ON THE SELF-DUALITY AND HOMOGENEITY OF ELLIPSOIDAL CONES

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ABSTRACT. The class of ellipsoidal cones, as an important prototype in closed convex cones, covers several practical instances such as second-order cone, circular cone and elliptic cone. In natural feature, it belongs to the category of nonsymmetric cones because it is non-self-dual under standard inner product. Nonetheless, it can be converted to a second-order cone, which is symmetric, by a transformation and vice versa. Is it possible to make an ellipsoidal cone to become self-dual by defining new setting of inner product? Is the class of ellipsoidal cones homogeneous? We provide affirmative answers for these two questions in this paper. As byproducts, its special cases such as circular cone and elliptic cone can be tackled likewise.

1. Introduction

Let V be a finite dimensional real Euclidean space, we denote by $\langle x, y\rangle_V$ the inner product of $x, y \in V$. A subset $C \subseteq V$ becomes a *cone* if $tx \in C$ for any $x \in C,\ t > 0$ and a *convex* set if $tx + (1-t)y \in C$ for any $x, y \in C,\ t \in (0,1)$. A closed convex cone $\Omega \subseteq V$ is *self-dual* if $\Omega^*_{\langle\cdot,\cdot\rangle_V} = \Omega$, where $\Omega^*_{\langle\cdot,\cdot\rangle_V}$ is the dual cone of Ω under the inner product $\langle\cdot,\cdot\rangle_V$, i.e.,

$$\Omega^*_{\langle\cdot,\cdot\rangle_V} := \{y \in V \mid \langle x, y\rangle_V \geq 0,\ \forall x \in \Omega\}.$$

Let $GL(V)$ be the group of all *automorphisms* of V. A closed convex cone $\Omega \subseteq V$ is *homogeneous* if for any $x, y \in \operatorname{int}\Omega$ (the interior of Ω), there exists an element $g \in Aut(\Omega)$ such that $gx = y$, where $Aut(\Omega)$ denotes the automorphism group of $\operatorname{int}\Omega$, i.e., $Aut(\Omega) := \{g \in GL(V) \mid g(\operatorname{int}\Omega) = \operatorname{int}\Omega\}$.

Let $Q \in \mathbb{R}^{n\times n}$ be a real-valued nonsingular symmetric matrix with a single negative eigenvalue $\lambda_n \in \mathbb{R}$ and its unit eigenvector $u_n \in \mathbb{R}^n$, the ellipsoidal cone is expressed in terms of the parameters (Q, u_n) as

$$\mathcal{K}_{\mathcal{E}} := \left\{x \in \mathbb{R}^n \mid x^T Q x \leq 0,\ u_n^T x \geq 0\right\}. \tag{1.1}$$

According to the eigenvalue decomposition on Q, there exist an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that $Q = U\Lambda U^T = \sum_{i=1}^n \lambda_i u_i u_i^T$,

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where $U := \big[u_1 \,|\, u_2 \,|\, \cdots \,|\, u_{n-1} \,|\, u_n\big]$, $\Lambda := \mathrm{diag}[\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n]$ with eigen-pairs (λ_i, u_i) for $i = 1, 2, \cdots, n$ satisfying the following two conditions:

$$\lambda_1 \geq \cdots \geq \lambda_{n-1} > 0 > \lambda_n, \quad \text{and} \quad u_i^T u_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.2}$$

Such cone has been previously studied in the literature [17, 16]. More specifically, Stern and Wolkowicz [17] characterizes the conditions of the spectrum of a given real-valued symmetric matrix based on the existence of a corresponding ellipsoidal cone. They also provide an equivalent description on exponential nonnegativity for second-order cone [16], which is related to the solution set of a linear autonomous system $\dot{\xi} = A\xi$. On the application side, the ellipsoidal cone is further used to model rendezvous of the multiple agents system and measure dispersion in directional datasets, see [3, 15] for more details.

Indeed, the class of ellipsoidal cones, as an important prototype in closed convex cones, covers several practical instances used in real world. For instances, if we set

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad u_n = e_n, \tag{1.3}$$

then the ellipsoidal cone $\mathcal{K}_\mathcal{E}$ reduces to the second-order cone [5, 6]:

$$\mathcal{K}^n := \left\{ (\bar{x}, x_n) \in \mathrm{I\!R}^{n-1} \times \mathrm{I\!R} \,|\, \|\bar{x}\| \leq x_n \right\}.$$

When Q and u_n are set as

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2\theta \end{bmatrix} \quad \text{and} \quad u_n = e_n, \tag{1.4}$$

then the ellipsoidal cone $\mathcal{K}_\mathcal{E}$ reduces to the circular cone [4, 19]:

$$\mathcal{L}_\theta := \left\{ (\bar{x}, x_n) \in \mathrm{I\!R}^{n-1} \times \mathrm{I\!R} \,|\, \|\bar{x}\| \leq x_n \tan\theta \right\}$$

When Q and u_n are taken as

$$Q = \begin{bmatrix} M^T M & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad u_n = e_n, \tag{1.5}$$

then the ellipsoidal cone $\mathcal{K}_\mathcal{E}$ reduces to the elliptic cone [2]:

$$\mathcal{K}^n_M := \left\{ (\bar{x}, x_n) \in \mathrm{I\!R}^{n-1} \times \mathrm{I\!R} \,|\, \|M\bar{x}\| \leq x_n \right\}.$$

Here $\|\cdot\|$ denotes the standard Euclidean norm, I_{n-1} means the identity matrix of order $n-1$, $\theta \in (0, \frac{\pi}{2})$, M is any nonsingular matrix of order $n-1$ and e_n is the n-th column vector of I_n. Hence, the ellipsoidal cone is a natural generalization of second-order cone, circular cone and elliptic cone, whose relations [13, Remark 1] are depicted in Fig. 1.

Not only the ellipsoidal cone includes the well known second-order cone, circular cone and elliptic cone as special cases, as mentioned above; but also the ellipsoidal cone and the second-order cone can be converted to each other, see [13, Theorem 2.1 and Theorem 2.2] or Section 2. Through this bridge, the class of ellipsoidal cones connect symmetric cones and nonsymmetric cones. It is well known that symmetric cones can be dealt with under the Euclidean Jordan Algebra (EJA) [5, 6, 4, 1, 9, 10], whereas there is no unified framework for the category of nonsymmetric cones yet. For symmetric cones, the self-duality and homogeneity are the two key components.

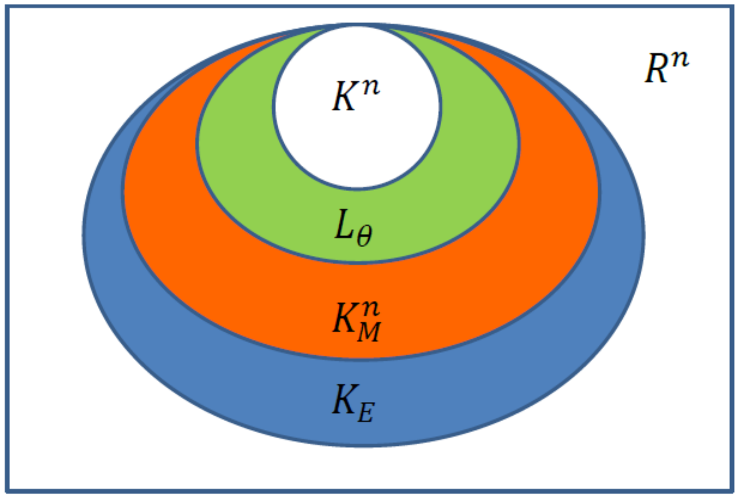

FIGURE 1. The relations among $\mathcal{K}^n$, $\mathcal{L}_\theta$, $\mathcal{K}_M^n$, and $\mathcal{K}_\mathcal{E}$.

In this paper, we address the issue regarding how to make an ellipsoidal cone to be self-dual and homogeneous[1]. Our contributions can be summarized as follows: (a) We provide new easy-calculate inner products associated with ellipsoidal cone and its special cases circular cone and elliptic cone, which make them to be self-dual. (b) Following an easy-check procedure, we show that ellipsoidal cone is homogeneous, see the proof of Theorem 3.1 below for more details. In some sense, we believe that such constructive analysis will be crucial for finding an unified framework for some families of nonsymmetric cone in the future.

2. Preliminaries

In this section, we recall some background materials about the ellipsoidal cone, including its interior, dual cone, and its connection to second-order cone. Most of them can be found in [13], we only extract some for our subsequent needs.

In what follows, the dual of ellipsoidal cone $\mathcal{K}_\mathcal{E}$ under the standard Euclidean inner product $\langle\cdot,\cdot\rangle$ defined on $\mathbb{R}^n$, denoted by $(\mathcal{K}_\mathcal{E})^*_{\langle\cdot,\cdot\rangle}$, is defined as

$$(\mathcal{K}_\mathcal{E})^*_{\langle\cdot,\cdot\rangle} := \{y \in \mathbb{R}^n \mid \langle x,y\rangle \geq 0,\ \forall x \in \mathcal{K}_\mathcal{E}\}.$$

[1]While finalizing a first version of this work, the authors became aware of a more general observation made in Güler [12], based on characteristic function of a cone. Following the discussion in [12], we know that if E is *isomorphic* to a self-dual cone in $\mathbb{R}^n$, then E is self-dual with respect to some inner product on $\mathbb{R}^n$. However, how to set such new inner product is not an easy task. On the other hand, ellipsoidal cone can be viewed as the image of second-order cone under an invertible linear transformation, therefore is homogenous due to theories on homogeneous cones [7, 8, 18]. In contrast to these qualitative analysis, we provide a quantitative approach to show the self-duality and homogeneity of ellipsoidal cones, in which more features and observations are obtained in this constructive way.

For any given vector $x \in \mathcal{K}_{\mathcal{E}}$, due to the orthogonal property of $\{u_i\}_{i=1}^n$, there exists a vector $\alpha := [\alpha_1, \cdots, \alpha_n]^T \in \mathbb{R}^n$ such that $x = U\alpha$, which implies

$$x^T Q x = \alpha^T U^T Q U \alpha = \alpha^T \Lambda \alpha = \sum_{i=1}^n \lambda_i \alpha_i^2 \quad \text{and} \quad u_n^T x = u_n^T \left(\sum_{i=1}^n \alpha_i u_i \right) = \alpha_n.$$

Hence, the set $\mathcal{K}_{\mathcal{E}}$ can be rewritten as follows:

$$\mathcal{K}_{\mathcal{E}} = U\Delta_\alpha \quad \text{where} \quad \Delta_\alpha := \left\{ \alpha \in \mathbb{R}^n \;\Big|\; \sum_{i=1}^n \lambda_i \alpha_i^2 \le 0,\ \alpha_n \ge 0 \right\}.$$

If we take $\lambda_i = 1$ for $i = 1, \cdots, n-1$ and $\lambda_n = -1$, the set Δ_α reduces to the second-order cone,

$$\mathcal{K}^n := \left\{ \alpha \in \mathbb{R}^n \;\Big|\; \sum_{i=1}^{n-1} \alpha_i^2 \le \alpha_n^2,\ \alpha_n \ge 0 \right\}. \tag{2.1}$$

For any $\alpha \in \Delta_\alpha$, in light of the relation (1.2) for $\{\lambda_i\}_{i=1}^n$, we have $\Delta_\alpha = D\mathcal{K}^n$, where D is a $n \times n$ diagonal matrix of the form

$$D := \operatorname{diag}\left[(\lambda_1)^{-1/2}, \cdots, (\lambda_{n-1})^{-1/2}, (-\lambda_n)^{-1/2} \right]. \tag{2.2}$$

In other words, we obtain $\mathcal{K}_{\mathcal{E}} = U\Delta_\alpha = UD\mathcal{K}^n = T\mathcal{K}^n$, where $T := UD$ is a nonsingular matrix in $\mathbb{R}^{n\times n}$. For simplicity, we denote by $|\Lambda| := \operatorname{diag}[|\lambda_1|, \cdots, |\lambda_n|]$, and hence $|\Lambda| = D^{-2}$. Similarly, we also obtain $(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle} = (T^T)^{-1}\mathcal{K}^n = UD^{-1}\mathcal{K}^n = T|\Lambda|\mathcal{K}^n$ and $(\mathcal{K}_{\mathcal{E}})^{**}_{\langle\cdot,\cdot\rangle} = \left((T|\Lambda|)^T\right)^{-1}\mathcal{K}^n = T\mathcal{K}^n = \mathcal{K}_{\mathcal{E}}$.

The next theorem sums up the aforementioned relations and presents a reformulation of the dual cone $(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle}$, whose proofs can be seen in [13, Theorem 2.1 and Theorem 2.2]. Such relations are also depicted in Fig. 2.

Theorem 2.1. *Let $\mathcal{K}_{\mathcal{E}}$ be an ellipsoidal cone given as in (1.1) and $\mathcal{K}^n$ be a second-order cone given as in (2.1). Then, the following relations hold.*

- **(a):** $\mathcal{K}_{\mathcal{E}} = T\mathcal{K}^n$ *and* $\mathcal{K}^n = T^{-1}\mathcal{K}_{\mathcal{E}}$*;*
- **(b):** $(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle} = T|\Lambda|\mathcal{K}^n$ *and* $\mathcal{K}^n = |\Lambda|^{-1}T^{-1}(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle}$*;*
- **(c):** $(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle} = T|\Lambda|T^{-1}\mathcal{K}_{\mathcal{E}}$ *and* $(\mathcal{K}_{\mathcal{E}})^{**}_{\langle\cdot,\cdot\rangle} = \mathcal{K}_{\mathcal{E}}$*;*
- **(d):** $(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle} = \left\{ y \in \mathbb{R}^n \,|\, y^T Q^{-1} y \le 0,\ u_n^T y \ge 0 \right\}$.

Remark 2.2. Applying Theorem 2.1(d), the duals of circular cone and elliptic cone under the standard Euclidean inner product $\langle\cdot,\cdot\rangle$, denoted by $(\mathcal{L}_\theta)^*_{\langle\cdot,\cdot\rangle}$ and $(\mathcal{K}^n_M)^*_{\langle\cdot,\cdot\rangle}$, are respectively given by

$$(\mathcal{L}_\theta)^*_{\langle\cdot,\cdot\rangle} := \left\{ (\bar{y}_{n-1}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{y}_{n-1}\| \le y_n \cot\theta \right\} = \mathcal{L}_{\frac{\pi}{2}-\theta},$$

$$(\mathcal{K}^n_M)^*_{\langle\cdot,\cdot\rangle} := \left\{ (\bar{y}_{n-1}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \;\Big|\; \left\| \left(M^{-1}\right)^T \bar{y}_{n-1} \right\| \le y_n \right\} = \mathcal{K}^n_{(M^{-1})^T},$$

where $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\bar{y}_{n-1} := (y_1, y_2, \cdots, y_{n-1})^T \in \mathbb{R}^{n-1}$. It follows from the above relations and Fig. 2 that circular cone, elliptic cone and ellipsoidal cone are obviously not self-dual under the standard Euclidean inner product, except their common special case, i.e., second-order cone.

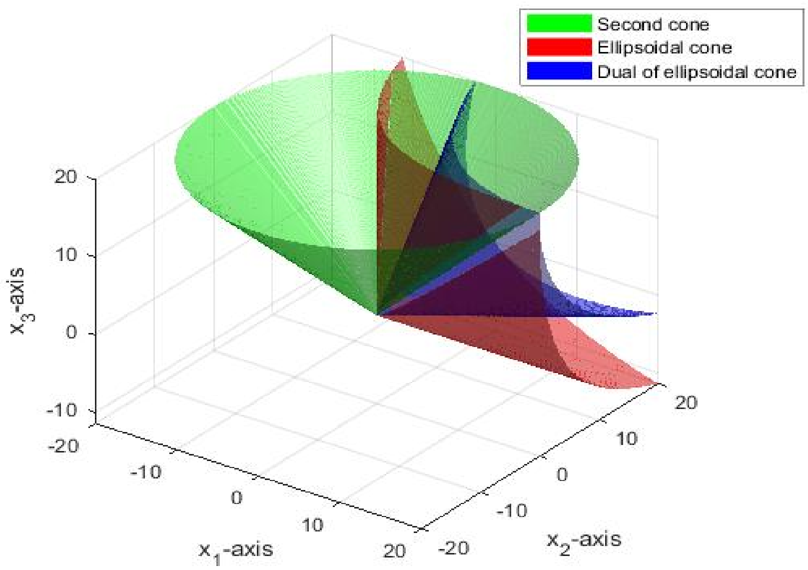

FIGURE 2. The graphs of a 3-dimensional ellipsoidal cone, its dual cone under the Euclidean inner product and a 3-dimensional second-order cone.

For convenience, we denote by $\operatorname{int}\mathcal{K}_{\mathcal{E}}$ the interior of ellipsoidal cone. It follows from Theorem 2.1 and [14, Theorem 6.6] that $\operatorname{int}\mathcal{K}_{\mathcal{E}} = T(\operatorname{int}\mathcal{K}^n)$. This together with the definition of $\mathcal{K}^n$ imply that $\operatorname{int}\mathcal{K}^n := \left\{\alpha \in \mathbb{R}^n \,\middle|\, \alpha^T Q_n \alpha < 0,\ e_n^T\alpha > 0\right\}$, where the matrix Q_n is defined as in (1.3), i.e.,

$$Q_n := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{2.3}$$

Then, for any given $x \in \mathcal{K}_{\mathcal{E}}$ and its corresponding vector $\alpha = T^{-1}x \in \mathcal{K}^n$, we obtain

$$\operatorname{int}\mathcal{K}_{\mathcal{E}} = \left\{x \in \mathbb{R}^n \,\middle|\, (T^{-1}x)^T Q_n (T^{-1}x) < 0,\ e_n^T T^{-1}x > 0\right\}.$$

From the definition of T, we also achieve two useful relations as follows:

$$\begin{aligned}&\left(T^{-1}\right)^T Q_n T^{-1} = \left(D^{-1}U^{-1}\right)^T Q_n D^{-1}U^{-1} = UD^{-1}Q_nD^{-1}U^T = U\Lambda U^T = Q,\\ &e_n^T T^{-1} = e_n^T(UD)^{-1} = e_n^T D^{-1}U^{-1} = e_n^T D^{-1}U^T = (-\lambda_n)^{1/2}u_n^T.\end{aligned}$$

With these, an explicit expression for the interior of $\mathcal{K}_{\mathcal{E}}$ is displayed in the following theorem.

Theorem 2.3. *Let $\mathcal{K}_{\mathcal{E}}$ be an ellipsoidal cone given as in (1.1). Then, the interior of $\mathcal{K}_{\mathcal{E}}$ can be expressed as*

$$\begin{aligned}\operatorname{int}\mathcal{K}_{\mathcal{E}} &= \left\{x \in \mathbb{R}^n \,\middle|\, (T^{-1}x)^T Q_n (T^{-1}x) < 0,\ e_n^T T^{-1}x > 0\right\}\\ &= \left\{x \in \mathbb{R}^n \,\middle|\, x^T Q x < 0,\ u_n^T x > 0\right\}.\end{aligned}$$

Remark 2.4. Likewise, we conclude from Theorem 2.3 that the interior of $\mathcal{L}_\theta$ and $\mathcal{K}^n_M$ are respectively described by

$$\begin{aligned}\operatorname{int}\mathcal{L}_\theta &:= \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}_{n-1}\| < x_n \tan\theta\right\},\\ \operatorname{int}\mathcal{K}^n_M &:= \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|M\bar{x}_{n-1}\| < x_n\right\}.\end{aligned}$$

where $\bar{x}_{n-1} := (x_1, x_2, \cdots, x_{n-1})^T \in \mathbb{R}^{n-1}$.

3. Main results

In this section, we first introduce a new inner product with respect to the ellipsoidal cone $\mathcal{K}_\mathcal{E}$ as in (1.1), which enables that the ellipsoidal cone is self-dual under this new setting. Then, we establish the homogeneity of the class of ellipsoidal cones.

In the sequel, the new inner product of $x, y \in \mathbb{R}^n$ is defined as follows:

$$\langle x, y\rangle_{\mathcal{K}_\mathcal{E}} := \frac{x^T|Q|y}{\|Qu_n\|} = -\frac{1}{\lambda_n}(U^Tx)^T|\Lambda|(U^Ty). \tag{3.1}$$

From the orthogonal property of $\{u_i\}_{i=1}^n$, there exist $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$ such that $x = U\alpha$ and $y = U\beta$. For simplicity, we write

$$\begin{aligned}&\Lambda := \operatorname{diag}[\bar{\Lambda}_{n-1}, \lambda_n] \in \mathbb{R}^{n\times n},\ \bar{\Lambda}_{n-1} := \operatorname{diag}[\lambda_1, \lambda_2, \cdots, \lambda_{n-1}] \in \mathbb{R}^{(n-1)\times(n-1)},\\ &\alpha := [\bar{\alpha}^T_{n-1}, \alpha_n]^T \in \mathbb{R}^n,\ \bar{\alpha}_{n-1} := [\alpha_1, \alpha_2, \cdots, \alpha_{n-1}]^T \in \mathbb{R}^{n-1},\\ &\beta := [\bar{\beta}^T_{n-1}, \beta_n]^T \in \mathbb{R}^n,\ \bar{\beta}_{n-1} := [\beta_1, \beta_2, \cdots, \beta_{n-1}]^T \in \mathbb{R}^{n-1}\end{aligned}$$

and $\bar{M}$ is a diagonal matrix of order $n-1$ with the i-th element $\sqrt{\lambda_i/(-\lambda_n)}$, where $i = 1, \cdots, n-1$. It is easy to verify that the matrix $\bar{M}$ also satisfies the equation

$$\bar{\Lambda}_{n-1} + \lambda_n \bar{M}^T\bar{M} = 0 \tag{3.2}$$

and the ellipsoidal cone $\mathcal{K}_\mathcal{E}$ can be recast as $\mathcal{K}_\mathcal{E} = U\mathcal{K}^n_{\bar{M}}$, where $\mathcal{K}^n_{\bar{M}}$ is an elliptic cone with the parameter $\bar{M}$.

After these discussions, we are ready to show that the ellipsoidal cone is self-dual under the new inner product.

Theorem 3.1. *Under the new inner product (3.1), the ellipsoidal cone $\mathcal{K}_\mathcal{E}$ given as in (1.1) is self-dual, i.e., $(\mathcal{K}_\mathcal{E})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_\mathcal{E}}} := \{x \in \mathbb{R}^n \,|\, \langle x, y\rangle_{\mathcal{K}_\mathcal{E}} \geq 0, \forall y \in \mathcal{K}_\mathcal{E}\} = \mathcal{K}_\mathcal{E}$.*

Proof First, we show the inclusion $\mathcal{K}_\mathcal{E} \subseteq (\mathcal{K}_\mathcal{E})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_\mathcal{E}}}$. Suppose that $x \in \mathcal{K}_\mathcal{E}$, we need to verify that $x \in (\mathcal{K}_\mathcal{E})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_\mathcal{E}}}$. For any $y \in \mathcal{K}_\mathcal{E}$, due to the orthogonal of $\{u_i\}_{i=1}^n$ and the fact $\mathcal{K}_\mathcal{E} = U\mathcal{K}^n_{\bar{M}}$, there exist $\alpha, \beta \in \mathcal{K}^n_{\bar{M}}$ such that $x = U\alpha$ and $y = U\beta$.

Hence, we obtain

$$\begin{aligned}
& \langle x,y\rangle_{\mathcal{K}_{\mathcal{E}}} \\
= \;& -\frac{1}{\lambda_n}(U^Tx)^T|\Lambda|(U^Ty) \\
= \;& -\frac{1}{\lambda_n}\alpha^T|\Lambda|\beta \\
= \;& \begin{bmatrix} \bar{\alpha}_{n-1} \\ \alpha_n \end{bmatrix}^T \begin{bmatrix} \frac{\bar{\Lambda}_{n-1}}{(-\lambda_n)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\beta}_{n-1} \\ \beta_n \end{bmatrix} \\
= \;& \begin{bmatrix} \bar{\alpha}_{n-1} \\ \alpha_n \end{bmatrix}^T \begin{bmatrix} \bar{M}^T\bar{M} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\beta}_{n-1} \\ \beta_n \end{bmatrix} \\
= \;& \left(\bar{M}\bar{\alpha}_{n-1}\right)^T \left(\bar{M}\bar{\beta}_{n-1}\right) + \alpha_n\beta_n \\
\geq \;& \left(\bar{M}\bar{\alpha}_{n-1}\right)^T \left(\bar{M}\bar{\beta}_{n-1}\right) + \left\|\bar{M}\bar{\alpha}_{n-1}\right\| \cdot \left\|\bar{M}\bar{\beta}_{n-1}\right\| \\
\geq \;& \left(\bar{M}\bar{\alpha}_{n-1}\right)^T \left(\bar{M}\bar{\beta}_{n-1}\right) + \left|\left(\bar{M}\bar{\alpha}_{n-1}\right)^T \left(\bar{M}\bar{\beta}_{n-1}\right)\right| \\
\geq \;& 0,
\end{aligned}$$

where the first inequality follows from the fact $\alpha, \beta \in \mathcal{K}^n_{\bar{M}}$ and the second one is obtained by Cauchy-Schwartz inequality. The above inequality says that $x \in (\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_{\mathcal{E}}}}$ and hence $\mathcal{K} \subseteq (\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_{\mathcal{E}}}}$.

Next, we prove that the reverse inclusion is also valid. Suppose that $x \in (\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_{\mathcal{E}}}}$. Similar to the above arguments, there exist $\alpha \in \mathbb{R}^n$ such that $x = U\alpha$. To proceed, we discuss two cases:

Case (a): If $\bar{\alpha}_{n-1} = 0$, then we have $x = \alpha_n u_n$. It suffices to verify that $\alpha_n \geq 0$. To see this, we choose $y = u_n$, which gives

$$y^TQy = u_n^TQu_n = u_n^T\left(\sum_{i=1}^n \lambda_i u_i u_i^T\right)u_n = \lambda_n < 0 \quad \text{and} \quad u_n^Ty = u_n^Tu_n = 1 > 0.$$

These imply that $y = u_n \in \mathcal{K}_{\mathcal{E}}$. Then, using definitions of $(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_{\mathcal{E}}}}$ and (3.1) yields

$$\begin{aligned}
& \langle x,y\rangle_{\mathcal{K}_{\mathcal{E}}} \\
= \;& -\frac{1}{\lambda_n}(U^Tx)^T|\Lambda|(U^Ty) \\
= \;& -\frac{1}{\lambda_n}\alpha^T|\Lambda|\beta \\
= \;& \begin{bmatrix} 0 \\ \alpha_n \end{bmatrix}^T \begin{bmatrix} \bar{M}^T\bar{M} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
= \;& \alpha_n \geq 0.
\end{aligned}$$

This shows $\alpha_n \geq 0$ and hence $x \in \mathcal{K}_{\mathcal{E}}$. Thus, we have proved $(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_{\mathcal{E}}}} \subseteq \mathcal{K}_{\mathcal{E}}$.

Case (b): If $\bar{\alpha}_{n-1} \neq 0$, then we set $y = U\beta$, where $\beta = (-\bar{\alpha}_{n-1}^T, \|\bar{M}\bar{\alpha}_{n-1}\|)^T \in \mathcal{K}^n_{\bar{M}}$. From the relation $\mathcal{K}_{\mathcal{E}} = U\mathcal{K}^n_{\bar{M}}$, we know that $y \in \mathcal{K}_{\mathcal{E}}$. using the fact $\langle x,y\rangle_{\mathcal{K}_{\mathcal{E}}} \geq 0$

and $\bar{\alpha}_{n-1} \neq 0$, we have

$$\langle x, y\rangle_{\mathcal{K}_\mathcal{E}} = \left(\bar{M}\bar{\alpha}_{n-1}\right)^T \left(\bar{M}\bar{\beta}_{n-1}\right) + \alpha_n\beta_n = \left\|\bar{M}\bar{\alpha}_{n-1}\right\| \left(\alpha_n - \|\bar{M}\bar{\alpha}_{n-1}\|\right) \geq 0,$$

which deduces that $\alpha_n - \|\bar{M}\bar{\alpha}_{n-1}\| \geq 0$, i.e., $\alpha \in \mathcal{K}^n_{\bar{M}}$ and $x = U\alpha \in \mathcal{K}_\mathcal{E}$. This shows $(\mathcal{K}_\mathcal{E})^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}_\mathcal{E}}} \subseteq \mathcal{K}_\mathcal{E}$. □

Similar to Theorem 3.1, for any $x := (\bar{x}^T_{n-1}, x_n)^T \in \mathbb{R}^n$ and $y := (\bar{y}^T_{n-1}, y_n)^T \in \mathbb{R}^n$, using (1.4) and (1.5), we define two types of inner products for the settings of circular cone and elliptic cone:

$$\langle x, y\rangle_{\mathcal{L}_\theta} := \cot^2\theta \cdot \bar{x}^T_{n-1}\bar{y}_{n-1} + x_n y_n, \tag{3.3}$$

$$\langle x, y\rangle_{\mathcal{K}^n_M} := (M\bar{x}_{n-1})^T(M\bar{y}_{n-1}) + x_n y_n. \tag{3.4}$$

Corollary 3.2. *Under the new inner products given as in (3.3) and (3.4), the circular cone $\mathcal{L}_\theta$ and the elliptic cone $\mathcal{K}^n_M$ are self-dual, respectively. In other words, we have*

$$(\mathcal{L}_\theta)^*_{\langle\cdot,\cdot\rangle_{\mathcal{L}_\theta}} := \{x \in \mathbb{R}^n \mid \langle x, y\rangle_{\mathcal{L}_\theta} \geq 0, \forall y \in \mathcal{L}_\theta\} = \mathcal{L}_\theta,$$
$$(\mathcal{K}^n_M)^*_{\langle\cdot,\cdot\rangle_{\mathcal{K}^n_M}} := \{x \in \mathbb{R}^n \mid \langle x, y\rangle_{\mathcal{K}^n_M} \geq 0, \forall y \in \mathcal{K}^n_M\} = \mathcal{K}^n_M.$$

Next, we establish the homogeneity of the class of ellipsoidal cones, which is another main result of this paper.

Theorem 3.3. *The ellipsoidal cone given as in (1.1) is homogeneous.*

Proof Our proof is inspired by [9, Chap I, pages 7-8] and we complete the arguments by six steps.
Step 1: The interior of ellipsoidal cone $\mathcal{K}_\mathcal{E}$ is given as

$$\operatorname{int}\mathcal{K}_\mathcal{E} = \left\{x \in \mathbb{R}^n \mid x^T(-Q)x > 0,\ u_n^T x > 0\right\},$$

which follows from Theorem 2.3.
Step 2: We introduce the corresponding bilinear form $[x, y] := x^T(-Q)y$ and rewrite $\operatorname{int}\mathcal{K}_\mathcal{E}$ as $\operatorname{int}\mathcal{K}_\mathcal{E} = \left\{x \in \mathbb{R}^n \mid [x, x] > 0,\ u_n^T x > 0\right\}$.
Step 3: Now, we define

$$\Xi := \left\{P \in \mathbb{R}^{n\times n} \mid P \text{ is nonsingular and } P^TQP = Q\right\}.$$

It is clear to see that Ξ is nonempty because $I_n \in \Xi$. For any $P_1, P_2 \in \Xi$, we have $P_1P_2 \in \Xi$ due to

$$(P_1P_2)^TQ(P_1P_2) = P_2^T(P_1^TQP_1)P_2 = P_2^TQP_2 = Q,$$

where the last two equations follow from the facts $P_1^TQP_1 = Q$ and $P_2^TQP_2 = Q$. Moreover, we know $P^{-1} \in \Xi$ for any $P \in \Xi$ by observing

$$(P^{-1})^TQ(P^{-1}) = (PQ^{-1}P^T)^{-1} = \left(P(P^TQP)^{-1}P^T\right)^{-1} = Q.$$

To sum up, we have shown that the set Ξ is a subgroup of $GL(\mathbb{R}^n)$ (the set of all nonsingular matrices of order n) and closed under matrix multiplication and inverse.
Step 4: We next construct a set of transformations as below

$$\Xi_{\mathcal{K}_\mathcal{E}} := \left\{TAT^{-1} \in \mathbb{R}^{n\times n} \mid A \text{ is nonsingular}, A^TQ_nA = Q_n \text{ and } A_{n,n} > 0\right\}, \tag{3.5}$$

where $T = UD$, D is defined as in (2.2), $A_{n,n}$ is the (n,n)-entry of A and the matrix Q_n is defined as in (2.3). We will show that this set $\Xi_{\mathcal{K}_\mathcal{E}}$ is indeed an automorphism group of $\operatorname{int}\mathcal{K}_\mathcal{E}$, that is, $Aut(\operatorname{int}\mathcal{K}_\mathcal{E}) = \Xi_{\mathcal{K}_\mathcal{E}}$.

(i) If $P \in \Xi_{\mathcal{K}_\mathcal{E}}$, then there exists a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ such that $P = TAT^{-1}$, $A^TQ_nA = Q_n$, $A_{n,n} > 0$ and hence the inverse of A now equals to $Q_n^{-1}A^TQ_n$. Besides these relations, we obtain $A^{-1}_{n,n} = A_{n,n} > 0$ and $(A^{-1})^TQ_nA^{-1} = (AQ_n^{-1}A^T)^{-1} = (A(A^TQ_nA)^{-1}A^T)^{-1} = Q_n$, which show that $P^{-1} = TA^{-1}T^{-1} \in \Xi_{\mathcal{K}_\mathcal{E}}$. Moreover, let $P_1, P_2 \in \Xi_{\mathcal{K}_\mathcal{E}}$, there exist two nonsingular matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ such that $P_i = TA_iT^{-1}$, $A_i^TQ_nA_i = Q_n$ and $(A_i)_{n,n} > 0$ for $i = 1, 2$ and $P_1P_2 = TA_1A_2T^{-1}$. Due to the properties of A_i $(i = 1, 2)$, we know that $(A_1A_2)^TQ_n(A_1A_2) = Q_n$. Then, it follows from [11, Proposition 7.1 and Proposition 7.6] that the matrix A in (3.5) has a polar decomposition of the form

$$A = \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (I_{n-1} + vv^T)^{1/2} & v \\ v^T & c \end{bmatrix}, \tag{3.6}$$

where S is an orthogonal matrix of order $n-1$ and $c = \sqrt{\|v\|^2 + 1}$, $v \in \mathbb{R}^{n-1}$. In particular, for the matrices A_1, A_2, there exist S_1, S_2, two orthogonal matrices of order $n-1$, and c_1, c_2, v_1, v_2 such that

$$\begin{aligned} A_1 &= \begin{bmatrix} S_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (I_{n-1} + v_1v_1^T)^{1/2} & v_1 \\ v_1^T & c_1 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} S_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (I_{n-1} + v_2v_2^T)^{1/2} & v_2 \\ v_2^T & c_2 \end{bmatrix}, \end{aligned}$$

with $c_i = \sqrt{\|v_i\|^2 + 1}$ for $i = 1, 2$. It is easy to verify that

$$\begin{aligned} & (A_1A_2)_{n,n} \\ = \; & v_1^TS_2v_2 + c_1c_2 \\ \geq \; & c_1c_2 - \|v_1\| \cdot \|S_2v_2\| \\ = \; & c_1c_2 - \|v_1\| \cdot \|v_2\| \\ \geq \; & \sqrt{\|v_1\|^2 + 1} \cdot \sqrt{\|v_2\|^2 + 1} - \|v_1\| \cdot \|v_2\| \\ > \; & 0, \end{aligned}$$

where the second equation uses the fact that S_2 is an orthogonal matrices of order $n-1$. From these results, we also obtain $P_1P_2 \in \Xi_{\mathcal{K}_\mathcal{E}}$. Hence, the set $\Xi_{\mathcal{K}_\mathcal{E}}$ is closed under matrix multiplication and inverse as well. Notice that $D^{-1}Q_nD^{-1} = \Lambda$. For any $x \in \mathbb{R}^n$ and $P \in \Xi_{\mathcal{K}_{\mathcal{E}}}$, we have

$$\begin{aligned}
& [Px, Px] \\
= \; & (Px)^T Q(Px) \\
= \; & x^T P^T Q P x \\
= \; & x^T (UDAD^{-1}U^T)^T (U\Lambda U^T)(UDAD^{-1}U^T)x \\
= \; & x^T UD^{-1}A^T DU^T (UD^{-1}Q_n D^{-1}U^T) UDAD^{-1}U^T x \\
= \; & x^T UD^{-1}A^T Q_n A D^{-1}U^T x \\
= \; & x^T UD^{-1}Q_n D^{-1}U^T x \\
= \; & x^T U\Lambda U^T x \\
= \; & x^T Q x \\
= \; & [x, x] > 0,
\end{aligned}$$

which shows that $\Xi_{\mathcal{K}_{\mathcal{E}}} \subseteq \Xi$ and hence $\Xi_{\mathcal{K}_{\mathcal{E}}}$ is a subgroup of Ξ.
(ii) On the other hand, for any $x \in \text{int}\mathcal{K}_{\mathcal{E}}$, there exist a element $\alpha = (\bar{\alpha}_{n-1}^T, \alpha_n)^T \in \text{int}\mathcal{K}^n_{\bar{M}}$ such that $x = U\alpha$ and $u_n^T x = u_n^T U\alpha = \alpha_n > 0$. For any $P \in \Xi_{\mathcal{K}_{\mathcal{E}}}$, due to the structure of A as in (3.6) and $T = UD$, we obtain

$$\begin{aligned}
& u_n^T(Px) \\
= \; & u_n^T UDAD^{-1}U^T U\alpha \\
= \; & e_n^T DAD^{-1}\alpha \\
= \; & e_n^T D \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (I_{n-1} + vv^T)^{1/2} & v \\ v^T & c \end{bmatrix} D^{-1}\alpha \\
= \; & e_n^T \begin{bmatrix} (\bar{\Lambda}_{n-1})^{-1/2} & 0 \\ 0 & (-\lambda_n)^{-1/2} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \qquad (3.7) \\
& \cdot \begin{bmatrix} (I_{n-1} + vv^T)^{1/2} & v \\ v^T & c \end{bmatrix} \begin{bmatrix} (\bar{\Lambda}_{n-1})^{1/2} & 0 \\ 0 & (-\lambda_n)^{1/2} \end{bmatrix} \alpha \\
= \; & v^T \bar{M} \bar{\alpha}_{n-1} + c \cdot \alpha_n \\
\geq \; & -\|v\| \cdot \|\bar{M}\alpha_{n-1}\| + c\alpha_n \\
= \; & \sqrt{\|v\|^2 + 1} \cdot \alpha_n - \|v\| \cdot \|\bar{M}\alpha_{n-1}\|, \qquad (3.8)
\end{aligned}$$

where the fifth equation is obtained from (3.2). If $v = 0$, then $u_n^T(Px) \geq \alpha_n > 0$; otherwise, i.e., $v \neq 0$, from the fact $\alpha = (\bar{\alpha}_{n-1}^T, \alpha_n)^T \in \text{int}\mathcal{K}^n_{\bar{M}}$, we obtain $u_n^T(Px) > \|v\| \cdot (\alpha_n - \|\bar{M}\alpha_{n-1}\|) > 0$. These relations together with (3.8) show that $Px \in \text{int}\mathcal{K}_{\mathcal{E}}$ for any given $x \in \text{int}\mathcal{K}_{\mathcal{E}}$ and $P \in \Xi_{\mathcal{K}_{\mathcal{E}}}$.
From (i) and (ii), we claim that the set $\Xi_{\mathcal{K}_{\mathcal{E}}}$ is indeed an automorphism group of $\text{int}\mathcal{K}_{\mathcal{E}}$, i.e., $Aut(\text{int}\,\mathcal{K}_{\mathcal{E}}) = \Xi_{\mathcal{K}_{\mathcal{E}}}$. In addition, it is clear that its dilation transformation $\tilde{\Xi}_{\mathcal{K}_{\mathcal{E}}} := \eta \cdot \Xi_{\mathcal{K}_{\mathcal{E}}}$ with $\eta > 0$ is an automorphism group of $\text{int}\mathcal{K}_{\mathcal{E}}$, too.
Step 5: The set $\Xi_{\mathcal{K}_{\mathcal{E}}}$ is nonempty. For example, we have two choices of A as below.
(a) $A = \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix}$, where S is an orthogonal matrix of order $n-1$. This subclass can be deduced from (3.6) when we set $v = 0$.

(b) $A = H_t$, where H_t are the hyperbolic rotations:

$$H_t = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-2} & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}, \ t \geq 0.$$

Here $\cosh t$ and $\sinh t$ are hyperbolic cosine and hyperbolic sine, which are respectively given by

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}.$$

We can easily verify that H_t satisfies the condition $H_t^T Q_n H_t = Q_n$ and $(H_t)_{n,n} = \cosh t > 0$ for any $t \geq 0$.

Step 6: In order to show that ellipsoidal cone is homogeneous, we only need to verify that for any given $x \in \mathrm{int}\mathcal{K}_{\mathcal{E}}$, there exists an element $\tilde{P} \in \tilde{\Xi}_{\mathcal{K}_{\mathcal{E}}}$ such that $\tilde{P}u_n = x$ for $u_n \in \mathrm{int}\mathcal{K}_{\mathcal{E}}$, which means that we need to find a positive scalar η and $P \in \Xi_{\mathcal{K}_{\mathcal{E}}}$ such that $x = \eta \cdot Pu_n$. From the relation $\mathrm{int}\mathcal{K}_{\mathcal{E}} = T(\mathrm{int}\mathcal{K}^n)$ and the structure of $\Xi_{\mathcal{K}_{\mathcal{E}}}$, the above requirements are equivalent to the following conditions: for any given $\alpha \in \mathrm{int}\mathcal{K}^n$, there exist a positive scalar η and a element $A \in \mathbb{R}^{n\times n}$ such that $A^T Q_n A = Q_n$, $A_{nn} > 0$ and $\alpha = (-\lambda_n)^{1/2}\eta \cdot Ae_n$, where e_n is the n-th column of I_n and $Te_n = (-\lambda_n)^{-1/2}u_n$, which reduces to the case in [9, Chap I, pages 7-8]. Thus, there exists S an orthogonal matrix of order $n-1$ such that $A = \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} H_t$ and $\eta = \left(\frac{\alpha^T(-Q_n)\alpha}{-\lambda_n}\right)^{1/2}$. Moreover, from the relation $x = T\alpha$ and $T = UD$, we obtain

$$x = \left(\frac{[x, x]}{-\lambda_n}\right)^{1/2} T \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} H_t T^{-1} u_n,$$

which shows that the ellipsoidal cone is homogeneous. □

As special cases of ellipsoidal cone, the homogeneities of circular cone and elliptic cone follow immediately by Theorem 3.3.

Corollary 3.4. *The circular cone $\mathcal{L}_\theta$ and the elliptic cone $\mathcal{K}^n_M$ are also homogeneous.*

4. Concluding remarks

In this paper, through introducing the new inner product (3.1) with respect to the ellipsoidal cone as in (1.1), we show its corresponding structural properties such as the self-duality and homogeneity. At the same time, due to the relation $\mathcal{K}_{\mathcal{E}} = T\mathcal{K}^n$ and [9, Chap I, pages 7-8], two interesting things are observed:

- If we set

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}, \quad U = I_n \quad \text{and} \quad u_n = e_n,$$

  then the new inner product (3.1) reduces to the standard Euclidean inner product.

- The set $\Xi_{\mathcal{K}_{\mathcal{E}}}$, the automorphism group of int$\mathcal{K}_{\mathcal{E}}$, can be characterized as the similarity transformation of the automorphism group of int$\mathcal{K}^n$ under the matrix parameter T. Let us denote the automorphism group of int$\mathcal{K}^n$ as

  $$\Xi_{\mathcal{K}^n} := \{A \in \mathbb{R}^{n\times n} \mid A \text{ is nonsingular}, A^T Q_n A = Q_n \text{ and } A_{n,n} > 0\},$$

  where $A_{n,n}$ is the (n,n)-entry of A and the matrix Q_n is defined as in (2.3). The relation of the automorphism group between int$\mathcal{K}_{\mathcal{E}}$ and int$\mathcal{K}^n$ is depicted as

  $$\begin{array}{ccc} \text{int}\mathcal{K}^n & \xrightarrow{T} & \text{int}\mathcal{K}_{\mathcal{E}} \\ {\scriptstyle Aut(\text{int}\mathcal{K}^n)}\downarrow & & \downarrow{\scriptstyle Aut(\text{int}\mathcal{K}_{\mathcal{E}})} \\ \Xi_{\mathcal{K}^n} & \xrightarrow{T\cdot * \cdot T^{-1}} & \Xi_{\mathcal{K}_{\mathcal{E}}} \end{array}$$

  where the notation "$Aut(\cdot)$" denotes the automorphism group of the given set.

To sum up, we believe that the analysis used in this paper will pave a way to tackling with other unfamiliar closed convex cones appeared in real world.

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