]$.*

(e) *ϕ^p_{FB} is Lipschitz continuous with $\kappa_1 = \sqrt{2} + 2^{(1/p - 1/2)}$ when $1 < p < 2$, and with $\kappa_2 = 1 + \sqrt{2}$ when $p \ge 2$. In other words, $|\phi^p_{\text{FB}}(w) - \phi^p_{\text{FB}}(w')| \le \kappa_1 \|w - w'\|$ when $1 < p < 2$ and $|\phi^p_{\text{FB}}(w) - \phi^p_{\text{FB}}(w')| \le \kappa_2 \|w - w'\|$ when $p \ge 2$ for all $w, w' \in \mathbb{R}^2$.*

Property 2.2. *Let $\phi^p_{\rm FB}$ be defined as in (4). Then, for any $\alpha > 0$, the following variants of $\phi^p_{\rm FB}$ are also NCP-functions.*

$$\begin{aligned}
\widetilde{\phi^p_{{\rm FB}-1}}(a,b) &= \phi^p_{\rm FB}(a,b) - \alpha(a)_+(b)_+, \\
\widetilde{\phi^p_{{\rm FB}-2}}(a,b) &= \phi^p_{\rm FB}(a,b) - \alpha((a)_+(b)_+)^2, \\
\widetilde{\phi^p_{{\rm FB}-3}}(a,b) &= \sqrt{[\phi^p_{\rm FB}(a,b)]^2 + \alpha\left((a)_+(b)_+\right)^2}, \\
\widetilde{\phi^p_{{\rm FB}-4}}(a,b) &= \sqrt{[\phi^p_{\rm FB}(a,b)]^2 + \alpha\left[(ab)_+\right]^2}.
\end{aligned}$$

Property 2.3. ([9, Lemma 2.2]) *Let $\phi^p_{\rm FB}$ be defined as in (4). Then, the generalized gradient $\partial\phi^p_{\rm FB}(a,b)$ of $\phi^p_{\rm FB}$ at a point (a,b) is equal to the set of all (v_a, v_b) such that*

$$(v_a, v_b) = \begin{cases} \left(\dfrac{{\rm sgn}(a)\cdot|a|^{p-1}}{\|(a,b)\|_p^{p-1}} - 1, \dfrac{{\rm sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}} - 1\right) & \text{if} \quad (a,b) \neq (0,0), \\ (\xi - 1, \zeta - 1) & \text{if} \quad (a,b) = (0,0), \end{cases}$$

where (ξ, ζ) is any vector satisfying $|\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1$.

Property 2.4. ([8, Propostion 3.2]) *Let $\phi^p_{\rm FB}$, $\psi^p_{\rm FB}$ be defined as in (4) and (5), respectively. Then, the following hold.*

(a) *$\psi^p_{\rm FB}$ is an NCP-function, i.e., it satisfies (1).*

(b) *$\psi^p_{\rm FB}(a,b) \geq 0$ for all $(a,b) \in \mathbb{R}^2$.*

(c) *$\psi^p_{\rm FB}$ is continuously differentiable everywhere.*

(d) *$\nabla_a\psi^p_{\rm FB}(a,b)\cdot\nabla_b\psi^p_{\rm FB}(a,b) \geq 0$ for all $(a,b) \in \mathbb{R}^2$. The equality holds if and only if $\phi^p_{\rm FB}(a,b) = 0$.*

(e) *$\nabla_a\psi^p_{\rm FB}(a,b) = 0 \Longleftrightarrow \nabla_b\psi^p_{\rm FB}(a,b) = 0 \Longleftrightarrow \phi^p_{\rm FB}(a,b) = 0$.*

3 The function $\phi^p_{\rm NR}$

In this section, we focus on the generalized NR function $\phi^p_{\rm NR}$ defined as in (6). Its continuous differentiability is studied in [7]. Here we further study the Lipschitz continuity and some property which is usually employed in derivative-free algorithm.

Proposition 3.1. ([7, Proposition 2.1]) *Let $\phi^p_{_{\rm NR}}$ be defined as in (6) with $p > 1$ being a positive odd integer. Then, $\phi^p_{_{\rm NR}}$ is an NCP-function.*

Proposition 3.2. ([7, Proposition 2.2]) *Let $\phi^p_{_{\rm NR}}$ be defined as in (6) with $p > 1$ being a positive odd integer, and let $p = 2k+1$ where $k \in \mathbb{N}$. Then, the following hold.*

(a) *An alternative expression of $\phi^p_{_{\rm NR}}$ is*

$$\phi^p_{_{\rm NR}}(a,b) = a^{2k+1} - \frac{1}{2}\left((a-b)^{2k+1} + (a-b)^{2k}|a-b|\right).$$

(b) *The function $\phi^p_{_{\rm NR}}$ is continuously differentiable with*

$$\nabla \phi^p_{_{\rm NR}}(a,b) = p \begin{bmatrix} a^{p-1} - (a-b)^{p-2}(a-b)_+ \\ (a-b)^{p-2}(a-b)_+ \end{bmatrix}.$$

(c) *The function $\phi^p_{_{\rm NR}}$ is twice continuously differentiable with*

$$\nabla^2 \phi^p_{_{\rm NR}}(a,b) = p(p-1) \begin{bmatrix} a^{p-2} - (a-b)^{p-3}(a-b)_+ & (a-b)^{p-3}(a-b)_+ \\ (a-b)^{p-3}(a-b)_+ & -(a-b)^{p-3}(a-b)_+ \end{bmatrix}.$$

Proposition 3.3. ([7, Proposition 2.4]) *Let $\phi^p_{_{\rm NR}}$ be defined as in (6) with $p > 1$ being a positive odd integer. Then, for any $\alpha > 0$, the following variants of $\phi^p_{_{\rm NR}}$ are also NCP-functions.*

$$\begin{aligned}
\widetilde{\phi^p_{_{\rm NR}-1}}(a,b) &= \phi^p_{_{\rm NR}}(a,b) + \alpha (a)_+(b)_+, \\
\widetilde{\phi^p_{_{\rm NR}-2}}(a,b) &= \phi^p_{_{\rm NR}}(a,b) + \alpha \left((a)_+(b)_+\right)^2, \\
\widetilde{\phi^p_{_{\rm NR}-3}}(a,b) &= [\phi^p_{_{\rm NR}}(a,b)]^2 + \alpha \left((ab)_+\right)^4, \\
\widetilde{\phi^p_{_{\rm NR}-4}}(a,b) &= [\phi^p_{_{\rm NR}}(a,b)]^2 + \alpha \left((ab)_+\right)^2.
\end{aligned}$$

Proposition 3.4. *Let $\phi^p_{_{\rm NR}}$ be defined as in (6) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) $\phi^p_{_{\rm NR}}(a,b) > 0 \iff a > 0,\ b > 0.$

(b) *$\phi^p_{_{\rm NR}}$ is positive homogeneous of degree p, i.e., $\phi^p_{_{\rm NR}}(\alpha w) = \alpha^p \phi^p_{_{\rm NR}}(w)$ for all $w \in \mathbb{R}^2$ and $\alpha \geq 0$.*

(c) *$\phi^p_{_{\rm NR}}$ is locally Lipschitz continuous, but not (globally) Lipschitz continuous.*

(d) ϕ^p_{NR} *is not α-Hölder continuous for any $\alpha \in (0,1]$, that is, the Hölder coefficient*

$$[\phi^p_{\text{NR}}]_{\alpha,\mathbb{R}^2} := \sup_{w \neq w'} \frac{|\phi^p_{\text{NR}}(w) - \phi^p_{\text{NR}}(w')|}{\|w - w'\|^\alpha}$$

is infinite.

(e) $$\nabla_a \phi^p_{\text{NR}}(a,b) \cdot \nabla_b \phi^p_{\text{NR}}(a,b) \begin{cases} > 0 & \text{on } \{(a,b) \mid a > b > 0 \text{ or } a > b > 2a\}, \\ = 0 & \text{on } \{(a,b) \mid a \leq b \text{ or } a > b = 2a \text{ or } a > b = 0\}, \\ < 0 & \text{otherwise.} \end{cases}$$

(f) $\nabla_a \phi^p_{\text{NR}}(a,b) \cdot \nabla_b \phi^p_{\text{NR}}(a,b) = 0$ *provided that* $\phi^p_{\text{NR}}(a,b) = 0$.

Proof. (a) This result has been mentioned in [7, Lemma 2.2].

(b) It is clear by definition of ϕ^p_{NR}.

(c) Since continuously differentiability implies locally Lipschitz continuity, it remains to show ϕ^p_{NR} is not Lipschitz continuous. Consider the restriction of ϕ^p_{NR} on the line $L := \{(a,b) \mid a = b > 0\}$. Note that for any $a > 0$, $\phi^p_{\text{NR}}(a,a) = a^p$, it suffices to show that $f(t) := t^p$ is not Lipschitz continuous. Indeed, for any $M > 0$, choosing $t = \max\{1, M\}$ and $t' = t + 1$ yields

$$\begin{aligned} \frac{|f(t) - f(t')|}{|t - t'|} &= (t+1)^p - t^p \\ &= (t+1)^{p-1} + (t+1)^{p-2}t + \cdots + t^{p-1} \\ &> p \cdot t^{p-1} \\ &> M. \end{aligned}$$

Hence, it follows that f is not Lipschitz continuous.

(d) As in the proof of part(c), we again restrict ϕ^p_{NR} on L and choose the same t. Hence, we also have

$$\frac{|f(t) - f(t')|}{|t - t'|^\alpha} > M$$

for any positive number M, that is, ϕ^p_{NR} is not α-Hölder continuous.

(e) According to Proposition 3.2, we know that

$$\begin{aligned} \nabla_a \phi^p_{\text{NR}}(a,b) \cdot \nabla_b \phi^p_{\text{NR}}(a,b) &= p^2 \cdot \left(a^{p-1} - (a-b)^{p-2}(a-b)_+\right)\left((a-b)^{p-2}(a-b)_+\right) \\ &= \begin{cases} p^2 \cdot \left(a^{p-1} - (a-b)^{p-1}\right)(a-b)^{p-1} & \text{if } \ a > b, \\ 0 & \text{if } \ a \leq b. \end{cases} \end{aligned}$$

When $a > b$, it is clear that $p^2 > 0$ and $(a-b)^{p-1} > 0$. Thus, we only consider the term $a^{p-1} - (a-b)^{p-1}$. Note that $p - 1$ is even, which implies

$$a^{p-1} = (a-b)^{p-1} \iff |a| = a - b \iff b = 0 \text{ or } b = 2a.$$

In addition to the case $a \le b$, there are two subcases $a > b = 0$ and $a > b = 2a$ such that $\nabla_a \phi^p_{\rm NR}(a,b) \cdot \nabla_b \phi^p_{\rm NR}(a,b) = 0$. On the other hand, we have

$$a^{p-1} > (a-b)^{p-1} \iff |a| > a - b \iff b > 0 \text{ or } b > 2a.$$

All the above says $\nabla_a \phi^p_{\rm NR}(a,b) \cdot \nabla_b \phi^p_{\rm NR}(a,b)$ is positive only when $a > b > 0$ or $a > b > 2a$. For the remainder case, it is not hard to verify $\nabla_a \phi^p_{\rm NR}(a,b) \cdot \nabla_b \phi^p_{\rm NR}(a,b) < 0$.

(f) It is clear from part(e). □

4 The function $\phi^p_{\rm S-NR}$

In this section, we focus on the function $\phi^p_{\rm S-NR}$ defined as in (7). As mentioned earlier, it is the symmetrization of $\phi^p_{\rm NR}$. As mentioned in [1], Chang *et al.* showed that it is not differentiable on the line $L = \{(a,b) \,|\, a = b\}$. However, it should be mildly modified since $\phi^p_{\rm S-NR}$ is differentiable at $(0,0)$. Here we further study the Lipschitz continuity, semismoothness, and some properties which are usually employed in derivative-free algorithm.

Proposition 4.1. ([1, Proposition 2.1]) *Let $\phi^p_{\rm S-NR}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\rm S-NR}$ is an NCP-function and is positive only on the first quadrant $\mathbb{R}^n_{++} := \{(a,b) \,|\, a > 0,\, b > 0\}$.*

Proposition 4.2. ([1, Proposition 2.2]) *Let $\phi^p_{\rm S-NR}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of $\phi^p_{\rm S-NR}$ is*

$$\phi^p_{\rm S-NR}(a,b) = \begin{cases} \phi^p_{\rm NR}(a,b) & \text{if} \quad a > b, \\ a^p = b^p & \text{if} \quad a = b, \\ \phi^p_{\rm NR}(b,a) & \text{if} \quad a < b. \end{cases}$$

(b) *The function $\phi^p_{\rm S-NR}$ is not differentiable. However, $\phi^p_{\rm S-NR}$ is continuously differentiable on the set $\Omega := \{(a,b) \,|\, a \neq b\}$ with*

$$\nabla \phi^p_{\rm S-NR}(a,b) = \begin{cases} p\,[\, a^{p-1} - (a-b)^{p-1},\; (a-b)^{p-1} \,]^T & \text{if} \quad a > b, \\ p\,[\, (b-a)^{p-1},\; b^{p-1} - (b-a)^{p-1} \,]^T & \text{if} \quad a < b. \end{cases}$$

In a more compact form,

$$\nabla \phi^p_{\rm S-NR}(a,b) = \begin{cases} p\,[\, \phi^{p-1}_{\rm NR}(a,b),\; (a-b)^{p-1} \,]^T & \text{if} \quad a > b, \\ p\,[\, (b-a)^{p-1},\; \phi^{p-1}_{\rm NR}(b,a) \,]^T & \text{if} \quad a < b. \end{cases}$$

(c) *The function $\phi^p_{\text{S-NR}}$ is twice continuously differentiable on the set $\Omega = \{(a,b) \,|\, a \neq b\}$ with*

$$\nabla^2 \phi^p_{\text{S-NR}}(a,b) = \begin{cases} p(p-1)\begin{bmatrix} a^{p-2} - (a-b)^{p-2} & (a-b)^{p-2} \\ (a-b)^{p-2} & -(a-b)^{p-2} \end{bmatrix} & \text{if} \quad a > b, \\ p(p-1)\begin{bmatrix} -(b-a)^{p-2} & (b-a)^{p-2} \\ (b-a)^{p-2} & b^{p-2} - (b-a)^{p-2} \end{bmatrix} & \text{if} \quad a < b. \end{cases}$$

In a more compact form,

$$\nabla^2 \phi^p_{\text{S-NR}}(a,b) = \begin{cases} p(p-1)\begin{bmatrix} \phi^{p-2}_{\text{NR}}(a,b) & (a-b)^{p-2} \\ (a-b)^{p-2} & -(a-b)^{p-2} \end{bmatrix} & \text{if} \quad a > b, \\ p(p-1)\begin{bmatrix} -(b-a)^{p-2} & (b-a)^{p-2} \\ (b-a)^{p-2} & \phi^{p-2}_{\text{NR}}(b,a) \end{bmatrix} & \text{if} \quad a < b. \end{cases}$$

Proposition 4.3. *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\text{S-NR}}$ is differentiable at $(0,0)$ with $\nabla \phi^p_{\text{S-NR}}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$*

Proof. First, we change the representation of $\phi^p_{\text{S-NR}}$ by polar coordinate, i.e.,

$$\begin{aligned} \phi^p_{\text{S-NR}}(a,b) &= \begin{cases} a^p - (a-b)^p & \text{if} \quad a \geq b, \\ b^p - (b-a)^p & \text{if} \quad a < b, \end{cases} \\ &= \begin{cases} r^p[\cos^p\theta - (\cos\theta - \sin\theta)^p] & \text{if} \quad \frac{-3\pi}{4} \leq \theta \leq \frac{\pi}{4}, \\ r^p[\sin^p\theta - (\sin\theta - \cos\theta)^p] & \text{if} \quad \frac{\pi}{4} < \theta < \frac{5\pi}{4}, \end{cases} \end{aligned}$$

We note that the parts $|\cos^p\theta - (\cos\theta - \sin\theta)^p|$ and $|\sin^p\theta - (\sin\theta - \cos\theta)^p|$ are bounded by some constant M_p which depends on p, hence we have

$$\frac{|\phi^p_{\text{S-NR}}(a,b) - \phi^p_{\text{S-NR}}(0,0)|}{\sqrt{a^2+b^2}} \leq \frac{M_p \cdot r^p}{r} = M_p \cdot r^{p-1} \to 0 \text{ as } r \to 0.$$

As $(a,b) \to (0,0)$, which implies $r \to 0$, we conclude that $\nabla \phi^p_{\text{S-NR}}(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. □

Note that $\phi^p_{\text{S-NR}}$ is indicated not differentiable on the line $L = \{(a,b) \,|\, a = b\}$ in [1, Proposition 2.2]. Here, we show that it is indeed differentiable at $(0,0)$ so that Proposition 4.3 can be viewed as an addendum to [1, Proposition 2.2].

Proposition 4.4. ([1, Proposition 2.3]) *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, for any $\alpha > 0$, the following variants of $\phi^p_{\text{S-NR}}$ are also NCP-functions.*

$$
\begin{aligned}
\widetilde{\phi}_1(a,b) &= \phi^p_{\text{S-NR}}(a,b) + \alpha(a)_+(b)_+, \\
\widetilde{\phi}_2(a,b) &= \phi^p_{\text{S-NR}}(a,b) + \alpha\left((a)_+(b)_+\right)^2, \\
\widetilde{\phi}_3(a,b) &= [\phi^p_{\text{S-NR}}(a,b)]^2 + \alpha\left((ab)_+\right)^4, \\
\widetilde{\phi}_4(a,b) &= [\phi^p_{\text{S-NR}}(a,b)]^2 + \alpha\left((ab)_+\right)^2.
\end{aligned}
$$

Proposition 4.5. *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) $\phi^p_{\text{S-NR}}(a,b) > 0 \iff a > 0,\ b > 0.$

(b) $\phi^p_{\text{S-NR}}$ *is positive homogeneous of degree p.*

(c) $\phi^p_{\text{S-NR}}$ *is not Lipschitz continuous.*

(d) $\phi^p_{\text{S-NR}}$ *is not α-Hölder continuous for any $\alpha \in (0,1]$.*

(e) $\nabla_a\phi^p_{\text{S-NR}}(a,b) \cdot \nabla_b\phi^p_{\text{S-NR}}(a,b) > 0$ *on* $\{(a,b)\,|\,a > b > 0\}\bigcup\{(a,b)\,|\,b > a > 0\}$.

(f) $\nabla_a\phi^p_{\text{S-NR}}(a,b) \cdot \nabla_b\phi^p_{\text{S-NR}}(a,b) = 0$ *provided that* $\phi^p_{\text{S-NR}}(a,b) = 0$ *and* $(a,b) \neq (0,0)$.

Proof. (a) It is clear from Proposition 4.1 or [1, Proposition 2.1]).

(b) It follows from the definition of $\phi^p_{\text{S-NR}}$.

(c)-(d) The proof is similar to Proposition 3.4(c)-(d).

(e) It is enough to verify the case for $a > b > 0$ because for $b > a > 0$, the inequality will hold automatically due to $\phi^p_{\text{S-NR}}$ having a symmetric surface. To see this, according to Proposition 4.2(b), we have

$$
\nabla_a\phi^p_{\text{S-NR}}(a,b) \cdot \nabla_b\phi^p_{\text{S-NR}}(a,b) = p^2 \cdot \left[a^{p-1} - (a-b)^{p-1}\right](a-b)^{p-1},
$$

which yields the desired result by Proposition 3.4(e).

(f) This result also follows from the proof of Proposition 3.4(e). $\square$

Next, we show the semismoothness of $\phi^p_{\text{S-NR}}$. In fact, each piecewise continuously differentiable function is semismooth. For the sake of completeness, we shall show this result according to the definition step by step, and hence we not only obtain the locally Lipschitz constant, generalized gradient, but also derive the "strongly" semismoothness. First, we need to check that it is strictly continuous (locally Lipschitz continuous). Note that $\phi^p_{\text{S-NR}}$ is not global Lipschitz continuous as shown in Proposition 4.5(c).

Lemma 4.1. *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\text{S-NR}}$ is strictly continuous (locally Lipschitz continuous).*

Proof. For any point $x = (a, b)$ with $a \neq b$, the continuous differentiability of $\phi^p_{\text{S-NR}}$ implies its locally Lipschitz continuity. It remains to show $\phi^p_{\text{S-NR}}$ is locally Lipschitz continuous on the line $L = \{(a, b) \mid a = b\}$.

To proceed the arguments, we present two inequalities that will be frequently used. Given any $x^0 = (a_0, a_0)$ and $\delta > 0$, let $N_\delta(x^0) := \{x \in \mathbb{R}^2 \mid \|x - x^0\| \leq \delta\}$. Then, for any $x = (x_1, x_2) \in N_\delta(x^0)$, we have two basic inequalities as follows:

$$|x_i| \leq \|x\| \leq \|x - x^0\| + \|x^0\| \leq \delta + \|x^0\| \quad \forall i = 1, 2. \tag{12}$$

$$|x_1 - x_2| \leq |x_1 - a_0| + |a_0 - x_2| \leq \|x - x^0\| + \|x^0 - x\| \leq 2\delta. \tag{13}$$

Now, for any $y, z \in N_\delta(x^0)$, we discuss four cases as below.

(i) For $y \in L$ and $z \in L$, we have

$$\begin{aligned}
\left|\phi^p_{\text{S-NR}}(y) - \phi^p_{\text{S-NR}}(z)\right| &= |y_1^p - z_1^p| \\
&= |y_1 - z_1| \cdot |y_1^{p-1} + y_1^{p-2} z_1 + \cdots + z_1^{p-1}| \\
&\leq \|y - z\| \cdot (|y_1|^{p-1} + |y_1|^{p-2} \cdot |z_1| + \cdots + |z_1|^{p-1}) \\
&\leq p(\delta + \|x^0\|)^{p-1} \|y - z\| \\
&= \kappa_1 \|y - z\|,
\end{aligned}$$

where $\kappa_1 := p(\delta + \|x^0\|)^{p-1}$ and the second inequality holds by (12).

(ii) For $y \notin L$ and $z \in L$ (or $y \in L$ and $z \notin L$), without loss of generality, we assume $y_1 > y_2$. Then, we have

$$\begin{aligned}
\left|\phi^p_{\text{S-NR}}(y) - \phi^p_{\text{S-NR}}(z)\right| &= |y_1^p - (y_1 - y_2)^p - z_1^p| \\
&\leq |y_1^p - z_1^p| + (y_1 - y_2)^p \\
&\leq \kappa_1 \|y - z\| + (y_1 - y_2)^{p-1} (|y_1 - z_1| + |z_1 - z_2| + |z_2 - y_2|) \\
&\leq \kappa_1 \|y - z\| + (2\delta)^{p-1} (\|y - z\| + \|z - y\|) \\
&= \kappa_2 \|y - z\|,
\end{aligned}$$

where $\kappa_2 := \kappa_1 + 2(2\delta)^{p-1}$ and the last inequality holds by (13).

(iii) For $y \notin L$, $z \notin L$ and y, z lie on the opposite side of L, i.e., $(y_1 - y_2)(z_1 - z_2) < 0$, without loss of generality, we assume $y_1 > y_2$ and $z_1 < z_2$. Since y, z lie on the opposite side of L, the line L and the segment $[y, z] := \{\lambda y + (1 - \lambda) z \,|\, \lambda \in [0, 1]\}$ must intersect at a point $w \in [y, z] \cap L$. Thus, we have

$$\begin{aligned}
\left|\phi^p_{\text{S-NR}}(y) - \phi^p_{\text{S-NR}}(z)\right| &\leq |\phi^p_{\text{S-NR}}(y) - \phi^p_{\text{S-NR}}(w)| + |\phi^p_{\text{S-NR}}(w) - \phi^p_{\text{S-NR}}(z)| \\
&\leq \kappa_2 \|y - w\| + \kappa_2 \|w - z\| \\
&\leq \kappa_2 \|y - z\| + \kappa_2 \|y - z\| \\
&= \kappa_3 \|y - z\|,
\end{aligned}$$

where $\kappa_3 := 2\kappa_2$ and the third inequality holds because $w \in [y, z]$.

(iv) For $y \notin L$, $z \notin L$ and y, z lie on the same side of L, i.e., $(y_1 - y_2)(z_1 - z_2) > 0$, without loss of generality, we assume $y_1 > y_2$ and $z_1 > z_2$. Then, we have

$$\begin{aligned}
\left|\phi^p_{\text{S-NR}}(y) - \phi^p_{\text{S-NR}}(z)\right| &= |(y_1^p - (y_1 - y_2)^p) - (z_1^p - (z_1 - z_2)^p)| \\
&\le |y_1^p - z_1^p| + |(y_1 - y_2)^p - (z_1 - z_2)^p| \\
&\le \kappa_1 \|y - z\| + 2p(2\delta)^{p-1}\|y - z\| \\
&= \kappa_4 \|y - z\|
\end{aligned}$$

where $\kappa_4 = \kappa_1 + 2p(2\delta)^{p-1}$ and the second part is estimated as follows:

$$\begin{aligned}
& |(y_1 - y_2)^p - (z_1 - z_2)^p| \\
= \;& |(y_1 - y_2) - (z_1 - z_2)| \cdot |(y_1 - y_2)^{p-1} + \cdots + (z_1 - z_2)^{p-1}| \\
\le \;& (|y_1 - z_1| + |y_2 - z_2|)(|y_1 - y_2|^{p-1} + \cdots + |z_1 - z_2|^{p-1}) \\
\le \;& (\|y - z\| + \|y - z\|)p(2\delta)^{p-1} \\
= \;& 2p(2\delta)^{p-1}\|y - z\|.
\end{aligned}$$

From all the above, by choosing $\kappa = \max\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$, we conclude that

$$\left|\phi^p_{\text{S-NR}}(y) - \phi^p_{\text{S-NR}}(z)\right| \le \kappa\|y - z\| \quad \text{for any} \quad y, z \in N_\delta(x^0).$$

This means that $\phi^p_{\text{S-NR}}$ is locally Lipschitz continuous at x^0. Then, the proof is complete. □

Proposition 4.6. *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, the generalized gradient of $\phi^p_{\text{S-NR}}$ is given by*

$$\partial\phi^p_{\text{S-NR}}(a, b) = \begin{cases} p\,[\,a^{p-1} - (a-b)^{p-1},\ (a-b)^{p-1}\,]^T & \text{if} \quad a > b, \\ \left\{p\,[\alpha a^{p-1}, (1-\alpha)a^{p-1}]^T \,|\, \alpha \in [0,1]\right\} & \text{if} \quad a = b, \\ p\,[\,(b-a)^{p-1},\ b^{p-1} - (b-a)^{p-1}\,]^T & \text{if} \quad a < b. \end{cases}$$

Proof. We have already seen the $\partial\phi^p_{\text{S-NR}}(a, b)$ when $a \neq b$ in [22]. For $a = b$, according to the definition of Clarke's generalized gradient, we claim that

$$\partial\phi^p_{\text{S-NR}}(a, a) = \text{conv}\left\{ \lim_{(a_i,b_i)\to(a,a)} \nabla\phi^p_{\text{S-NR}}(a_i, b_i) \,\Big|\, \phi^p_{\text{S-NR}} \text{ is differentiable at } (a_i, b_i) \in \mathbb{R}^2 \right\}.$$

To see this, we discuss three cases as below.

(i) If $a_i > b_i$, for any $i \ge n$ and sufficiently large n, then

$$\lim_{(a_i,b_i)\to(a,a)} \nabla\phi^p_{\text{S-NR}}(a_i, b_i) = \lim_{(a_i,b_i)\to(a,a)} p \begin{bmatrix} a_i^{p-1} - (a_i - b_i)^{p-1} \\ (a_i - b_i)^{p-1} \end{bmatrix} = p \begin{bmatrix} a^{p-1} \\ 0 \end{bmatrix}.$$

(ii) If $a_i < b_i$, for any $i \geq n$ and sufficiently large n, then

$$\lim_{(a_i,b_i)\to(a,a)} \nabla\phi^p_{\text{S-NR}}(a_i,b_i) = \lim_{(a_i,b_i)\to(a,a)} p\begin{bmatrix} (b_i-a_i)^{p-1} \\ b_i^{p-1}-(b_i-a_i)^{p-1} \end{bmatrix} = p\begin{bmatrix} 0 \\ a^{p-1} \end{bmatrix}.$$

(iii) For the remainder case, $\nabla\phi^p_{\text{S-NR}}(a_i,b_i)$ has no limit as $(a_i,b_i)\to(a,a)$.

From all the above, we conclude that

$$\partial\phi^p_{\text{S-NR}}(a,a) = \text{conv}\left\{ p\begin{bmatrix} a^{p-1} \\ 0 \end{bmatrix},\ p\begin{bmatrix} 0 \\ a^{p-1} \end{bmatrix} \right\} = \left\{ p\begin{bmatrix} \alpha a^{p-1} \\ (1-\alpha)a^{p-1} \end{bmatrix} \,\middle|\, \alpha\in[0,1] \right\}.$$

Thus, the desired result follows. □

Lemma 4.2. *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\text{S-NR}}$ is a directional differentiable function.*

Proof. For any point $x = (a,b)$ with $a \neq b$, the continuous differentiability of $\phi^p_{\text{S-NR}}$ implies the directional differentiability. Thus, it remains to show $\phi^p_{\text{S-NR}}$ is directional differentiable on the line $L = \{(a,b) \mid a = b\}$.

To proceed, given any $x = (a,a)$, $h = (h_1,h_2)$ and $t > 0$, we discuss three cases as below.

(i) If $h_1 = h_2$, then

$$\begin{aligned} & \lim_{t\to 0^+} \frac{\phi^p_{\text{S-NR}}(x+th)-\phi^p_{\text{S-NR}}(x)}{t} \\ = & \lim_{t\to 0^+} \frac{(a+th_1)^p - a^p}{t} \\ = & \lim_{t\to 0^+} \frac{a^p + pa^{p-1}th_1 + \sum_{k=2}^{p}\binom{p}{k}a^{p-k}t^k h_1^k - a^p}{t} \\ = & \lim_{t\to 0^+} \left(pa^{p-1}h_1 + \sum_{k=2}^{p}\binom{p}{k}a^{p-k}t^{k-1}h_1^k \right) \\ = & \ pa^{p-1}h_1. \end{aligned}$$

(ii) If $h_1 > h_2$, then

$$\begin{aligned} & \lim_{t\to 0^+} \frac{\phi^p_{\text{S-NR}}(x+th)-\phi^p_{\text{S-NR}}(x)}{t} \\ = & \lim_{t\to 0^+} \frac{(a+th_1)^p - (th_1-th_2)^p - a^p}{t} \\ = & \lim_{t\to 0^+} \frac{a^p + pa^{p-1}th_1 + \sum_{k=2}^{p}\binom{p}{k}a^{p-k}t^k h_1^k - t^p(h_1-h_2)^p - a^p}{t} \\ = & \lim_{t\to 0^+} \left(pa^{p-1}h_1 + \sum_{k=2}^{p}\binom{p}{k}a^{p-k}t^{k-1}h_1^k - t^{p-1}(h_1-h_2)^p \right) \\ = & \ pa^{p-1}h_1. \end{aligned}$$

(iii) If $h_1 < h_2$, then

$$\begin{aligned}
& \lim_{t\to 0^+} \frac{\phi^p_{\text{S-NR}}(x+th) - \phi^p_{\text{S-NR}}(x)}{t} \\
= & \lim_{t\to 0^+} \frac{(a+th_2)^p - (th_2 - th_1)^p - a^p}{t} \\
= & \lim_{t\to 0^+} \frac{a^p + pa^{p-1}th_2 + \sum_{k=2}^{p}\binom{p}{k}a^{p-k}t^k h_2^k - t^p(h_2-h_1)^p - a^p}{t} \\
= & \lim_{t\to 0^+} \left(pa^{p-1}h_2 + \sum_{k=2}^{p}\binom{p}{k}a^{p-k}t^{k-1}h_2^k - t^{p-1}(h_2-h_1)^p \right) \\
= & \ pa^{p-1}h_2.
\end{aligned}$$

To sum up, the definition of directional differentiability is checked. Then, the proof is complete. □

Proposition 4.7. *Let $\phi^p_{\text{S-NR}}$ be defined as in (7) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\text{S-NR}}$ is a semismooth function. Moreover, $\phi^p_{\text{S-NR}}$ is strongly semismooth.*

Proof. We shall directly show $\phi^p_{\text{S-NR}}$ is strongly semismooth. Note that $\phi^p_{\text{S-NR}}$ is twice continuously differentiable at any point $x = (a, b)$ with $a \neq b$, which implies the strongly semismoothness of $\phi^p_{\text{S-NR}}$ at x. It remains to show $\phi^p_{\text{S-NR}}$ is strongly semismooth on the line $L = \{(a, b) \mid a = b\}$.

For any $x = (a, a)$, $h = (h_1, h_2)$, $V \in \partial\phi^p_{\text{S-NR}}(x + h)$ and $h \to 0$, we have the following inequality while $\|h\| \leq 1$:

$$\|h\|^p \leq \|h\|^2 \quad \text{for any } p \geq 2.$$

To prove the strong semismoothness of $\phi^p_{\text{S-NR}}$, we will apply this inequality and verify (11) by discussing three cases as below.

(i) If $h_1 = h_2$, then for any $\alpha \in [0, 1]$

$$\begin{aligned}
& \left|\phi^p_{\text{S-NR}}(x+h) - \phi^p_{\text{S-NR}}(x) - Vh\right| \\
= & \left|(a+h_1)^p - a^p - p\left[\alpha a^{p-1}, (1-\alpha)a^{p-1}\right]\begin{bmatrix} h_1 \\ h_1 \end{bmatrix}\right| \\
= & \left|a^p + pa^{p-1}h_1 + \sum_{k=2}^{p}\binom{p}{k}a^{p-k}h_1^k - a^p - pa^{p-1}h_1\right| \\
\leq & \ M_1(|h_1|^2 + \cdots + |h_1|^p) \\
\leq & \ M_1(\|h\|^2 + \cdots + \|h\|^p) \\
\leq & \ (p-1)M_1\|h\|^2,
\end{aligned}$$

where $M_1 = \max\left\{\binom{p}{k}|a|^{p-k} \mid k = 2, 3, \cdots, p\right\}$ and the last inequality holds when $\|h\| \leq 1$.

(ii) If $h_1 > h_2$, then

$$\begin{aligned}
& \left|\phi^p_{\text{S-NR}}(x+h) - \phi^p_{\text{S-NR}}(x) - Vh\right| \\
= & \left|(a+h_1)^p - (h_1-h_2)^p - a^p - p\left[(a+h_1)^{p-1} - (h_1-h_2)^{p-1}, (h_1-h_2)^{p-1}\right]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right| \\
= & \left|(a+h_1)^p - (h_1-h_2)^p - a^p - p(a+h_1)^{p-1}h_1 + p(h_1-h_2)^p\right| \\
= & \left|(a+h_1)^p - a^p - p\left(a^{p-1} + \sum_{k=1}^{p-1}\binom{p-1}{k}a^{p-1-k}h_1^k\right)h_1 + (p-1)(h_1-h_2)^p\right| \\
\leq & \underbrace{\left|(a+h_1)^p - a^p - pa^{p-1}h_1\right|}_{\Xi_1} + p\underbrace{\left|\sum_{k=1}^{p-1}\binom{p-1}{k}a^{p-1-k}h_1^{k+1}\right|}_{\Xi_2} + (p-1)\underbrace{\left|(h_1-h_2)^p\right|}_{\Xi_3}.
\end{aligned}$$

As $h \to 0$, we have the following estimations for each Ξ_i.

- $\Xi_1 \leq (p-1)M_1\|h\|^2$ by case (i).
- $\Xi_2 \leq \sum_{k=1}^{p-1}\binom{p-1}{k}|a|^{p-1-k}|h_1|^{k+1} \leq M_2(|h_1|^2 + \cdots + |h_1|^p) \leq (p-1)M_2\|h\|^2$, where $M_2 = \max\left\{\binom{p-1}{k}|a|^{p-1-k} \mid k = 1, 2, \cdots, p-1\right\}$.
- $\Xi_3 \leq \sum_{k=0}^{p}\binom{p}{k}|h_1|^{p-k}|h_2|^k \leq M_3(\|h\|^p + \cdots + \|h\|^p) \leq (p+1)M_3\|h\|^2$, where $M_3 = \max\left\{\binom{p}{k} \mid k = 0, 1, \cdots, p\right\}$.

Hence, we conclude that

$$\left|\phi^p_{\text{S-NR}}(x+h) - \phi^p_{\text{S-NR}}(x) - Vh\right| \leq M\|h\|^2,$$

where $M = (p-1)M_1 + p(p-1)M_2 + (p-1)(p+1)M_3$.

(iii) If $h_1 < h_2$, the argument is similar to the case (ii).

All the above together with Lemmas 4.1-4.2 prove that $\phi^p_{\text{S-NR}}$ is strongly semismooth. □

5 The function $\psi^p_{\text{S-NR}}$

In this section, we focus on the function $\psi^p_{\text{S-NR}}$ defined as in (8). As mentioned earlier, it is the second symmetrization of ϕ^p_{NR}. Moreover, it is differentiable and possesses the symmetric surface as shown in [1]. Here we further study the Lipschitz continuity, and some property which is usually employed in derivative-free algorithm.

Proposition 5.1. ([1, Proposition 3.1]) *Let $\psi^p_{_{\rm S-NR}}$ be defined as in (8) with $p > 1$ being a positive odd integer. Then, $\psi^p_{_{\rm S-NR}}$ is an NCP-function and is positive on the set*

$$\Omega' = \{(a, b) \,|\, ab \neq 0\} \cup \{(a, b) \,|\, a < b = 0\} \cup \{(a, b) \,|\, 0 = a > b\}.$$

Proposition 5.2. ([1, Proposition 3.2]) *Let $\psi^p_{_{\rm S-NR}}$ be defined as in (8) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) *An alternative expression of $\phi^p_{_{\rm S-NR}}$ is*

$$\psi^p_{_{\rm S-NR}}(a,b) = \begin{cases} \phi^p_{_{\rm NR}}(a,b)b^p & \text{if} \quad a > b, \\ a^p b^p = a^{2p} & \text{if} \quad a = b, \\ \phi^p_{_{\rm NR}}(b,a)a^p & \text{if} \quad a < b. \end{cases}$$

(b) *The function $\psi^p_{_{\rm S-NR}}$ is continuously differentiable with*

$$\nabla\psi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p\,[\,a^{p-1}b^p - (a-b)^{p-1}b^p,\ a^p b^{p-1} - (a-b)^p b^{p-1} + (a-b)^{p-1}b^p\,]^T & \text{if} \quad a > b, \\ p\,[\,a^{p-1}b^p, a^p b^{p-1}\,]^T = pa^{2p-1}[1\ ,1\,]^T & \text{if} \quad a = b, \\ p\,[\,a^{p-1}b^p - (b-a)^p a^{p-1} + (b-a)^{p-1}a^p,\ a^p b^{p-1} - (b-a)^{p-1}a^p\,]^T & \text{if} \quad a < b. \end{cases}$$

In a more compact form,

$$\nabla\psi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p\,[\,\phi^{p-1}_{_{\rm NR}}(a,b)b^p,\ \phi^p_{_{\rm NR}}(a,b)b^{p-1} + (a-b)^{p-1}b^p\,]^T & \text{if} \quad a > b, \\ p\,[\,a^{2p-1}, a^{2p-1}\,]^T & \text{if} \quad a = b, \\ p\,[\,\phi^p_{_{\rm NR}}(b,a)a^{p-1} + (b-a)^{p-1}a^p,\ \phi^{p-1}_{_{\rm NR}}(b,a)a^p\,]^T & \text{if} \quad a < b. \end{cases}$$

(c) *The function $\psi^p_{_{\rm S-NR}}$ is twice continuously differentiable with*

$$\nabla^2\psi^p_{_{\rm S-NR}}(a,b) = \begin{cases} p\begin{bmatrix} (p-1)[a^{p-2} - (a-b)^{p-2}]b^p & \begin{array}{c}(p-1)(a-b)^{p-2}b^p \\ +p[a^{p-1} - (a-b)^{p-1}]b^{p-1}\end{array} \\ \begin{array}{c}(p-1)(a-b)^{p-2}b^p \\ +p[a^{p-1} - (a-b)^{p-1}]b^{p-1}\end{array} & \begin{array}{c}(p-1)[a^p - (a-b)^p]b^{p-2} \\ +2p(a-b)^{p-1}b^{p-1} \\ -(p-1)(a-b)^{p-2}b^p\end{array} \end{bmatrix} & \text{if} \quad a > b, \\ p\begin{bmatrix} (p-1)a^{p-2}b^p & pa^{p-1}b^{p-1} \\ pa^{p-1}b^{p-1} & (p-1)a^p b^{p-2} \end{bmatrix} & \text{if} \quad a = b, \\ p\begin{bmatrix} \begin{array}{c}(p-1)[b^p - (b-a)^p]a^{p-2} \\ +2p(b-a)^{p-1}a^{p-1} \\ -(p-1)(b-a)^{p-2}a^p\end{array} & \begin{array}{c}(p-1)(b-a)^{p-2}a^p \\ +p[b^{p-1} - (b-a)^{p-1}]a^{p-1}\end{array} \\ \begin{array}{c}(p-1)(b-a)^{p-2}a^p \\ +p[b^{p-1} - (b-a)^{p-1}]a^{p-1}\end{array} & (p-1)[b^{p-2} - (b-a)^{p-2}]a^p \end{bmatrix} & \text{if} \quad a < b. \end{cases}$$

Proposition 5.3. [1, Proposition 3.3] *Let $\psi^p_{\text{S-NR}}$ be defined as in (8) with $p > 1$ being a positive odd integer. Then, for any $\alpha > 0$, the following variants of $\psi^p_{\text{S-NR}}$ are also NCP-functions.*

$$\begin{aligned}
\widetilde{\psi}_1(a,b) &= \psi^p_{\text{S-NR}}(a,b) + \alpha(a)_+(b)_+. \\
\widetilde{\psi}_2(a,b) &= \psi^p_{\text{S-NR}}(a,b) + \alpha\left((a)_+(b)_+\right)^2. \\
\widetilde{\psi}_3(a,b) &= \psi^p_{\text{S-NR}}(a,b) + \alpha\left((ab)_+\right)^4. \\
\widetilde{\psi}_4(a,b) &= \psi^p_{\text{S-NR}}(a,b) + \alpha\left((ab)_+\right)^2.
\end{aligned}$$

Proposition 5.4. *Let $\psi^p_{\text{S-NR}}$ be defined as in (8) with $p > 1$ being a positive odd integer. Then, the following hold.*

(a) $\psi^p_{\text{S-NR}}(a,b) \geq 0$ *for all* $(a,b) \in \mathbb{R}^2$.

(b) $\psi^p_{\text{S-NR}}$ *is positive homogeneous of degree* $2p$.

(c) $\psi^p_{\text{S-NR}}$ *is locally Lipschitz continuous, but not Lipschitz continuous.*

(d) $\psi^p_{\text{S-NR}}$ *is not* α*-Hölder continuous for any* $\alpha \in (0,1]$.

(e) $\nabla_a \psi^p_{\text{S-NR}}(a,b) \cdot \nabla_b \psi^p_{\text{S-NR}}(a,b) > 0$ *on the first quadrant* $\mathbb{R}^2_{++}$.

(f) $\psi^p_{\text{S-NR}}(a,b) = 0 \iff \nabla \psi^p_{\text{S-NR}}(a,b) = 0$. *In particular, we have* $\nabla_a \psi^p_{\text{S-NR}}(a,b) \cdot \nabla_b \psi^p_{\text{S-NR}}(a,b) = 0$ *provided that* $\psi^p_{\text{S-NR}}(a,b) = 0$.

Proof. (a) This inequality follows from Proposition 5.1 or [1, Proposition 3.1].

(b) It is clear by the definition of $\psi^p_{\text{S-NR}}$.

(c)-(d) The proof is similar to Proposition 3.4(c)-(d).

(e) For convenience, we denote $\Lambda := \nabla_a \psi^p_{\text{S-NR}}(a,b) \cdot \nabla_b \psi^p_{\text{S-NR}}(a,b)$. Then, we proceed the proof by discussing three cases. For $a > b > 0$, we have

$$\Lambda = p^2 b^{2p-1} \cdot \left(a^{p-1} - (a-b)^{p-1}\right)\left(a^p - (a-b)^p + (a-b)^{p-1} b\right).$$

Note that $a > a - b > 0$ and $b > 0$, therefore we prove $\Lambda > 0$. Similarly, when $b > a > 0$, we also have $\Lambda > 0$. For the third case $a = b > 0$, it is clear that $\Lambda = p^2 a^{4p-2} > 0$.

(f) Note that $\psi^p_{\text{S-NR}}$ is an NCP-function, i.e.,

$$\psi^p_{\text{S-NR}}(a,b) = 0 \quad \iff \quad a \geq 0,\ b \geq 0,\ ab = 0.$$

From Proposition 5.2(b), we know$\nabla \psi^p_{\text{S-NR}}(a,b) = 0$ either when $a \geq b = 0$ or $b \geq a = 0$. Conversely, we suppose $\nabla \psi^p_{\text{S-NR}}(a,b) = 0$. For $a = b$,

$$\nabla \psi^p_{\text{S-NR}}(a,b) = p a^{2p-1} [1\ ,1]^T = 0 \implies a = b = 0,$$

this proves that $\psi^p_{\text{S-NR}}(a,b)=0$. For $a>b$, we know from Proposition 5.2(b) that

$$\begin{aligned} a^{p-1}b^p-(a-b)^{p-1}b^p &= 0, \\ a^p b^{p-1}-(a-b)^p b^{p-1}+(a-b)^{p-1}b^p &= 0. \end{aligned} \tag{14}$$

Note that it is clear to see $b=0$ satisfies the system (14). Assume $b\neq 0$, the system (14) becomes

$$a^{p-1}-(a-b)^{p-1}=0, \tag{15}$$

$$a^p-(a-b)^p+(a-b)^{p-1}b=0. \tag{16}$$

From (15), we obtain $(a-b)^{p-1}=a^{p-1}$. Then, substituting it into the equation (16) yields

$$a^p-(a-b)a^{p-1}+a^{p-1}b=0.$$

This implies $a^{p-1}b=0$. Thus, we obtain $a=0$. Again, by (15), we obtain $(-b)^{p-1}=0$. This leads to a contradiction since we assume $b\neq 0$. Therefore, for $a>b$, we obtain that b must be zero, and hence $\psi^p_{\text{S-NR}}(a,b)=0$. Similarly, when $a<b$, we also have $\psi^p_{\text{S-NR}}(a,b)=0$. In summary, we conclude that $\psi^p_{\text{S-NR}}(a,b)=0$ if and only if $\nabla\psi^p_{\text{S-NR}}(a,b)=0$. □

6 The function $\phi^p_{\text{D-FB}}$

In this section, we move to another discrete generalization of FB function $\phi^p_{\text{D-FB}}$ defined as in (9). The differentiability of $\phi^p_{\text{D-FB}}$ is studied in [22], here we investigate some other properties from view point of an algorithm.

Proposition 6.1. ([22, Proposition 3.1]) *Let $\phi^p_{\text{D-FB}}$ be defined as in (9) where $p>1$ is a positive odd integer. Then, we have*

(a) *$\phi^p_{\text{D-FB}}$ is a NCP-function;*

(b) *$\phi^p_{\text{D-FB}}$ is positive homogeneous of degree p.*

Note that the function ϕ^p_{FB} is a convex function, whereas $\phi^p_{\text{D-FB}}$ is not. This can be checked by the following:

$$2\sqrt{2}-8=\phi^3_{\text{D-FB}}(1,1)>\frac{1}{2}\phi^3_{\text{D-FB}}(0,0)+\frac{1}{2}\phi^3_{\text{D-FB}}(2,2)=0+\frac{1}{2}(2^{\frac{9}{2}}-2^6)$$

Next, we look at the gradient about this function.

Proposition 6.2. *Let $\phi^p_{_{\rm D-FB}}$ be defined as in (9) where $p>1$ being a positive odd integer. Then, the followings hold.*

(a) *$\phi^p_{_{\rm D-FB}}$ is continuously differentiable with*

$$\nabla \phi^p_{_{\rm D-FB}}(a,b) = p \left[\begin{array}{c} a(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \\ b(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \end{array} \right].$$

(b) *$\phi^p_{_{\rm D-FB}}$ is twice continuously differentiable with $\nabla^2 \phi^p_{_{\rm D-FB}}(0,0) = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$, and for $(a,b) \neq (0,0)$,*

$$\nabla^2 \phi^p_{_{\rm D-FB}}(a,b) = \left[\begin{array}{cc} \dfrac{\partial^2 \phi^p_{_{\rm D-FB}}}{\partial a^2} & \dfrac{\partial^2 \phi^p_{_{\rm D-FB}}}{\partial a \partial b} \\ \dfrac{\partial^2 \phi^p_{_{\rm D-FB}}}{\partial b \partial a} & \dfrac{\partial^2 \phi^p_{_{\rm D-FB}}}{\partial b^2} \end{array} \right], \qquad (17)$$

where

$$\begin{aligned} \frac{\partial^2 \phi^p_{_{\rm D-FB}}}{\partial a^2} &= p\left\{[(p-1)a^2+b^2](\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}\right\}, \\ \frac{\partial^2 \phi^p_{_{\rm D-FB}}}{\partial a \partial b} &= p[(p-2)ab(\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}] = \frac{\partial^2 \phi^p_{_{\rm D-FB}}}{\partial b \partial a}, \\ \frac{\partial^2 \phi^p_{_{\rm D-FB}}}{\partial b^2} &= p\left\{[a^2+(p-1)b^2](\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}\right\}. \end{aligned}$$

Proof. Ma *et al.* [22, Proposition 3.2] have shown that $\phi^p_{_{\rm D-FB}}$ is continuously differentiable when $p>1$ and twice continuously differentiable when $p>3$. It remains to show that $\phi^p_{_{\rm D-FB}}$ is twice continuously differentiable whenever $p=3$. In fact, for $(a,b)\neq(0,0)$, $\phi^3_{_{\rm D-FB}}$ is twice continuously differentiable with $\nabla^2\phi^3_{_{\rm D-FB}}$ satisfying (17). We only need to claim $\nabla^2 \phi^3_{_{\rm D-FB}}(0,0) = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$, and $\nabla^2\phi^3_{_{\rm D-FB}}$ is continuous at $(0,0)$.

First, we note that

$$\nabla \phi^3_{_{\rm D-FB}}(a,b) - \nabla \phi^3_{_{\rm D-FB}}(0,0) = 3 \left[\begin{array}{c} a(\sqrt{a^2+b^2}) - (a+b)^2 \\ b(\sqrt{a^2+b^2}) - (a+b)^2 \end{array} \right],$$

and

$$\begin{aligned} \left\| \left[\begin{array}{c} a(\sqrt{a^2+b^2}) - (a+b)^2 \\ b(\sqrt{a^2+b^2}) - (a+b)^2 \end{array} \right] \right\| &\leq \left\| (\sqrt{a^2+b^2}) \left[\begin{array}{c} a \\ b \end{array} \right] \right\| + \left\| (a+b)^2 \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \right\| \\ &= a^2+b^2+\sqrt{2}(a+b)^2 \\ &= (1+\sqrt{2}) \underbrace{(a^2+b^2)}_{\rm (i)} + 2 \underbrace{\sqrt{2}ab}_{\rm (ii)}. \end{aligned}$$

(i) $\dfrac{(a^2+b^2)}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2} \to 0$ as $(a,b) \to (0,0)$.

(ii) $\dfrac{|\sqrt{2}ab|}{\sqrt{a^2+b^2}} \le \dfrac{\sqrt{2}|ab|}{\sqrt{2|ab|}} = \sqrt{|ab|} \to 0$ as $(a,b) \to (0,0)$, where the inequality holds by arithmetic-geometric mean inequality.

Hence, we have

$$\lim_{(a,b)\to(0,0)} \frac{\|\nabla \phi^3_{\text{D-FB}}(a,b) - \nabla \phi^3_{\text{D-FB}}(0,0)\|}{\sqrt{a^2+b^2}} = 0,$$

i.e., $\phi^3_{\text{D-FB}}$ is twice differentiable and $\nabla^2 \phi^3_{\text{D-FB}}(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Secondly, we claim each second partial derivative is continuous at $(0,0)$. For

$$\frac{\partial^2 \phi^3_{\text{D-FB}}}{\partial a^2} = 3\left(\frac{2a^2+b^2}{\sqrt{a^2+b^2}} - 2(a+b)\right) = 3\left(\sqrt{a^2+b^2} + \frac{a^2}{\sqrt{a^2+b^2}} - 2(a+b)\right),$$

it is clear that $\sqrt{a^2+b^2} \to 0$, $a+b \to 0$ as $(a,b) \to (0,0)$. And the second term $\frac{a^2}{\sqrt{a^2+b^2}}$ also tends to zero because

$$\frac{a^2}{\sqrt{a^2+b^2}} = |a| \cdot \frac{|a|}{\sqrt{a^2+b^2}} \le |a| \to 0.$$

Hence, $\frac{\partial^2 \phi^3_{\text{D-FB}}}{\partial a^2}$ is continuous at $(0,0)$. For $\frac{\partial^2 \phi^3_{\text{D-FB}}}{\partial b^2}$, the proof is similar. For

$$\frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a \partial b} = 3\left(\frac{ab}{\sqrt{a^2+b^2}} - 2(a+b)\right) = \frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial b \partial a},$$

it is obvious that $\frac{\partial^2 \phi^p_{\text{D-FB}}}{\partial a \partial b}$ tends to zero, where the first term tends to zero by (ii).

Therefore, we obtain $\phi^3_{\text{D-FB}}$ is twice continuously differentiable at $(0,0)$, and we complete the proof. □

Proposition 6.3. ([22]) *Let $\phi^p_{\text{D-FB}}$ be defined as in (9). Then, for any $\alpha > 0$, the following variants of $\phi^p_{\text{D-FB}}$ are also NCP-functions.*

$$\begin{aligned}
\varphi_1(a,b) &= \phi^p_{\text{D-FB}}(a,b) - \alpha(a)_+(b)_+, \\
\varphi_2(a,b) &= \phi^p_{\text{D-FB}}(a,b) - \alpha\left((a)_+(b)_+\right)^2, \\
\varphi_3(a,b) &= [\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha\left((ab)_+\right)^4, \\
\varphi_4(a,b) &= [\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha\left((ab)_+\right)^2.
\end{aligned}$$

Proposition 6.4. *Let $\phi^p_{_{\rm D-FB}}$ be defined as in (9) where $p>1$ being a positive odd integer. Then, the following hold.*

(a) $\phi^p_{_{\rm D-FB}}(a,b) < 0 \iff a>0,\ b>0.$

(b) $\phi^p_{_{\rm D-FB}}$ *is locally Lipschitz continuous, but not Lipschitz continuous.*

(c) $\phi^p_{_{\rm D-FB}}$ *is not α-Hölder continuous for any $\alpha \in (0,1]$.*

(d) $\nabla_a \phi^p_{_{\rm D-FB}}(a,b) \cdot \nabla_b \phi^p_{_{\rm D-FB}}(a,b) > 0$ *on the first quadrant $\mathbb{R}^2_{++}$.*

(e) $\nabla_a \phi^p_{_{\rm D-FB}}(a,b) \cdot \nabla_b \phi^p_{_{\rm D-FB}}(a,b) = 0$ *provided that $\phi^p_{_{\rm D-FB}}(a,b)=0$.*

Proof. (a) It follows from [22, Lemma 3.1] immediately.

(b)-(c) The arguments are similar to Proposition 3.4(c)-(d).

(d) According to Proposition 6.2, we have

$$\begin{aligned}
& \nabla_a \phi^p_{_{\rm D-FB}}(a,b) \cdot \nabla_b \phi^p_{_{\rm D-FB}}(a,b) \\
= \ & p\left[a(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1}\right] \cdot p\left[b(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1}\right] \\
= \ & p^2\left[ab(a^2+b^2)^{p-2} + (a+b)^{2p-2} - (a+b)^{p-1}(\sqrt{a^2+b^2})^{p-2} \cdot (a+b)\right] \\
= \ & p^2\left[ab(a^2+b^2)^{p-2} + (a+b)^{2p-2} - (a+b)^{p}(\sqrt{a^2+b^2})^{p-2}\right] \\
= \ & p^2\left[ab(a^2+b^2)^{p-2} + (a+b)^{p}\left((a+b)^{p-2} - (\sqrt{a^2+b^2})^{p-2}\right)\right].
\end{aligned}$$

Since $a>0$, $b>0$ and $p-2$ is also an odd number, the term $(a+b)^{p-2} - (\sqrt{a^2+b^2})^{p-2}$ is always positive by part(a). This clearly implies the desired result.

(e) From Proposition 6.1, we know $\phi^p_{_{\rm D-FB}}$ is an NCP-function, which implies

$$\phi^p_{_{\rm D-FB}}(a,b) = 0 \iff a \geq 0,\ b \geq 0,\ ab = 0.$$

When $a \geq 0$ and $b = 0$, we have $\nabla_a \phi^p_{_{\rm D-FB}}(a,0) = a(\sqrt{a^2})^{p-2} - a^{p-1} = a^{p-1} - a^{p-1} = 0$. Similarly, when $b \geq 0$ and $a = 0$, we have $\nabla_b \phi^p_{_{\rm D-FB}}(0,b) = 0$. In summary, we conclude $\nabla_a \phi^p_{_{\rm D-FB}}(a,b) \cdot \nabla_b \phi^p_{_{\rm D-FB}}(a,b) = 0$ provided that $\phi^p_{_{\rm D-FB}} = 0$. □

7 Conclusion

In this paper, we further study some new NCP-functions which are discrete types of extensions of the well known Fisher-Burmeister and Natural-Residual functions. We explore more properties about these functions. It is observed that for such discrete NCP-functions, they possess continuous differentiability surprisingly. However, at the same instant, they lose the Lipschitz continuity unfortunately. In general, the Newton method may not be applicable even though we have the differentiability for some new complementarity functions because the Jacobian at a degenerate solution may be singular (see [19, 20]). Nonetheless, there are some other applicable algorithms like quasi-Newton methods and neural network methods, which heavily rely on differentiability. Moreover, we can reformulate NCP and SOCCP as nonsmooth equations or unconstrained minimization, for which merit function approach, nonsmooth function approach, smoothing function approach, and regularization approach can be studied. All the proposed new complementarity functions in this paper can be employed in these approaches.

To close the conclusion, we point out there is another way to achieve $\phi^p_{_{\rm D-FB}}$ and $\phi^p_{_{\rm NR}}$ which was proposed in [16]. More specifically, it is a construction based on monotone transformations to create new NCP-functions from the existing ones. The construction is stated as below.

Remark 7.1. ([16, Lemma 15]) *Assume that ϕ is continuous and $\phi(a,b) = f_1(a,b) - f_2(a,b)$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a strictly monotone increasing and continuous function. Then ϕ is an NCP function if and only if $\psi_\theta(a,b) = \theta(f_1(a,b)) - \theta(f_2(a,b))$ is an NCP-function.*

In light of this, we let the function $\theta = \theta_p$ be $\theta_p(t) = \text{sign}(t)|t|^p$, where "sign$(t)$" is the sign function and $p \geq 1$. For Fischer-Burmeister function, we choose $f_1(a,b) = \sqrt{a^2+b^2}$, $f_2(a,b) = a+b$, and for Natural-Residual function, we choose $f_1(a,b) = a$, $f_2(a,b) = (a-b)_+$, then it can be verified that both $\phi^p_{_{\rm D-FB}}$ and $\phi^p_{_{\rm NR}}$ (only with odd integer p) can be obtained from the function ψ_{θ_p}. In other words, the function ψ_{θ_p} includes both them as special cases, from which we may view it as a "continuous generalization". However, we prefer to treating them as "discrete generalization". We elaborate more about our original idea as below. First, for the function $\phi^p_{_{\rm NR}}(a,b) = a^p - (a-b)^p_+$, as remarked in [7], the parameter p must be an odd integer to ensure that the generalization is also an NCP-function. This means that the main idea to create our new families of functions relies on "discrete generalization", not on the "continuous generalization". That is why we call it "discrete generalization". On the other hand, if we consider the FB-function $\phi_{_{\rm FB}}(a,b) = \sqrt{a^2+b^2} - (a+b)$. When plugging $p=2$ into θ_p, we obtain a corresponding NCP-function

$$\psi_{\theta_2}(a,b) = a^2 + b^2 - \text{sign}(a+b)(a+b)^2,$$

which doesn't coincide with the form

$$\phi^p_{_{\rm D-FB}}(a,b) = \left(\sqrt{a^2+b^2}\right)^2 - (a+b)^2.$$

Thus, the functions $\phi^p_{_{\rm D-FB}}$ and $\phi^p_{_{\rm NR}}$ only with positive odd integer p can be retrieved from the way proposed in [16]. Again, it requires p to be a positive odd integer to guarantee that both $\phi^p_{_{\rm D-FB}}$ and $\phi^p_{_{\rm NR}}$ are NCP-functions. In view of all the above, we still call them discrete-type families of NCP-functions. Another thing to mention is that both $\phi^p_{_{\rm D-FB}}$ and $\phi^p_{_{\rm NR}}$ are not only NCP-functions, but also complementarity functions for SOCCP (see [22] for more details).

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