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THE HÖLDER CONTINUTIY OF VECTOR-VALUED FUNCTION ASSOCIATED WITH SECOND-ORDER CONE

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Abstract: Let $\mathcal{K}^n$ be the Lorentz/second-order cone in $\mathbb{R}^n$. For any function f from $\mathbb{R}$ to $\mathbb{R}$, one can define a corresponding vector-valued function $f^{\rm soc}(x)$ on $\mathbb{R}^n$ by applying f to the spectral values of the spectral decomposition of $x \in \mathbb{R}^n$ with respect to $\mathcal{K}^n$. It was shown by J.-S. Chen, X. Chen and P. Tseng in [5] that this vector-valued function inherits from f the properties of continuity, Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, as well as semismoothness. In this note, we further show that the Hölder continuity of this vector-valued function is also inherited from f. Such property will be useful in designing solution methods for second-order cone programming and second-order cone complementarity problem.

Key words: *Hölder continuity, second-order cone*

Mathematics Subject Classification: *26B05, 26B35, 90C33, 65K05*

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1 Introduction

The second-order cone (SOC) in $\mathbb{R}^n$, also called the Lorentz cone, is defined to be

$$\mathcal{K}^n := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. If $n = 1$, $\mathcal{K}^1$ is the set of nonnegative reals $\mathbb{R}_+$. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we can decompose x as

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{1.1}$$

where $\lambda_1(x)$, $\lambda_2(x)$ and $u_x^{(1)}$, $u_x^{(2)}$ are the *spectral values* and the associated *spectral vectors* of x, with respect to $\mathcal{K}^n$, given by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|, \tag{1.2}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{x_2}{\|x_2\|}\right) & \text{if } x_2 \neq 0, \\ \frac{1}{2}\left(1, (-1)^i w\right) & \text{if } x_2 = 0, \end{cases} \tag{1.3}$$

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for $i = 1, 2$, with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition (1.1) is unique. With this spectral decomposition, for any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}^n$ was considered (see [5, 9]):

$$f^{\text{soc}}(x) = f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} \qquad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{1.4}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition (1.4) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. The above definition (1.4) is analogous to one associated with the semidefinite cone $\mathcal{S}^n_+$, see [13].

In the paper [5], there had already studied the continuity and differentiability properties of the vector-valued function f^{soc}. In particular, it was shown that the properties of continuity, Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, and (ρ-order) semismoothness are each inherited by f^{soc} from f. These results parallel those obtained recently in [7] for matrix-valued functions and are useful in the design and analysis of smoothing and nonsmooth methods for solving second-order cone programs (SOCP) and second-order cone complementarity problems (SOCCP), see [3, 6, 10, 11] and references therein. In this note, we further show that the Hölder continuity of this vector-valued function f^{soc} is also inherited from f. We suspect that some useful inequalities can be derived by applying this property which will help towards more subtle analysis in SOCP and SOCCP. For example, $f(t) = \sqrt{t}$ is not Lipschitz continuous on $[0, \infty)$ and hence its corresponding SOC-function $f^{\text{soc}}(x) = x^{1/2}$ is not Lipschitz continuous. However, $f(t) = \sqrt{t}$ is Hölder continuous with exponent $\alpha \leq \frac{1}{2}$ which says $f^{\text{soc}}(x) = x^{1/2}$ is Hölder continuous by the main result established in this note. Another example is that $f(t) = t^{1/p}$ with $p > 1$ is Hölder continuous on $[0, \infty)$ with exponent $\alpha = \frac{1}{p}$, hence its corresponding SOC-function $f^{\text{soc}}(x) = x^{1/p}$ is Hölder continuous. These are new discoveries, which we believe they will be useful in some other contexts, for instance, exploring more nice properties for Fischer-Burmeister function [4, 9] and generalized Fischer-Burmeister function [12], defined in the second-order cone case.

2 Preliminary

In this section, we review some background materials that will be used for proving our main result. We first present some facts by looking at definition of spectral decomposition of x given as in (1.1)-(1.3). For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, it can be easily observed that

$$|\lambda_1(x) - \lambda_2(x)| = 2\|x_2\|, \quad \|u_x^{(i)}\| = \|u_y^{(i)}\| = \frac{1}{\sqrt{2}}, \quad u_x^{(1)} - u_y^{(1)} = -(u_x^{(2)} - u_y^{(2)}). \tag{2.1}$$

Moreover, we have the following lemmas which describe the relations among $|\lambda_i(x) - \lambda_i(y)|$, $\|u_x^{(i)} - u_y^{(i)}\|$ and $\|x - y\|$.

Lemma 2.1 ([2, Lemma 3.1]). *Let $\lambda_1(x) \leq \lambda_2(x)$ be the spectral values of $x \in \mathbb{R}^n$ and $\lambda_1(y) \leq \lambda_2(y)$ be the spectral values of $y \in \mathbb{R}^n$. Then we have*

$$|\lambda_1(x) - \lambda_1(y)|^2 + |\lambda_2(x) - \lambda_2(y)|^2 \leq 2\|x - y\|^2, \tag{2.2}$$

and hence, $|\lambda_i(x) - \lambda_i(y)| \leq \sqrt{2}\|x - y\|$, $\forall i = 1, 2$.

Lemma 2.2 ([2, Lemma 3.2]). *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$.*

(a) *If $x_2 \neq 0, y_2 \neq 0$, then we have*

$$\|u_x^{(i)} - u_y^{(i)}\| \leq \frac{1}{\|x_2\|}\|x_2 - y_2\| \quad \forall i = 1, 2, \tag{2.3}$$

where $u_x^{(i)}$, $u_y^{(i)}$ are the unique spectral vectors of x and y, respectively.

(b) *If either $x_2 = 0$ or $y_2 = 0$, then we can choose $u_x^{(i)}$, $u_y^{(i)}$ such that the left hand side of inequality (2.3) is zero.*

We note that inequality (2.3) was stated as $\|u_x^{(i)} - u_y^{(i)}\| \leq \frac{1}{\|x_2\|}\|x - y\|$ in [2, Lemma 3.2], however, from the proof therein inequality (2.3) at present form is also true (actually is more sharp). Next, we introduce the definition of Hölder continuity of a mapping.

Definition 2.3. Let f be a real-valued function defined on $D \subseteq \mathbb{R}^n$ and $0 < \alpha \leq 1$. The function f is said to be Hölder continuous with exponent α in D if the quantity

$$[f]_{\alpha,D} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha} \tag{2.4}$$

is finite. When $D = \mathbb{R}^n$, we abbreviate $[f]_{\alpha,D}$ as $[f]_\alpha$.

The above definition for real-valued function can be generalized to a mapping $f : X \to Y$ where (X, d_X) and (Y, d_Y) are metric spaces. In other words, a mapping $f : X \to Y$ is called Hölder continuous with exponent α if there exists a constant $M > 0$ such that

$$d_Y(f(x), f(y)) \leq M \cdot d_X(x, y)^\alpha \quad \forall x, y \in X.$$

In view of this, we say the vector-valued SOC-function $f^{\mathrm{soc}} : \mathbb{R}^n \to \mathbb{R}^n$ is Hölder continuous with exponent α if the quantity

$$[f^{\mathrm{soc}}]_\alpha = \sup_{x \neq y} \frac{\|f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y)\|}{\|x - y\|^\alpha} \tag{2.5}$$

is finite.

3 Main Result

In this section, we present the main result of this note, which says the Hölder continuity of the vector-valued function f^{soc} is inherited from f.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and f^{soc} be its corresponding SOC-function defined as in (1.4). Then, f is Hölder continuous with exponent $\alpha \in (0, 1]$ if and only if f^{soc} is Hölder continuous with exponent $\alpha \in (0, 1]$.*

Proof. "$\Longrightarrow$" Suppose f is Hölder continuous with exponent $\alpha \in (0,1]$. To prove $f^{\rm soc}$ is Hölder continuous with exponent $\alpha \in (0,1]$, we discuss two cases.

Case(1): If $x_2 \neq 0$ and $y_2 \neq 0$, without loss of generality, we may assume that $\|x_2\| \geq \|y_2\|$. Then, we have

$$\begin{aligned}
& \left\| f^{\rm soc}(x) - f^{\rm soc}(y) \right\| \\
= & \left\| f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} - f(\lambda_1(y))u_y^{(1)} - f(\lambda_2(y))u_y^{(2)} \right\| \\
= & \Big\| f(\lambda_1(x)) \left[u_x^{(1)} - u_y^{(1)} \right] + f(\lambda_2(x)) \left[u_x^{(2)} - u_y^{(2)} \right] \\
& + \left[f(\lambda_1(x)) - f(\lambda_1(y)) \right] u_y^{(1)} + \left[f(\lambda_2(x)) - f(\lambda_2(y)) \right] u_y^{(2)} \Big\| \qquad (3.1) \\
\leq & \left\| f(\lambda_1(x)) \left[u_x^{(1)} - u_y^{(1)} \right] + f(\lambda_2(x)) \left[u_x^{(2)} - u_y^{(2)} \right] \right\| \\
& + |f(\lambda_1(x)) - f(\lambda_1(y))| \cdot \|u_y^{(1)}\| + |f(\lambda_2(x)) - f(\lambda_2(y))| \cdot \|u_y^{(2)}\|.
\end{aligned}$$

For convenience, we denote

$$\begin{aligned}
I_1 &:= \left\| f(\lambda_1(x)) \left[u_x^{(1)} - u_y^{(1)} \right] + f(\lambda_2(x)) \left[u_x^{(2)} - u_y^{(2)} \right] \right\| \\
I_2 &:= |f(\lambda_1(x)) - f(\lambda_1(y))| \cdot \|u_y^{(1)}\| \\
I_3 &:= |f(\lambda_2(x)) - f(\lambda_2(y))| \cdot \|u_y^{(2)}\|
\end{aligned}$$

and do estimations for them separatively. First, we analyze I_1 as below.

$$\begin{aligned}
I_1 &= \left\| f(\lambda_1(x)) \left[(u_x^{(1)} - u_y^{(1)}) \right] + f(\lambda_2(x)) \left[(u_x^{(2)} - u_y^{(2)}) \right] \right\| \\
&= \left\| f(\lambda_1(x)) \left[(u_x^{(1)} - u_y^{(1)}) \right] - f(\lambda_2(x)) \left[(u_x^{(1)} - u_y^{(1)}) \right] \right\| \\
&= \left\| \left[f(\lambda_1(x)) - f(\lambda_2(x)) \right] \cdot \left[(u_x^{(1)} - u_y^{(1)}) \right] \right\| \\
&\leq |f(\lambda_1(x)) - f(\lambda_2(x))| \cdot \left\| (u_x^{(1)} - u_y^{(1)}) \right\| \\
&\leq [f]_\alpha \cdot |\lambda_1(x) - \lambda_2(x)|^\alpha \cdot \frac{\|x_2 - y_2\|}{\|x_2\|} \\
&= [f]_\alpha \cdot (2\|x_2\|)^\alpha \cdot \frac{\|x_2 - y_2\|}{\|x_2\|} \\
&= [f]_\alpha \cdot 2^\alpha \left(\frac{\|x_2 - y_2\|}{\|x_2\|} \right)^{1-\alpha} \cdot \|x_2 - y_2\|^\alpha \\
&= [f]_\alpha \cdot 2^\alpha \cdot \left\| \frac{x_2}{\|x_2\|} - \frac{y_2}{\|x_2\|} \right\|^{1-\alpha} \cdot \|x_2 - y_2\|^\alpha \\
&\leq [f]_\alpha \cdot 2^\alpha \cdot 2^{1-\alpha} \cdot \|x_2 - y_2\|^\alpha \\
&\leq [f]_\alpha \cdot 2 \cdot \|x - y\|^\alpha
\end{aligned}$$

where the second and fourth equalities are from the fact (2.1), the second inequality is due to (2.3) in Lemma 2.2(a) and definition of Hölder continuity, the fourth equality is by (2.1) again, and the third inequality is true by applying triangle inequality. Estimation of I_2

follows from

$$\begin{aligned} I_2 &= |f(\lambda_1(x)) - f(\lambda_1(y))| \cdot \|u_y^{(1)}\| \\ &\le [f]_\alpha \cdot |\lambda_1(x) - \lambda_1(y)|^\alpha \cdot \frac{1}{\sqrt{2}} \\ &\le [f]_\alpha \cdot \left(\sqrt{2}\|x-y\|\right)^\alpha \frac{1}{\sqrt{2}} \\ &= [f]_\alpha \cdot 2^{\frac{\alpha-1}{2}} \cdot \|x-y\|^\alpha \end{aligned}$$

where the first inequality is due to the fact (2.1) and definition of Hölder continuity, and the second inequality holds because of Lemma 2.1. Similarly, we also have

$$I_3 \le [f]_\alpha \cdot 2^{\frac{\alpha-1}{2}} \cdot \|x-y\|^\alpha.$$

Now, we plug all the above estimations for I_1, I_2 and I_3 back into (3.1), which yield

$$\begin{aligned} & \|f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y)\| \\ \le\ & [f]_\alpha \cdot 2 \cdot \|x-y\|^\alpha + [f]_\alpha \cdot 2^{\frac{\alpha-1}{2}} \cdot \|x-y\|^\alpha + [f]_\alpha \cdot 2^{\frac{\alpha-1}{2}} \cdot \|x-y\|^\alpha \\ =\ & [f]_\alpha \cdot \left[2 + 2^{\frac{\alpha+1}{2}}\right] \cdot \|x-y\|^\alpha. \end{aligned}$$

Thus, f^{soc} is Hölder continuous with exponent α in this case.

Case(2): If either $x_2 = 0$ or $y_2 = 0$, we can take $u_x^{(i)} = u_y^{(i)}$. Therefore, we obtain

$$\begin{aligned} & \left\|f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y)\right\| \\ =\ & \left\|f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} - f(\lambda_1(y))u_y^{(1)} - f(\lambda_2(y))u_y^{(2)}\right\| \\ =\ & \left\|\left[f(\lambda_1(x)) - f(\lambda_1(y))\right] u_x^{(1)} + \left[f(\lambda_2(x)) - f(\lambda_2(y))\right] u_x^{(2)}\right\| \\ \le\ & |f(\lambda_1(x)) - f(\lambda_1(y))| \cdot \|u_x^{(1)}\| + |f(\lambda_2(x)) - f(\lambda_2(y))| \cdot \|u_x^{(2)}\| \\ \le\ & [f]_\alpha \cdot |\lambda_1(x) - \lambda_1(y)|^\alpha \cdot \frac{1}{\sqrt{2}} + [f]_\alpha \cdot |\lambda_2(x) - \lambda_2(y)|^\alpha \cdot \frac{1}{\sqrt{2}} \\ \le\ & [f]_\alpha \cdot \left(\sqrt{2}\|x-y\|\right)^\alpha \cdot \frac{1}{\sqrt{2}} + [f]_\alpha \cdot \left(\sqrt{2}\|x-y\|\right)^\alpha \cdot \frac{1}{\sqrt{2}} \\ =\ & [f]_\alpha \cdot 2^{\frac{\alpha+1}{2}} \cdot \|x-y\|^\alpha \end{aligned}$$

where the third inequality is due to the fact (2.1) and definition of Hölder continuity, and the second inequality holds because of Lemma 2.1. Thus, we also prove that f^{soc} is Hölder continuous with exponent α in this case.

"$\Longleftarrow$" Suppose f^{soc} is Hölder continuous with exponent α. Then, for any $\xi, \zeta \in \mathbb{R}$, we have

$$\begin{aligned} |f(\xi) - f(\zeta)| &= \left\|f^{\mathrm{soc}}(\xi e) - f^{\mathrm{soc}}(\zeta e)\right\| \\ &\le [f^{\mathrm{soc}}]_\alpha \cdot \|\xi e - \zeta e\|^\alpha \\ &= [f^{\mathrm{soc}}]_\alpha \cdot |\xi - \zeta|^\alpha \end{aligned}$$

where $e = (1, 0, \cdots, 0) \in \mathbb{R}^n$. Thus, the Hölder continuity of f follows. □

We want to point out that one direction in Theorem 3.1 for matrix-valued function associated with $\mathcal{S}^n_+$ was recently proved in [14], and Lipschitz continuity of such matrix-valued function, corresponding to $\alpha = 1$, was also shown in [1, Chapter X]. It is well known that both $\mathcal{K}^n$ and $\mathcal{S}^n_+$ belong to symmetric cones [8]. Hence, one may naturally ask whether such property holds for functions associated with general symmetric cones or not. We are not clear about this yet. The main difficulty comes from that the spectral values and spectral vectors associated with symmetric cones have no explicit formulas though similar spectral decomposition exists. Nonetheless, it is definitely an interesting future direction to try to answer this question.

References

[1] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[2] J.-S. Chen, Alternative proofs for some results of vector-valued functions associated withsecond-order cone, *Journal of Nonlinear and Convex Analysis* 6 (2005) 297–325.

[3] J.-S. Chen, The convex and monotone functions associated withsecond-order cone, *Optimization* 55 (2006) 363–385.

[4] J.-S. Chen and P. Tseng, An unconstrained smooth minimization reformulation of second-order cone complementarity problem, *Mathematical Programming* 104 (2005) 293–327.

[5] J.-S. Chen, X. Chen and P. Tseng, Analysis of nonsmooth vector-valued functions associated with second-order cones, *Mathematical Programming* 101 (2004) 95–117.

[6] J.-S. Chen, X. Chen, S.-H. Pan and J. Zhang, Some characterizations for SOC-monotone and SOC-convex functions, *Journal of Global Optimization* 45 (2009) 259–279.

[7] X. Chen, H. Qi and P. Tseng, Analysis of nonsmooth symmetric-matrix-valued functions with applications to semidefinite complementarity problems, *SIAM Journal on Optimization* 13 (2003) 960–985.

[8] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford Mathematical Monographs, Oxford University Press, New York, 1994.

[9] M. Fukushima, Z.-Q. Luo and P. Tseng, Smoothing functions for second-order-cone complementarity problems, *SIAM Journal on Optimization* 12 (2002) 436–460.

[10] S.-H. Pan and J.-S. Chen, A class of interior proximal-like algorithms for convex second-order programming, *SIAM Journal on Optimization* 19 (2008) 883–910.

[11] S.-H. Pan and J.-S. Chen, Proximal-like algorithm using the quasi D-function for convex second-order cone programming, *Journal of Optimization Theory and Applications* 138 (2008) 95–113.

[12] S.-H. Pan, S. Kum, Y. Lim and J.-S. Chen, A generalized Fischer-Burmeister function for the second-order cone complementarity problem, submitted manuscript, 2009.

[13] D. Sun and J. Sun, Semismooth matrix valued functions, *Mathematics of Operations Research* 27 (2002) 150–169.

[14] T. Wihler, On the Hölder continuity of matrix functions for normal matrices, *Journal of Inequalities in Pure and Applied Mathematics* 10 (2009) 1–5.

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