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SMOOTH ANALYSIS ON CONE FUNCTION ASSOCIATED WITH ELLIPSOIDAL CONE

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Abstract. As an important prototype in closed convex cones, ellipsoidal cone covers several practical instances such as second-order cone, circular cone and elliptic cone. In virtue of a recent study on its decomposition expression, we present a symmetric type of ellipsoidal cone function and show that this vector-valued function inherits some smooth properties from its corresponding scalar function, particularly in continuity, directional differentiability, differentiability and continuous differentiability. We believe that these results will play important roles on further analysis and study about conic programming problems associated with ellipsoidal cone.

1. Introduction

Consider the *ellipsoidal cone* [2, 21, 22, 23] with the form

$$\mathcal{K}_{\mathcal{E}} := \left\{ x \in \mathbb{R}^n \,|\, x^T Q x \leq 0,\ u_n^T x \geq 0 \right\}, \tag{1.1}$$

where $Q \in \mathbb{R}^{n\times n}$ is a real-valued nonsingular symmetric matrix and $(\lambda_i, u_i) \in \mathbb{R}\times\mathbb{R}^n$ denotes its i-th eigenpair. In addition, these pairs satisfy the underline relations:

$$\lambda_1 \geq \cdots \geq \lambda_{n-1} > 0 > \lambda_n \quad \text{and} \quad \begin{cases} u_i^T u_j = 1, & \text{if } i = j, \\ u_i^T u_j = 0, & \text{if } i \neq j. \end{cases}$$

Under the standard Euclidean inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ defined on $\mathbb{R}^n$, its dual cone $(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle}$ has an explicit expression (due to [16, Theorem 2.1]) as follows:

$$(\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle} := \{ y \in \mathbb{R}^n \mid y^T Q^{-1} y \leq 0,\ u_n^T y \geq 0 \}. \tag{1.2}$$

It is not difficult to see that the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$ defined as in (1.1) can be viewed as a type of nonsymmetric cones.

During the past two decades, conic programming have been extensively studied [4, 6, 7, 8, 9, 12, 13, 14, 20], particularly in three types of closed convex cones, i.e., the nonnegative octant $\mathbb{R}^n_+$, second-order cone $\mathcal{K}^n$ and positive semi-definite cone $\mathcal{S}^n_+$. All of these cones are fully addressed and some fundamental topics such as projection, spectral decomposition, cone function and cone-convexity have been studied. A fascinating feature among them is to unify all these results under the framework of Euclidean Jordan Algebra defined on symmetric cones, we refer to the

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monograph [10] for more details. A natural question is how to extend these observations on symmetric cones suitable for nonsymmetric counterparts? Recently, Miao, Lu and Chen [18] look into the first three items in the setting of some nonsymmetric cases such as circular cone, p-order cone, geometric cone, exponential cone and power cone, in which the lack of explicit projection formulae onto these cones (except for the circular cone case) become the main hurdle for non-symmetric cone optimization problems and cause some unpleasant consequences. For example, the classical Moreau decomposition in convex analysis cannot be used directly; the associated decomposition expressions and cone functions are correspondingly missing. These observations motivate us to focus on algebraic properties of nonsymmetric cones and to provide a systematical study on their analytic features.

As an important prototype, several famous instances can be generated from ellipsoidal cones by different choices of parameters (Q, u_n). For instances, let us take

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2\theta \end{bmatrix} \text{ or } \begin{bmatrix} M^TM & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad u_n = e_n,$$

where I_{n-1} denotes the identity matrix of order $n-1$, $\theta \in (0, \frac{\pi}{2})$, M is any nonsingular matrix of order $n-1$ and e_n is the n-th column vector of I_n. In these cases, the ellipsoidal cone respectively reduces to the second-order cone [5, 8]:

$$\mathcal{K}^n := \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}_{n-1}\| \leq x_n\right\},$$

the circular cone [3, 24]:

$$\mathcal{L}_\theta := \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}_{n-1}\| \leq x_n \tan\theta\right\}$$

and the elliptic cone [1]:

$$\mathcal{K}^n_M := \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|M\bar{x}_{n-1}\| \leq x_n\right\}.$$

Therefore, ellipsoidal cone is a natural generalization of second-order cone, circular cone and elliptic cone.

For algebraic properties of ellipsoidal cones, there have been several literatures in recent studies. More specifically, Lu and Chen [16] discuss its self-duality and positive homogeneity, in which the authors observe that ellipsoidal cone can become self-dual by introducing a new inner product and the associated automorphism group can be characterized as the similarity transformation of its special counterpart in the second-order cone setting with an appropriate nonsingular matrix. Furthermore, they also provide an investigation on its variational geometry, projection expression and decomposition, see [17] for more details. In particular, the decomposition of the given point associated with the ellipsoidal cone is characterized in [17, Theorem 8]. For completeness, we restate it as follows. Due to the eigenvalue decomposition, we rewrite $Q \in \mathbb{R}^{n\times n}$ as the form $Q = U\Lambda U^T$ with an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$, where $U := [\bar{U}_{n-1}, u_n] \in \mathbb{R}^{n\times n}$, $\bar{U}_{n-1} := [u_1, u_2, \ldots, u_{n-1}] \in \mathbb{R}^{n\times(n-1)}$ and $\Lambda := \mathrm{diag}(\bar{\Lambda}_{n-1}, \lambda_n) \in \mathbb{R}^{n\times n}$, $\bar{\Lambda}_{n-1} := \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \in \mathbb{R}^{(n-1)\times(n-1)}$.

Theorem 1.1 (Decomposition). [17, Theorem 8] *Let $\mathcal{K}_\mathcal{E} \in \mathbb{R}^n$ be an ellipsoidal cone defined as in (1.1) and $(\mathcal{K}_\mathcal{E})^*_{\langle\cdot,\cdot\rangle}$ be its dual cone defined as in (1.2). For any* *given $x \in \mathbb{R}^n$, it has the following decomposition:*

$$x = \begin{cases} s_{I_a}^{(1)}(x) \cdot v_{I_a}^{(1)}(x) + s_{I_a}^{(2)}(x) \cdot v_{I_a}^{(2)}(x) & \text{if } \bar{U}_{n-1}^T x \neq 0, \\ s_{I_b}^{(1)}(x) \cdot v_{I_b}^{(1)}(x) + s_{I_b}^{(2)}(x) \cdot v_{I_b}^{(2)}(x) & \text{if } \bar{U}_{n-1}^T x = 0, \end{cases}$$

where $s_{I_a}^{(1)}(x), s_{I_a}^{(2)}(x), s_{I_b}^{(1)}(x), s_{I_b}^{(2)}(x)$ and $v_{I_a}^{(1)}(x), v_{I_a}^{(2)}(x), v_{I_b}^{(1)}(x), v_{I_b}^{(2)}(x)$ are respectively given by

$$\begin{aligned}
s_{I_a}^{(1)}(x) &:= u_n^T x + \|\bar{M}\bar{U}_{n-1}^T x\|, & v_{I_a}^{(1)}(x) &:= \frac{1}{2} \cdot \left(\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} + u_n \right) \in \mathcal{K}_{\mathcal{E}}, \\
s_{I_a}^{(2)}(x) &:= u_n^T x - \|\bar{M}\bar{U}_{n-1}^T x\|, & v_{I_a}^{(2)}(x) &:= \frac{1}{2} \cdot \left(-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} + u_n \right) \in \mathcal{K}_{\mathcal{E}}, \\
s_{I_b}^{(1)}(x) &:= u_n^T x, & v_{I_b}^{(1)}(x) &:= \frac{1}{2} \cdot \left(\frac{\bar{U}_{n-1} w}{\|\bar{M}w\|} + u_n \right) \in \mathcal{K}_{\mathcal{E}}, \\
s_{I_b}^{(2)}(x) &:= u_n^T x, & v_{I_b}^{(2)}(x) &:= \frac{1}{2} \cdot \left(-\frac{\bar{U}_{n-1} w}{\|\bar{M}w\|} + u_n \right) \in \mathcal{K}_{\mathcal{E}}
\end{aligned}$$

with any given nonzero vector $w \in \mathbb{R}^{n-1}$ and a diagonal matrix $\bar{M}$ looks like

$$\begin{aligned}
\bar{M} &:= \left[\frac{\bar{U}_{n-1}^T (Q - \lambda_n u_n u_n^T) \bar{U}_{n-1}}{(-\lambda_n)} \right]^{1/2} \\
&= \mathit{diag}\left(\sqrt{\frac{\lambda_1}{(-\lambda_n)}}, \sqrt{\frac{\lambda_2}{(-\lambda_n)}}, \ldots, \sqrt{\frac{\lambda_{n-1}}{(-\lambda_n)}} \right).
\end{aligned} \tag{1.3}$$

Theorem 1.1 indicates that by denoting

$$\begin{aligned}
&\left(\lambda_I^{(1)}(x), \lambda_I^{(2)}(x), u_I^{(1)}(x), u_I^{(2)}(x)\right) \\
&:= \begin{cases} \left(s_{I_a}^{(1)}(x), s_{I_a}^{(2)}(x), v_{I_a}^{(1)}(x), v_{I_a}^{(2)}(x)\right) & \text{if } \bar{U}_{n-1}^T x \neq 0, \\ \left(s_{I_b}^{(1)}(x), s_{I_b}^{(2)}(x), v_{I_b}^{(1)}(x), v_{I_b}^{(2)}(x)\right) & \text{if } \bar{U}_{n-1}^T x = 0, \end{cases}
\end{aligned} \tag{1.4}$$

the decomposition formula now can be rewritten as follows:

$$x = \lambda_I^{(1)}(x) \cdot u_I^{(1)}(x) + \lambda_I^{(2)}(x) \cdot u_I^{(2)}(x), \quad \forall x \in \mathbb{R}^n. \tag{1.5}$$

For any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}_{\mathcal{E}}$ is considered:

$$f_I^{EC}(x) = f(\lambda_I^{(1)}(x)) \cdot u_I^{(1)}(x) + f(\lambda_I^{(2)}(x)) \cdot u_I^{(2)}(x), \quad \forall x \in \mathbb{R}^n. \tag{1.6}$$

If f is defined only on the subset of $\mathbb{R}$, then f_I^{EC} is defined on the corresponding subset of $\mathbb{R}^n$. Notice that the expression (1.6) is well-defined whether $\bar{U}_{n-1}^T x \neq 0$ or $\bar{U}_{n-1}^T x = 0$. In the sequel, we call this function *a symmetric type of ellipsoidal cone function*, due to the fact that the vectors $u_I^{(1)}(x)$, $u_I^{(2)}(x)$ in (1.4) are both contained in $\mathcal{K}_{\mathcal{E}}$. For any given $x \in \mathbb{R}^n$, $\lambda_I^{(i)}(x), u_I^{(i)}(x)$ $(i = 1, 2)$ are the spectral values and the spectral vectors of x, respectively.

In this paper, we aim to study smooth properties of the vector-valued function f_I^{EC}, particularly in continuity, directional differentiability, differentiability, continuous differentiability inherited by f_I^{EC} from f. As a byproduct, we also establish these results for some special cases of ellipsoidal cone such as second-order cone, circular cone and elliptic cone.

The rest of this paper is organized as follows. In Section 2, we present some technical lemmas used in the sequel. The main conclusions will be established in Section 3. We next discuss some special examples in Section 4. Finally, some concluding remarks are drawn.

1.1. **Notation and terminology.** In what follows, we review some basic concepts about vector-valued functions. For the mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, we say F to be continuous at $x \in \mathbb{R}^n$ if

$$F(y) \to F(x) \quad \text{as} \quad y \to x,$$

and F is continuous if F is continuous at every $x \in \mathbb{R}^n$. Similarly, we say F is directionally differentiable at $x \in \mathbb{R}^n$ if

$$F'(x;h) = \lim_{t\downarrow 0} \frac{F(x+th) - F(x)}{t}$$

exists for all $h \in \mathbb{R}^n$ and F is directionally differentiable if F is directionally differentiable at every $x \in \mathbb{R}^n$. Moreover, F is differentiable (in the Fréchet sense) at $x \in \mathbb{R}^n$ if there exists a linear mapping $\mathcal{D}F : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$F(x+h) - F(x) - \mathcal{D}F(x)h = o(\|h\|).$$

We call $\mathcal{D}F(x)$ the Jacobian of F at $x \in \mathbb{R}^n$. Furthermore, if F is differentiable at every $x \in \mathbb{R}^n$ and $\mathcal{D}F(x)$ is also continuous, then F is continuous differentiable. For a differentiable mapping $g : \mathbb{R}^n \to \mathbb{R}$, the gradient of g with respect to the variable $x \in \mathbb{R}^n$ is denoted by $\nabla_x g$.

2. Preliminaries

Before establishing smooth analytic properties of f_I^{EC}, we need the following technical lemmas.

Lemma 2.1 (Perturbation of spectral values)**.** *Let $\lambda_I^{(i)}(x)$ $(i = 1, 2)$ be the spectral values of $x \in \mathbb{R}^n$ and $m_I^{(i)}(y)$ $(i = 1, 2)$ be the spectral values of $y \in \mathbb{R}^n$. Then, we have*

$$\left|\lambda_I^{(i)}(x) - m_I^{(i)}(y)\right| \leq \left(1 + \|\bar{M}\bar{U}_{n-1}^T\|\right) \cdot \|x-y\|, \quad \forall i = 1,2, \tag{2.1}$$

where $\|\bar{M}\bar{U}_{n-1}^T\|$ is the induced matrix norm on the space $\mathbb{R}^{(n-1)\times n}$ of $\bar{M}\bar{U}_{n-1}^T$ as follows:

$$\|\bar{M}\bar{U}_{n-1}^T\| := \sup\{\|\bar{M}\bar{U}_{n-1}^T x\| \mid \|x\| = 1,\ x \in \mathbb{R}^n\}.$$

Proof. The proof can be obtained easily by simple calculation. Note that

$$\begin{aligned}
& |\lambda_I^{(i)}(x) - m_I^{(i)}(y)| \\
\leq\ & \left|u_n^T(x-y)\right| + \left|\|\bar{M}\bar{U}_{n-1}^T x\| - \|\bar{M}\bar{U}_{n-1}^T y\|\right| \\
\leq\ & \|u_n\| \cdot \|x-y\| + \|\bar{M}\bar{U}_{n-1}^T(x-y)\| \\
\leq\ & \left(1 + \|\bar{M}\bar{U}_{n-1}^T\|\right) \cdot \|x-y\|, \quad \forall i = 1,2,
\end{aligned}$$

where the second inequality follows from the facts $|a^T b| \leq \|a\| \cdot \|b\|$, $|\|a\| - \|b\|| \leq \|a-b\|$, $\forall a, b \in \mathbb{R}^n$. $\square$

Lemma 2.2 (Perturbation of spectral vectors). *Let $u_I^{(i)}(x)$ $(i = 1, 2)$ be the spectral vectors of $x \in \mathbb{R}^n$ and $p_I^{(i)}(y)$ $(i = 1, 2)$ be the spectral vectors of $y \in \mathbb{R}^n$.*

(a) *If $\bar{U}_{n-1}^T x \neq 0$, $\bar{U}_{n-1}^T y \neq 0$, then we have*

$$\left\| u_I^{(i)}(x) - p_I^{(i)}(y) \right\| \leq \frac{1}{2} \left(\frac{\|\bar{U}_{n-1}\bar{U}_{n-1}^T\|}{\|\bar{M}\bar{U}_{n-1}^T x\|} + \frac{\|\bar{U}_{n-1}\bar{M}^{-1}\| \cdot \|\bar{M}\bar{U}_{n-1}^T\|}{\|\bar{M}\bar{U}_{n-1}^T x\|} \right) \cdot \|x - y\| \tag{2.2}$$

for any $i = 1, 2$. In this case, $u_I^{(i)}(x)$ and $p_I^{(i)}(y)$ are the unique spectral vectors of x and y, respectively.

(b) *If either $\bar{U}_{n-1}^T x = 0$ or $\bar{U}_{n-1}^T y = 0$, then we can choose $u_I^{(i)}(x)$, $p_I^{(i)}(y)$ such that the left hand side of the above inequality (2.2) is zero.*

Proof. (a) If $\bar{U}_{n-1}^T x \neq 0$, $\bar{U}_{n-1}^T y \neq 0$, according to the decomposition in Theorem 1.1 and (1.4), we obtain

$$\begin{aligned}
u_I^{(1)}(x) &= \frac{1}{2} \cdot \left(\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} + u_n \right), \quad u_I^{(2)}(x) = \frac{1}{2} \cdot \left(-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} + u_n \right), \\
p_I^{(1)}(y) &= \frac{1}{2} \cdot \left(\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T y}{\|\bar{M}\bar{U}_{n-1}^T y\|} + u_n \right), \quad p_I^{(2)}(y) = \frac{1}{2} \cdot \left(-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T y}{\|\bar{M}\bar{U}_{n-1}^T y\|} + u_n \right).
\end{aligned}$$

From the above, we see that $u_I^{(i)}(x)$, $p_I^{(i)}(y)$ $(i = 1, 2)$ are unique. In addition, we have

$$\begin{aligned}
&\|u_I^{(i)}(x) - p_I^{(i)}(y)\| \\
\leq\ & \frac{1}{2} \left\| \frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} - \frac{\bar{U}_{n-1}\bar{U}_{n-1}^T y}{\|\bar{M}\bar{U}_{n-1}^T y\|} \right\| \\
\leq\ & \frac{1}{2} \left\| \frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x - \bar{U}_{n-1}\bar{U}_{n-1}^T y}{\|\bar{M}\bar{U}_{n-1}^T x\|} + \frac{\bar{U}_{n-1}\bar{U}_{n-1}^T y}{\|\bar{M}\bar{U}_{n-1}^T x\|} - \frac{\bar{U}_{n-1}\bar{U}_{n-1}^T y}{\|\bar{M}\bar{U}_{n-1}^T y\|} \right\| \\
\leq\ & \frac{1}{2} \left(\frac{\|\bar{U}_{n-1}\bar{U}_{n-1}^T\|}{\|\bar{M}\bar{U}_{n-1}^T x\|} \cdot \|x - y\| + \frac{\|\bar{U}_{n-1}\bar{M}^{-1}\|}{\|\bar{M}\bar{U}_{n-1}^T x\|} \cdot \left| \|\bar{M}\bar{U}_{n-1}^T y\| - \|\bar{M}\bar{U}_{n-1}^T x\| \right| \right) \\
\leq\ & \frac{1}{2} \left(\frac{\|\bar{U}_{n-1}\bar{U}_{n-1}^T\|}{\|\bar{M}\bar{U}_{n-1}^T x\|} \cdot \|x - y\| + \frac{\|\bar{U}_{n-1}\bar{M}^{-1}\|}{\|\bar{M}\bar{U}_{n-1}^T x\|} \cdot \|\bar{M}\bar{U}_{n-1}^T (y - x)\| \right) \\
\leq\ & \frac{1}{2} \left(\frac{\|\bar{U}_{n-1}\bar{U}_{n-1}^T\|}{\|\bar{M}\bar{U}_{n-1}^T x\|} + \frac{\|\bar{U}_{n-1}\bar{M}^{-1}\| \cdot \|\bar{M}\bar{U}_{n-1}^T\|}{\|\bar{M}\bar{U}_{n-1}^T x\|} \right) \cdot \|x - y\|, \quad \forall i = 1, 2.
\end{aligned}$$

(b) It is clear that we can choose the same spectral vectors for x and y from the relation (1.4), since either $\bar{U}_{n-1}^T x = 0$ or $\bar{U}_{n-1}^T y = 0$. $\square$

Lemma 2.3 (Gradients)**.** *Let* $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{s\times n}$ *and* $x \in \mathbb{R}^n$*. If* $Bx \neq 0$*, then we have*

$$\nabla_x\left(\frac{Ax}{\|Bx\|}\right) = \frac{1}{\|Bx\|}\left[I_n - \frac{(B^TB)(xx^T)}{\|Bx\|^2}\right]A^T, \tag{2.3}$$

$$\nabla_x(\|Bx\|) = \frac{1}{\|Bx\|}B^TBx. \tag{2.4}$$

Proof. Let us rewrite

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{s1} & b_{s2} & \dots & b_{sn} \end{bmatrix} = \begin{bmatrix} B_{\cdot,1} & B_{\cdot,2} & \dots & B_{\cdot,n} \end{bmatrix} \in \mathbb{R}^{s\times n}$$

with $a_i \in \mathbb{R}^n$ $(i = 1, 2, \dots, n)$ and $B_{\cdot,j} \in \mathbb{R}^s$ $(j = 1, 2, \dots, n)$. Therefore, we have

$$\nabla_x\left(\frac{Ax}{\|Bx\|}\right) = \left[\nabla_x\left(\frac{a_1^Tx}{\|Bx\|}\right), \nabla_x\left(\frac{a_2^Tx}{\|Bx\|}\right), \dots, \nabla_x\left(\frac{a_n^Tx}{\|Bx\|}\right)\right],$$

where $\nabla_x\left(\frac{a_i^Tx}{\|Bx\|}\right)$ $(i = 1, 2, \dots, n)$ is defined as follows:

$$\nabla_x\left(\frac{a_i^Tx}{\|Bx\|}\right) = \begin{bmatrix} \frac{\partial}{\partial x_1}\left(\frac{a_i^Tx}{\|Bx\|}\right) \\ \frac{\partial}{\partial x_2}\left(\frac{a_i^Tx}{\|Bx\|}\right) \\ \vdots \\ \frac{\partial}{\partial x_n}\left(\frac{a_i^Tx}{\|Bx\|}\right) \end{bmatrix} \in \mathbb{R}^n.$$

By direct calculation, we have

$$\frac{\partial}{\partial x_j}\left(\frac{a_i^Tx}{\|Bx\|}\right) = a_{ij}\frac{1}{\|Bx\|} - (a_i^Tx)\frac{(Bx)^TB_{\cdot,j}}{\|Bx\|^3}, \quad j = 1, 2, \dots, n.$$

Consequently, we obtain

$$\begin{aligned} & \nabla_x\left(\frac{Ax}{\|Bx\|}\right) \\ = \ & \frac{1}{\|Bx\|}A^T - \frac{1}{\|Bx\|^3}[(Ax)(Bx)^TB]^T \\ = \ & \frac{1}{\|Bx\|}\left[I_n - \frac{(B^TB)(xx^T)}{\|Bx\|^2}\right]A^T, \end{aligned}$$

which shows that Eq. (2.3) holds. On the other hand, similar to the first part, we obtain

$$\frac{\partial}{\partial x_j}(\|Bx\|) = \frac{(Bx)^T B_{\cdot,j}}{\|Bx\|}, \quad j = 1, 2, \ldots, n,$$

and therefore the gradient of $\|Bx\|$ with respect to x is given by

$$\nabla_x(\|Bx\|) = \begin{bmatrix} \frac{\partial}{\partial x_1}(\|Bx\|) \\ \frac{\partial}{\partial x_2}(\|Bx\|) \\ \vdots \\ \frac{\partial}{\partial x_n}(\|Bx\|) \end{bmatrix} = \begin{bmatrix} \frac{(Bx)^T B_{\cdot,1}}{\|Bx\|} \\ \frac{(Bx)^T B_{\cdot,2}}{\|Bx\|} \\ \vdots \\ \frac{(Bx)^T B_{\cdot,n}}{\|Bx\|} \end{bmatrix} = \frac{1}{\|Bx\|} B^T Bx,$$

which implies that Eq. (2.4) is true. □

3. Smooth analysis

In this section, we aim to show the properties of continuity and differentiability between the scalar function f and its associated cone function f_I^{EC}. Now, after the above preparations, we are ready to present our first main result about continuity between f and f_I^{EC}.

Theorem 3.1 (Continuous)**.** *For any given function $f : \mathbb{R} \to \mathbb{R}$, let f_I^{EC} be its corresponding cone function defined as in (1.6). Then, the following statements are equivalent to each other.*

(a) *f is continuous at $\lambda_I^{(i)}(x)$ $(i = 1, 2)$.*

(b) *f_I^{EC} is continuous at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(i)}(x)$ $(i = 1, 2)$.*

Proof. (a) ⇒ (b) Suppose f is continuous at $\lambda_I^{(i)}(x)$ $(i = 1, 2)$. For any fixed $x \in \mathbb{R}^n$ and $y \to x$, let x and y be decomposed as

$$x = \lambda_I^{(1)}(x) \cdot u_I^{(1)}(x) + \lambda_I^{(2)}(x) \cdot u_I^{(2)}(x), \quad y = m_I^{(1)}(y) \cdot p_I^{(1)}(y) + m_I^{(2)}(y) \cdot p_I^{(2)}(y).$$

Then, we consider the following two cases:

Case (a): If $\bar{U}_{n-1}^T x \neq 0$, then we have

$$\begin{aligned} & f_I^{EC}(y) - f_I^{EC}(x) \\ = \; & f(m_I^{(1)}(y)) \cdot p_I^{(1)}(y) + f(m_I^{(2)}(y)) \cdot p_I^{(2)}(y) \\ & - f(\lambda_I^{(1)}(x)) \cdot u_I^{(1)}(x) - f(\lambda_I^{(2)}(x)) \cdot u_I^{(2)}(x) \\ = \; & f(m_I^{(1)}(y)) \cdot (p_I^{(1)}(y) - u_I^{(1)}(x)) + (f(m_I^{(1)}(y)) - f(\lambda_I^{(1)}(x))) \cdot u_I^{(1)}(x) \\ & + f(m_I^{(2)}(y)) \cdot (p_I^{(2)}(y) - u_I^{(2)}(x)) + (f(m_I^{(2)}(y)) - f(\lambda_I^{(2)}(x))) \cdot u_I^{(2)}(x). \end{aligned} \tag{3.1}$$

Since f is continuous at $\lambda_I^{(i)}(x)$ $(i = 1, 2)$ and the inequality (2.1) in Lemma 2.1, we obtain

$$f(m_I^{(i)}(y)) \to f(\lambda_I^{(i)}(x)) \; (i = 1, 2) \quad \text{as} \quad y \to x.$$

According to the relation (2.2) in Lemma 2.2, we also know

$$\|p_I^{(i)}(y) - u_I^{(i)}(x)\| \to 0 \ (i = 1,2) \quad \text{as} \quad y \to x.$$

Moreover, the equation (3.1) and the boundedness of $f(m_I^{(i)}(y)), u_I^{(i)}(x)$ yield that

$$f_I^{EC}(y) \to f_I^{EC}(x) \quad \text{as} \quad y \to x.$$

Therefore, f_I^{EC} is continuous at $x \in \mathbb{R}^n$.

Case (b): If $\bar{U}_{n-1}^T x = 0$, we can arrange that x, y have the same vector parts, regardless of $\bar{U}_{n-1}^T y$ is equal to zero or not. At the same time, we obtain

$$f_I^{EC}(y) - f_I^{EC}(x) = (f(m_I^{(1)}) - f(\lambda_I^{(1)})) \cdot u_I^{(1)} + (f(m_I^{(2)}) - f(\lambda_I^{(2)})) \cdot u_I^{(2)}.$$

By similar arguments as Case (a), we know that f_I^{EC} is continuous at $x \in \mathbb{R}^n$.

(b) $\Rightarrow$ (a) The proof for this direction is straightforward and has a similar arguments for [5, Proposition 2]. $\square$

Theorem 3.2 (Directionally Differentiable)**.** *For any given function $f : \mathbb{R} \to \mathbb{R}$, let f_I^{EC} be its corresponding cone function defined as in (1.6). Then, the following statements are equivalent to each other.*

(a) *f is directionally differentiable at $\lambda_I^{(i)}(x)$ $(i = 1,2)$.*

(b) *f_I^{EC} is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(i)}(x)$ $(i = 1,2)$.*

Proof. (a) $\Rightarrow$ (b) Suppose f is directionally differentiable at $\lambda_I^{(i)}(x)$ $(i = 1,2)$. We divide our proof into the following two cases:

Case (a): If $\bar{U}_{n-1}^T x \neq 0$, then we have

$$f(x) = f(\lambda_I^{(1)}(x)) \cdot u_I^{(1)}(x) + f(\lambda_I^{(2)}(x)) \cdot u_I^{(2)}(x),$$

where the scalars $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ and the vectors $u_I^{(1)}(x), u_I^{(2)}(x)$ are given by

$$\begin{aligned} \lambda_I^{(1)}(x) &= u_n^T x + \|\bar{M} U_{n-1}^T x\|, & \lambda_I^{(2)}(x) &= u_n^T x - \|\bar{M} U_{n-1}^T x\|, \\ u_I^{(1)}(x) &= \frac{1}{2} \cdot \left(\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} + u_n \right), & u_I^{(2)}(x) &= \frac{1}{2} \cdot \left(-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} + u_n \right). \end{aligned}$$

Due to the nonsingularity of $\bar{M}$ defined as in (1.3), we obtain $\bar{M}\bar{U}_{n-1}^T x \neq 0$. From Lemma 2.3, we know that $\lambda_I^i(x), u_I^{(i)}(x)$ $(i = 1,2)$ are Fréchet-differentiable with respect to the variable x, i.e.,

$$\begin{aligned} \nabla_x \lambda_I^{(1)}(x) &= u_n + \frac{1}{\|\bar{M}\bar{U}_{n-1}^T x\|} (\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T) x, \\ \nabla_x \lambda_I^{(2)}(x) &= u_n - \frac{1}{\|\bar{M}\bar{U}_{n-1}^T x\|} (\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T) x, \\ \nabla_x u_I^{(1)}(x) &= \frac{1}{2\|\bar{M}\bar{U}_{n-1}^T x\|} \left[I_n - \frac{(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)(xx^T)}{\|\bar{M}\bar{U}_{n-1}^T x\|^2} \right] (\bar{U}_{n-1}\bar{U}_{n-1}^T), \\ \nabla_x u_I^{(2)}(x) &= -\frac{1}{2\|\bar{M}\bar{U}_{n-1}^T x\|} \left[I_n - \frac{(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)(xx^T)}{\|\bar{M}\bar{U}_{n-1}^T x\|^2} \right] (\bar{U}_{n-1}\bar{U}_{n-1}^T). \end{aligned}$$

To show that f_I^{EC} is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(i)}(x)$ $(i = 1, 2)$, we only need to verify the directional differentiability of the composition functions $f(\lambda_I^{(i)}(x))$ $(i = 1, 2)$ with respect to $x \in \mathbb{R}^n$ and then use the product rule and the chain rule on f_I^{EC}.

Since f is directionally differentiable at $\lambda_I^{(1)}(x)$, then it is easy to see that

$$\begin{aligned}
\lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x) + t\cdot 1) - f(\lambda_I^{(1)}(x))}{t} &= f'(\lambda_I^{(1)}(x); 1),\\
\lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x) - t\cdot 1) - f(\lambda_I^{(1)}(x))}{t} &= f'(\lambda_I^{(1)}(x); -1),\\
\lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x) + o(t)) - f(\lambda_I^{(1)}(x))}{t} &= 0.
\end{aligned}$$

Using the fact that $\lambda_I^{(1)}(x)$ is Fréchet-differentiable at x, we obtain

$$\lambda_I^{(1)}(x + th) = \lambda_I^{(1)}(x) + t \cdot h^T \nabla_x \lambda_I^{(1)}(x) + o(t).$$

Let $y = h^T \nabla_x \lambda_I^{(1)}(x) + \frac{o(t)}{t}$, then $y \to h^T \nabla_x \lambda_I^{(1)}(x)$ as $t \to 0^+$. If $h^T \nabla_x \lambda_I^{(1)}(x) < 0$, then $y < 0$ as t is sufficiently close to 0 and we obtain

$$\begin{aligned}
& \lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x + th)) - f(\lambda_I^{(1)}(x))}{t}\\
= & \lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x) + ty) - f(\lambda_I^{(1)}(x))}{t}\\
= & \lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x) - (-ty)\cdot 1) - f(\lambda_I^{(1)}(x))}{(-ty)} \cdot (-y)\\
= & \lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x) - (-ty)\cdot 1) - f(\lambda_I^{(1)}(x))}{(-ty)} \cdot \lim_{t\to 0^+} (-y)\\
= & f'(\lambda_I^{(1)}(x); -1) \cdot (-h^T \nabla_x \lambda_I^{(1)}(x))\\
= & f'(\lambda_I^{(1)}(x); h^T \nabla_x \lambda_I^{(1)}(x)),
\end{aligned}$$

where the last equation follows from the positive homogeneous property of directionally differentiable functions. On the other hand, we can also deduce a similar result

$$\lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x + th)) - f(\lambda_I^{(1)}(x))}{t} = f'(\lambda_I^{(1)}(x); h^T \nabla_x \lambda_I^{(1)}(x))$$

under the case $h^T \nabla_x \lambda_I^{(1)}(x) \geq 0$. Therefore, the directional differentiability of $f(\lambda_I^{(1)}(x))$ with respect to $x \in \mathbb{R}^n$ is fulfilled and so does $f(\lambda_I^{(2)}(x))$ by repeating the above procedure. Consequently, we obtain

$$\begin{aligned}
& (f_I^{EC})'(x; h)\\
= & f'(\lambda_I^{(1)}(x); h^T \nabla_x \lambda_I^{(1)}(x)) \cdot u_I^{(1)}(x) + f'(\lambda_I^{(2)}(x); h^T \nabla_x \lambda_I^{(2)}(x)) \cdot u_I^{(2)}(x)\\
& + f(\lambda_I^{(1)}(x)) \cdot (\nabla_x u_I^{(1)}(x))^T h + f(\lambda_I^{(2)}(x)) \cdot (\nabla_x u_I^{(2)}(x))^T h,
\end{aligned} \tag{3.2}$$

where the terms $h^T\nabla_x\lambda_I^{(i)}(x)$, $(\nabla_x u_I^{(i)}(x))^T h$, $(i=1,2)$ are defined as follows:

$$
\begin{aligned}
h^T\nabla_x\lambda_I^{(1)}(x) &= u_n^T h + \frac{1}{\|\bar{M}\bar{U}_{n-1}^T x\|} x^T(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)h,\\
h^T\nabla_x\lambda_I^{(2)}(x) &= u_n^T h - \frac{1}{\|\bar{M}\bar{U}_{n-1}^T x\|} x^T(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)h,\\
(\nabla_x u_I^{(1)}(x))^T h &= \frac{1}{2\|\bar{M}\bar{U}_{n-1}^T x\|}(\bar{U}_{n-1}\bar{U}_{n-1}^T)\left[I_n - \frac{(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{\|\bar{M}\bar{U}_{n-1}^T x\|^2}\right]h,\\
(\nabla_x u_I^{(2)}(x))^T h &= -\frac{1}{2\|\bar{M}\bar{U}_{n-1}^T x\|}(\bar{U}_{n-1}\bar{U}_{n-1}^T)\left[I_n - \frac{(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{\|\bar{M}\bar{U}_{n-1}^T x\|^2}\right]h.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
& f'(\lambda_I^{(1)}(x); h^T\nabla_x\lambda_I^{(1)}(x))\cdot u_I^{(1)}(x) + f'(\lambda_I^{(2)}(x); h^T\nabla_x\lambda_I^{(2)}(x))\cdot u_I^{(2)}(x)\\
= \; & \frac{1}{2}\cdot\Bigg(f'(\lambda_I^{(1)}(x); u_n^T h + \frac{1}{\|\bar{M}\bar{U}_{n-1}^T x\|}x^T(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)h)\\
& - f'(\lambda_I^{(2)}(x); u_n^T h - \frac{1}{\|\bar{M}\bar{U}_{n-1}^T x\|}x^T(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)h)\Bigg)\cdot\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|}\\
& + \frac{1}{2}\cdot\Bigg(f'(\lambda_I^{(1)}(x); u_n^T h + \frac{1}{\|\bar{M}\bar{U}_{n-1}^T x\|}x^T(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)h)\\
& + f'(\lambda_I^{(2)}(x); u_n^T h - \frac{1}{\|\bar{M}\bar{U}_{n-1}^T x\|}x^T(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)h)\Bigg)\cdot u_n \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
& f(\lambda_I^{(1)}(x))\cdot(\nabla_x u_I^{(1)}(x))^T h + f(\lambda_I^{(2)}(x))\cdot(\nabla_x u_I^{(2)}(x))^T h\\
= \; & \frac{f(\lambda_I^{(1)}(x)) - f(\lambda_I^{(2)}(x))}{2\|\bar{M}\bar{U}_{n-1}^T x\|}\cdot(\bar{U}_{n-1}\bar{U}_{n-1}^T)\left[I_n - \frac{(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{\|\bar{M}\bar{U}_{n-1}^T x\|^2}\right]h\\
= \; & \frac{f(\lambda_I^{(1)}(x)) - f(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)}\cdot(\bar{U}_{n-1}\bar{U}_{n-1}^T)\left[I_n - \frac{(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{\|\bar{M}\bar{U}_{n-1}^T x\|^2}\right]h, \qquad (3.4)
\end{aligned}
$$

where the last equation uses the relation $\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x) = 2\|\bar{M}\bar{U}_{n-1}^T x\|$. From (3.2), we can rewrite $(f_I^{EC})'(x;h)$ in a more compact form as below:

$$
\begin{aligned}
& (f_I^{EC})'(x;h)\\
= \; & f'(\lambda_I^{(1)}(x); h^T\nabla_x\lambda_I^{(1)}(x))\cdot u_I^{(1)}(x) + f'(\lambda_I^{(2)}(x); h^T\nabla_x\lambda_I^{(2)}(x))\cdot u_I^{(2)}(x)\\
& + \frac{f(\lambda_I^{(1)}(x)) - f(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)}\cdot(\bar{U}_{n-1}\bar{U}_{n-1}^T)\left[I_n - \frac{(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{\|\bar{M}\bar{U}_{n-1}^T x\|^2}\right]h. \qquad (3.5)
\end{aligned}
$$

Case (b): If $\bar{U}_{n-1}^T x = 0$, then we have

$$f_I^{EC}(x) = f(\lambda_I^{(1)}(x))\cdot u_I^{(1)}(x) + f(\lambda_I^{(2)}(x))\cdot u_I^{(2)}(x),$$

where the scalars $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ and the vectors $u_I^{(1)}(x), u_I^{(2)}(x)$ are given by

$$\begin{aligned}
&\lambda_I^{(1)}(x) = u_n^T x,\ \lambda_I^{(2)}(x) = u_n^T x,\\
&u_I^{(1)}(x) = \tfrac{1}{2}\cdot\left(\tfrac{\bar{U}_{n-1}w}{\|\bar{M}w\|} + u_n\right),\ u_I^{(2)}(x) = \tfrac{1}{2}\cdot\left(-\tfrac{\bar{U}_{n-1}w}{\|\bar{M}w\|} + u_n\right),
\end{aligned}$$

where w is any given nonzero vector in $\mathbb{R}^{n-1}$. If $\bar{U}_{n-1}^T h \neq 0$, then $\bar{U}_{n-1}^T(x+th) \neq 0$ and $f_I^{EC}(x+th)$ has the following decomposition

$$f_I^{EC}(x+th) = f(\lambda_I^{(1)}(x+th))\cdot u_I^{(1)}(x+th) + f(\lambda_I^{(2)}(x+th))\cdot u_I^{(2)}(x+th),$$

where the scalars $\lambda_I^{(1)}(x+th), \lambda_I^{(2)}(x+th)$ and the vectors $u_I^{(1)}(x+th), u_I^{(2)}(x+th)$ satisfy the following relations

$$\begin{aligned}
\lambda_I^{(1)}(x+th) &= u_n^T(x+th) + \|\bar{M}\bar{U}_{n-1}^T(x+th)\| = \lambda_I^{(1)}(x) + t\lambda_I^{(1)}(h),\\
\lambda_I^{(2)}(x+th) &= u_n^T(x+th) - \|\bar{M}\bar{U}_{n-1}^T(x+th)\| = \lambda_I^{(2)}(x) + t\lambda_I^{(2)}(h),\\
u_I^{(1)}(x+th) &= \frac{1}{2}\cdot\left(\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T(x+th)}{\|\bar{M}\bar{U}_{n-1}^T(x+th)\|} + u_n\right) := u_I^{(1)}(h),\\
u_I^{(1)}(x+th) &= \frac{1}{2}\cdot\left(-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T(x+th)}{\|\bar{M}\bar{U}_{n-1}^T(x+th)\|} + u_n\right) := u_I^{(2)}(h).
\end{aligned}$$

In addition, if we choose $w = \bar{U}_{n-1}^T h$, then $u_I^{(1)}(x) = u_I^{(1)}(x+th) = u_I^{(1)}(h)$ and $u_I^{(2)}(x) = u_I^{(2)}(x+th) = u_I^{(2)}(h)$, which show that

$$\begin{aligned}
&\frac{f_I^{EC}(x+th) - f_I^{EC}(x)}{t}\\
=\ &\frac{f(\lambda_I^{(1)}(x) + t\lambda_I^{(1)}(h)) - f(\lambda_I^{(1)}(x))}{t}\cdot u_I^{(1)}(h)\\
&+\frac{f(\lambda_I^{(2)}(x) + t\lambda_I^{(2)}(h)) - f(\lambda_I^{(2)}(x))}{t}\cdot u_I^{(2)}(h).
\end{aligned}$$

Therefore, the following relation is fulfilled under the directionally differentiability of f at $\lambda_I^{(i)}(x)$ $(i = 1, 2)$:

$$(f_I^{EC})'(x;h) = f'(\lambda_I^{(1)}(x);\lambda_I^{(1)}(h))\cdot u_I^{(1)}(h) + f'(\lambda_I^{(2)}(x);\lambda_I^{(2)}(h))\cdot u_I^{(2)}(h). \tag{3.6}$$

On the other hand, if $\bar{U}_{n-1}^T h = 0$, then $\bar{U}_{n-1}^T(x+th) = 0$. In this case, we know

$$f_I^{EC}(x+th) = f(\lambda_I^{(1)}(x+th))\cdot u_I^{(1)}(x+th) + f(\lambda_I^{(2)}(x+th))\cdot u_I^{(2)}(x+th),$$

where the scalars $\lambda_I^{(1)}(x+th), \lambda_I^{(2)}(x+th)$ and the vectors $u_I^{(1)}(x+th), u_I^{(2)}(x+th)$ now can be rewritten as follows:

$$\begin{aligned}
\lambda_I^{(1)}(x+th) &= u_n^T(x+th) = \lambda_I^{(1)}(x) + t\lambda_I^{(1)}(h),\\
\lambda_I^{(2)}(x+th) &= u_n^T(x+th) = \lambda_I^{(2)}(x) + t\lambda_I^{(2)}(h),\\
u_I^{(1)}(x+th) &= \frac{1}{2}\cdot\left(\frac{\bar{U}_{n-1}\eta}{\|\bar{M}\eta\|} + u_n\right) := u_I^{(1)},\\
u_I^{(1)}(x+th) &= \frac{1}{2}\cdot\left(-\frac{\bar{U}_{n-1}\eta}{\|\bar{M}\eta\|} + u_n\right) := u_I^{(2)},
\end{aligned}$$

where η is any given nonzero vector in $\mathbb{R}^{n-1}$. In addition, if we choose $w = \eta \neq 0$, then $u_I^{(1)}(x) = u_I^{(1)}(x+th) = u_I^{(1)}, u_I^{(2)}(x) = u_I^{(2)}(x+th) = u_I^{(2)}$. Similarly, we have

$$(f_I^{EC})'(x;h) = f'(\lambda_I^{(1)}(x);\lambda_I^{(1)}(h)) \cdot u_I^{(1)} + f'(\lambda_I^{(2)}(x);\lambda_I^{(2)}(h)) \cdot u_I^{(2)}. \tag{3.7}$$

In summary, we show that f_I^{EC} is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(i)}(x)$ $(i = 1, 2)$.

(b) $\Rightarrow$ (a) The proof for this direction is trivial by adapting the arguments for [5, Proposition 3]. $\square$

Theorem 3.3 (Differentiable). *For any given function $f : \mathbb{R} \to \mathbb{R}$, let f_I^{EC} be its corresponding cone function defined as in (1.6). Then, the following statements are equivalent to each other.*

(a) *f is differentiable at $\lambda_I^{(i)}(x)$ $(i = 1, 2)$.*

(b) *f_I^{EC} is differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(i)}(x)$ $(i = 1, 2)$.*

Moreover, the corresponding Jacobian of f_I^{EC} at x is defined as follows:

$$\begin{aligned}
&\mathcal{D}f_I^{EC}(x) \\
= \; &(\bar{U}_{n-1}\bar{U}_{n-1}^T)\left[\frac{f'(\lambda_I^{(1)}(x)) - f'(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)}(xu_n^T)\right. \\
&+2 \cdot \frac{f'(\lambda_I^{(1)}(x)) + f'(\lambda_I^{(2)}(x))}{(\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x))^2}(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T) \\
&\left.+\frac{f(\lambda_I^{(1)}(x)) - f(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)}\left(I_n - 4 \cdot \frac{(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{(\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x))^2}\right)\right] \\
&+(u_n u_n^T)\left[\frac{f'(\lambda_I^{(1)}(x)) - f'(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)}(u_n x^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)\right. \\
&\left.+\frac{f'(\lambda_I^{(1)}(x)) + f'(\lambda_I^{(2)}(x))}{2}I_n\right]
\end{aligned} \tag{3.8}$$

if $\bar{U}_{n-1}^T x \neq 0$; otherwise,

$$\mathcal{D}f_I^{EC}(x) = f'(u_n^T x)I_n. \tag{3.9}$$

Proof. (a) $\Rightarrow$ (b) The proof for this direction can be adapted from Theorem 3.2, in which we only need to use "differentiable" to replace "directionally differentiable". At the same time, we know that $f'(\lambda_I^{(i)}(x), \cdot)$ $(i = 1, 2)$ are linear, in other words,

$$f'(\lambda_I^{(i)}(x), a+b) = f'(\lambda_I^{(i)}(x))a + f'(\lambda_I^{(i)}(x))b, \quad \forall a, b \in \mathbb{R}, \tag{3.10}$$

since f is differentiable at $\lambda_I^{(i)}(x)$ $(i = 1, 2)$.

Next, the remaining part will be verified under the following two cases:

Case (a): If $\bar{U}_{n-1}^T x \neq 0$, according to the relations (3.3)-(3.5) and (3.10), then we have

$$\begin{aligned}
& (f_I^{EC})'(x;h) \\
= \ & \frac{1}{2}\Bigg[(f'(\lambda_I^{(1)}(x)) - f'(\lambda_I^{(2)}(x)))u_n^T h \\
& + \frac{f'(\lambda_I^{(1)}(x)) + f'(\lambda_I^{(2)}(x))}{\|\bar{M}\bar{U}_{n-1}^T x\|} x^T(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)h\Bigg] \frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} \\
& + \frac{1}{2}\Bigg[(f'(\lambda_I^{(1)}(x)) + f'(\lambda_I^{(2)}(x)))u_n^T h \\
& + \frac{f'(\lambda_I^{(1)}(x)) - f'(\lambda_I^{(2)}(x))}{\|\bar{M}\bar{U}_{n-1}^T x\|} x^T(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)h\Bigg] u_n \\
& + \frac{f(\lambda_I^{(1)}(x)) - f(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)} \cdot (\bar{U}_{n-1}\bar{U}_{n-1}^T)\left[I_n - \frac{(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{\|\bar{M}\bar{U}_{n-1}^T x\|^2}\right] h \\
= \ & (\bar{U}_{n-1}\bar{U}_{n-1}^T)\Bigg[\frac{f'(\lambda_I^{(1)}(x)) - f'(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)}(xu_n^T) \\
& + 2\cdot\frac{f'(\lambda_I^{(1)}(x)) + f'(\lambda_I^{(2)}(x))}{(\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x))^2}(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T) \\
& + \frac{f(\lambda_I^{(1)}(x)) - f(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)}\left(I_n - 4\cdot\frac{(xx^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{(\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x))^2}\right)\Bigg] h \\
& + (u_n u_n^T)\Bigg[\frac{f'(\lambda_I^{(1)}(x)) - f'(\lambda_I^{(2)}(x))}{\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x)}(u_n x^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T) \\
& + \frac{f'(\lambda_I^{(1)}(x)) + f'(\lambda_I^{(2)}(x))}{2} I_n\Bigg] h,
\end{aligned}$$

where the last equation follows from the fact $\lambda_I^{(1)}(x) - \lambda_I^{(2)}(x) = 2\|\bar{M}\bar{U}_{n-1}^T x\|$. The above relation shows that $(f_I^{EC})'(x;h) = \mathcal{D}f_I^{EC}(x)h$, where $\mathcal{D}f_I^{EC}(x)$ is defined as in (3.8).

Case (b): If $\bar{U}_{n-1}^T x = 0$, then $\lambda_I^{(1)}(x) = \lambda_I^{(2)}(x) = u_n^T x$. In addition, if $\bar{U}_{n-1}^T h \neq 0$, similar to the above discussion in Case (a), we obtain

$$\begin{aligned}
(f_I^{EC})'(x;h) &= f'(\lambda_I^{(1)}(x))\lambda_I^{(1)}(h)\cdot u_I^{(1)}(h) + f'(\lambda_I^{(2)}(x))\lambda_I^{(2)}(h)\cdot u_I^{(2)}(h) \\
&= f'(u_n^T x)(u_n^T h + \|\bar{M}\bar{U}_{n-1}^T h\|)\cdot\frac{1}{2}\left(\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T h}{\|\bar{M}\bar{U}_{n-1}^T h\|} + u_n\right) \\
&\quad + f'(u_n^T x)(u_n^T h - \|\bar{M}\bar{U}_{n-1}^T h\|)\cdot\frac{1}{2}\left(-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T h}{\|\bar{M}\bar{U}_{n-1}^T h\|} + u_n\right);
\end{aligned}$$

otherwise, we have

$$\begin{aligned}
&(f_I^{EC})'(x;h)\\
= \;& f'(\lambda_I^{(1)}(x))\lambda_I^{(1)}(h)\cdot u_I^{(1)} + f'(\lambda_I^{(2)}(x))\lambda_I^{(2)}(h)\cdot u_I^{(2)}\\
= \;& f'(u_n^T x)(u_n^T h)\cdot \frac{1}{2}\left(\frac{\bar{U}_{n-1}\eta}{\|\bar{M}\eta\|} + u_n\right) + f'(u_n^T x)(u_n^T h)\cdot \frac{1}{2}\left(-\frac{\bar{U}_{n-1}\eta}{\|\bar{M}\eta\|} + u_n\right).
\end{aligned}$$

By direct calculation, in both cases the following relations hold:

$$(f_I^{EC})'(x;h) = f'(u_n^T x)(u_n u_n^T + \bar{U}_{n-1}\bar{U}_{n-1}^T)h = f'(u_n^T x)I_n h,$$

where the last equation uses the fact $u_n u_n^T + \bar{U}_{n-1}\bar{U}_{n-1}^T = I_n$. Therefore, the relation (3.9) is fulfilled under this case.

(b) $\Rightarrow$ (a) Let f_I^{EC} be differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(i)}(x)$ $(i = 1, 2)$. By contradiction, without loss of generality, assume that f is not differentiable at $\lambda_I^{(1)}(x)$, the following limits

$$\lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x)+t) - f(\lambda_I^{(1)}(x))}{t},$$

$$\lim_{t\to 0^-} \frac{f(\lambda_I^{(1)}(x)+t) - f(\lambda_I^{(1)}(x))}{t}$$

either are unequal or one of them does not exist. Now, we choose

$$\begin{aligned}
x &= \lambda_I^{(1)}(x)\cdot u_I^{(1)}(x) + \lambda_I^{(2)}(x)\cdot u_I^{(2)}(x),\\
h &= 1\cdot u_I^{(1)}(x) + 0\cdot u_I^{(2)}(x).
\end{aligned}$$

Then, we know $x + th = (\lambda_I^{(1)}(x) + t)\cdot u_I^{(1)}(x) + \lambda_I^{(2)}(x)\cdot u_I^{(2)}(x)$ and $f_I^{EC}(x + th) = f(\lambda_I^{(1)}(x) + t)\cdot u_I^{(1)}(x) + f(\lambda_I^{(2)}(x))\cdot u_I^{(2)}(x)$, which implies

$$\begin{aligned}
\lim_{t\to 0^+} \frac{f_I^{EC}(x+th) - f_I^{EC}(x)}{t} &= \lim_{t\to 0^+} \frac{f(\lambda_I^{(1)}(x)+t) - f(\lambda_I^{(1)}(x))}{t}\cdot u_I^{(1)}(x),\\
\lim_{t\to 0^-} \frac{f_I^{EC}(x+th) - f_I^{EC}(x)}{t} &= \lim_{t\to 0^-} \frac{f(\lambda_I^{(1)}(x)+t) - f(\lambda_I^{(1)}(x))}{t}\cdot u_I^{(1)}(x).
\end{aligned}$$

It follows that these two limits either are unequal or one of them does not exist, which contradicts with the assumption that f_I^{EC} is differentiable at $x \in \mathbb{R}^n$.

Theorem 3.4 (Continuously Differentiable). *For any given function $f : \mathbb{R} \to \mathbb{R}$, let f_I^{EC} be its corresponding cone function defined as in (1.6). Then, the following statements are equivalent to each other.*

(a) *f is continuously differentiable at $\lambda_I^{(i)}(x)$ $(i = 1, 2)$.*

(b) *f_I^{EC} is continuously differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(i)}(x)$ $(i = 1, 2)$.*

Proof. (a) $\Rightarrow$ (b) Suppose f is continuously differentiable at $\lambda_I^{(i)}(x)$ $(i = 1, 2)$. If $\bar{U}_{n-1}^T x \neq 0$, it follows from (3.8) that $\mathcal{D}f_I^{EC}$ is continuous at $x \in \mathbb{R}^n$. We only need to verify that $\mathcal{D}f_I^{EC}$ is continuous at every $x \in \mathbb{R}^n$ with $\bar{U}_{n-1}^T x = 0$. In this case, we know

$$\lambda_I^{(1)}(x) = \lambda_I^{(2)}(x) = u_n^T x. \tag{3.11}$$

Let y^ν be any sequence converging to x. If $\bar{U}_{n-1}^T y^\nu = 0$, from (3.9) we obtain

$$\lim_{y^\nu \to x, \bar{U}_{n-1}^T y^\nu = 0} \mathcal{D}f_I^{EC}(y^\nu) = \lim_{y^\nu \to x, \bar{U}_{n-1}^T y^\nu = 0} f'(u^T y^\nu) I_n = \mathcal{D}f_I^{EC}(x); \tag{3.12}$$

otherwise, i.e., $\bar{U}_{n-1}^T y^\nu \neq 0$. According to the relation (3.8), we have

$$\begin{aligned}
& \mathcal{D}f_I^{EC}(y^\nu) \\
= \; & (\bar{U}_{n-1}\bar{U}_{n-1}^T)\left[\frac{f'(\lambda_I^{(1)}(y^\nu)) - f'(\lambda_I^{(2)}(y^\nu))}{\lambda_I^{(1)}(y^\nu) - \lambda_I^{(2)}(y^\nu)}(y^\nu u_n^T)\right. \\
& +2\cdot\frac{f'(\lambda_I^{(1)}(y^\nu)) + f'(\lambda_I^{(2)}(y^\nu))}{(\lambda_I^{(1)}(y^\nu) - \lambda_I^{(2)}(y^\nu))^2}(y^\nu (y^\nu)^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T) \\
& \left.+\frac{f(\lambda_I^{(1)}(y^\nu)) - f(\lambda_I^{(2)}(y^\nu))}{\lambda_I^{(1)}(y^\nu) - \lambda_I^{(2)}(y^\nu)}\left(I_n - 4\cdot\frac{(y^\nu (y^\nu)^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{(\lambda_I^{(1)}(y^\nu) - \lambda_I^{(2)}(y^\nu))^2}\right)\right] \\
& +(u_n u_n^T)\left[\frac{f'(\lambda_I^{(1)}(y^\nu)) - f'(\lambda_I^{(2)}(y^\nu))}{\lambda_I^{(1)}(y^\nu) - \lambda_I^{(2)}(y^\nu)}(u_n (y^\nu)^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)\right. \\
& \left.+\frac{f'(\lambda_I^{(1)}(y^\nu)) + f'(\lambda_I^{(2)}(y^\nu))}{2} I_n\right]
\end{aligned}$$

In addition, the following relation holds when $y^\nu \to x$ and $\bar{U}_{n-1}^T y^\nu \neq 0$:

$$\begin{aligned}
& \lim_{y^\nu \to x, \bar{U}_{n-1}^T y^\nu \neq 0} \frac{f'(\lambda_I^{(1)}(y^\nu)) - f'(\lambda_I^{(2)}(y^\nu))}{\lambda_I^{(1)}(y^\nu) - \lambda_I^{(2)}(y^\nu)}(\bar{U}_{n-1}\bar{U}_{n-1}^T)(y^\nu u_n^T) \\
= \; & \lim_{y^\nu \to x, \bar{U}_{n-1}^T y^\nu \neq 0} \frac{f'(\lambda_I^{(1)}(y^\nu)) - f'(\lambda_I^{(2)}(y^\nu))}{2}(\bar{U}_{n-1}\bar{M}^{-1})\frac{\bar{M}\bar{U}_{n-1}^T y^\nu}{\|\bar{M}\bar{U}_{n-1}^T y^\nu\|}u_n^T \\
= \; & 0,
\end{aligned}$$

where the last equation follows from the differentiability of f at $\lambda_I^{(i)}(x)$, $f'(\lambda_I^{(i)}(y^\nu)) \to f'(\lambda_I^{(i)}(x))$ $(i = 1, 2)$, $\lambda_I^{(1)}(y^\nu) - \lambda_I^{(2)}(y^\nu) = 2\|\bar{M}\bar{U}_{n-1}^T y^\nu\|$, $\dfrac{\bar{M}\bar{U}_{n-1}^T y^\nu}{\|\bar{M}\bar{U}_{n-1}^T y^\nu\|}$ is bounded and (3.11). For simplicity, we assume that

$$\lim_{y^\nu \to x, \bar{U}_{n-1}^T y^\nu \neq 0} \frac{\bar{M}\bar{U}_{n-1}^T y^\nu}{\|\bar{M}\bar{U}_{n-1}^T y^\nu\|} \to \xi \in \mathbb{R}^{n-1} \quad \text{with} \quad \|\xi\| = 1.$$

Similarly, we obtain

$$\begin{aligned}
&\lim_{y^\nu\to x,\bar{U}_{n-1}^T y^\nu\neq 0}\Bigg[2\cdot\frac{f'(\lambda_I^{(1)}(y^\nu))+f'(\lambda_I^{(2)}(y^\nu))}{(\lambda_I^{(1)}(y^\nu)-\lambda_I^{(2)}(y^\nu))^2}\\
&\qquad\cdot(\bar{U}_{n-1}\bar{U}_{n-1}^T)(y^\nu(y^\nu)^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)\Bigg]\\
=\;& f'(u_n^Tx)\bar{U}_{n-1}\bar{M}^{-1}\xi\xi^T\bar{M}\bar{U}_{n-1}^T,\\
&\lim_{y^\nu\to x,\bar{U}_{n-1}^T y^\nu\neq 0}\Bigg[\frac{f(\lambda_I^{(1)}(y^\nu))-f(\lambda_I^{(2)}(y^\nu))}{\lambda_I^{(1)}(y^\nu)-\lambda_I^{(2)}(y^\nu)}\\
&\qquad\cdot(\bar{U}_{n-1}\bar{U}_{n-1}^T)\left(I_n-4\cdot\frac{(y^\nu(y^\nu)^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)}{(\lambda_I^{(1)}(y^\nu)-\lambda_I^{(2)}(y^\nu))^2}\right)\Bigg]\\
=\;& f'(u_n^Tx)\bar{U}_{n-1}\bar{U}_{n-1}^T-f'(u_n^Tx)\bar{U}_{n-1}\bar{M}^{-1}\xi\xi^T\bar{M}\bar{U}_{n-1}^T,\\
&\lim_{y^\nu\to x,\bar{U}_{n-1}^T y^\nu\neq 0}(u_nu_n^T)\Bigg[\frac{f'(\lambda_I^{(1)}(y^\nu))-f'(\lambda_I^{(2)}(y^\nu))}{\lambda_I^{(1)}(y^\nu)-\lambda_I^{(2)}(y^\nu)}(u_n(y^\nu)^T)(\bar{U}_{n-1}\bar{M}^T\bar{M}\bar{U}_{n-1}^T)\\
&\qquad+\frac{f'(\lambda_I^{(1)}(y^\nu))+f'(\lambda_I^{(2)}(y^\nu))}{2}I_n\Bigg]\\
=\;&\lim_{y^\nu\to x,\bar{U}_{n-1}^T y^\nu\neq 0}(u_nu_n^T)\Bigg[\frac{f'(\lambda_I^{(1)}(y^\nu))-f'(\lambda_I^{(2)}(y^\nu))}{2}u_n\left(\frac{\bar{M}\bar{U}_{n-1}^Ty^\nu}{\|\bar{M}\bar{U}_{n-1}^Ty^\nu\|}\right)^T\bar{M}\bar{U}_{n-1}^T\\
&\qquad+\frac{f'(\lambda_I^{(1)}(y^\nu))+f'(\lambda_I^{(2)}(y^\nu))}{2}I_n\Bigg]\\
=\;& f'(u_n^Tx)u_nu_n^T.
\end{aligned}$$

Summing up these equations, we obtain

$$\lim_{y^\nu\to x,\bar{U}_{n-1}^T y^\nu\neq 0}\mathcal{D}f_I^{EC}(y^\nu)=f'(u_n^Tx)(u_nu_n^T+\bar{U}_{n-1}\bar{U}_{n-1}^T)=f'(u_n^Tx)I_n=\mathcal{D}f_I^{EC}(x).$$

This together with (3.12) imply that $\mathcal{D}f_I^{EC}$ is continuous at every $x\in\mathbb{R}^n$ with $\bar{U}_{n-1}^Tx=0$.

(b) $\Rightarrow$ (a) Suppose f_I^{EC} is continuously differentiable at $x\in\mathbb{R}^n$ with spectral values $\lambda_I^{(i)}(x)$ $(i=1,2)$. From Theorem 3.3, f is differentiable at the neighborhoods around $\lambda_I^{(i)}(x)$ $(i=1,2)$. If $\bar{U}_{n-1}^Tx=0$, then

$$\lambda_I^{(1)}(x)=\lambda_I^{(2)}(x)=u_n^Tx,\quad \mathcal{D}f_I^{EC}(x)=f'(u_n^Tx)I_n. \tag{3.13}$$

For any $h\in\mathbb{R}^{n-1}$ and $\bar{U}_{n-1}^Th=0$, then $\bar{U}_{n-1}^T(x+h)=0$ and hence $\mathcal{D}f_I^{EC}(x+h)=f'(u_n^T(x+h))I_n$. Since $\mathcal{D}f_I^{EC}(x)$ is continuous at x, then $\lim_{h\to 0}\mathcal{D}f_I^{EC}(x+h)=\mathcal{D}f_I^{EC}(x)$, which implies $\lim_{h\to 0}f'(u_n^T(x+h))=f'(u_n^Tx)$. This together with (3.13) show that $f'(x)$ is continuous at $\lambda_I^{(i)}(x)$ $(i=1,2)$. On the other hand, similar to the one for [5, Proposition 5], through adapting its proof, the same result is also fulfilled under this case. $\square$

4. Examples

According to Theorem 1.1, in this section we investigate on some properties of three special cases for ellipsoidal cone in the following examples.

Example 4.1. Consider the second-order cone

$$\mathcal{K}^n := \left\{ (\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}\| \leq x_n \right\}.$$

In this case, we know

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}, \; \bar{U}_{n-1} = \bar{E}_{n-1}, \; u_n = e_n, \; \bar{M} = I_{n-1}, \; \lambda_n = -1,$$

where $\bar{E}_{n-1} := [e_1, e_2, \ldots, e_{n-1}] \in \mathbb{R}^{n\times(n-1)}$. With respect to the second-order cone $\mathcal{K}^n$, we can decompose $x \in \mathbb{R}^n$ as

$$x = \lambda_I^{(1)}(x) \cdot u_I^{(1)}(x) + \lambda_I^{(2)}(x) \cdot u_I^{(2)}(x)$$

with $\lambda_I^{(i)}(x), u_I^{(i)}(x)$ $(i = 1, 2)$ given by

$$\begin{aligned} \lambda_I^{(i)}(x) &= x_n + (-1)^{i+1}\|\bar{x}_{n-1}\|, \\ u_I^{(i)}(x) &= \begin{cases} \frac{1}{2}\begin{bmatrix} (-1)^{i+1}\dfrac{\bar{x}_{n-1}}{\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} & \text{if } \bar{x}_{n-1} \neq 0, \\ \frac{1}{2}\begin{bmatrix} (-1)^{i+1}\dfrac{w}{\|w\|} \\ 1 \end{bmatrix} & \text{if } \bar{x}_{n-1} = 0, \end{cases} \end{aligned} \tag{4.1}$$

where w is any given nonzero vector in $\mathbb{R}^{n-1}$. It is easy to see that the above decomposition reduces to the classical decomposition expression associated with the second-order cone [5, 11]. Let $f : \mathbb{R} \to \mathbb{R}$ be a scalar function, the corresponding second-order cone function is given by

$$f^{soc}(x) := f(\lambda_I^{(1)}(x)) \cdot u_I^{(1)}(x) + f(\lambda_I^{(2)}(x)) \cdot u_I^{(2)}(x),$$

where $\lambda_I^{(i)}(x)$, $u_I^{(i)}(x)$ $(i = 1, 2)$ are defined as in (4.1). Similar to the above theorems, we can obtain the following relations between f and f^{soc}, which is also found in [5, Section 5].

Theorem 4.2 (Second-order Cone Case)**.** *For any given function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}^n$, let $\lambda_I^{(i)}(x), u_I^{(i)}(x)$ $(i = 1, 2)$ be defined as in (4.1). The following statements hold:*

(a) *f is continuous at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{soc} is continuous at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

(b) *f is directionally differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{soc} is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

(c) *f is differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{soc} is differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

(d) *f is continuously differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{soc} is continuously differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

Example 4.3. Consider the circular cone

$$\mathcal{L}_\theta := \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}_{n-1}\| \le x_n \tan\theta\right\}$$

In this case, we know

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2\theta \end{bmatrix}, \; \bar{U}_{n-1} = \bar{E}_{n-1}, \; u_n = e_n, \; \bar{M} = I_{n-1}, \; \lambda_n = -\tan^2\theta.$$

With respect to the circular cone $\mathcal{L}_\theta$, we can decompose $x \in \mathbb{R}^n$ as

$$x = \lambda_I^{(1)}(x) \cdot u_I^{(1)}(x) + \lambda_I^{(2)}(x) \cdot u_I^{(2)}(x)$$

with $\lambda_I^{(i)}(x), u_I^{(i)}(x)$ $(i = 1, 2)$ given by

$$\begin{aligned} \lambda_I^{(i)}(x) &= x_n + (-1)^{i+1} \cot\theta \|\bar{x}_{n-1}\|, \\ u_I^{(i)}(x) &= \begin{cases} \frac{1}{2}\begin{bmatrix} (-1)^{i+1}\dfrac{\bar{x}_{n-1}}{\cot\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} & \text{if } \bar{x}_{n-1} \neq 0, \\ \frac{1}{2}\begin{bmatrix} (-1)^{i+1}\dfrac{w}{\cot\theta\|w\|} \\ 1 \end{bmatrix} & \text{if } \bar{x}_{n-1} = 0, \end{cases} \end{aligned} \tag{4.2}$$

where w is any given nonzero vector in $\mathbb{R}^{n-1}$. Notice that the above decomposition is different with the existing decomposition expression associated with the circular cone, see [24, Theorem 3.1] for more details. Let $f : \mathbb{R} \to \mathbb{R}$ be a scalar function, the corresponding circular cone function is given by

$$f^{circ}(x) := f(\lambda_I^{(1)}(x)) \cdot u_I^{(1)}(x) + f(\lambda_I^{(2)}(x)) \cdot u_I^{(2)}(x),$$

where $\lambda_I^{(i)}(x)$, $u_I^{(i)}(x)$ $(i = 1, 2)$ are defined as in (4.2). Consequently, we can also obtain the similar relations between f and f^{cir} in the following theorem.

Theorem 4.4 (Circular Cone Case)**.** *For any given function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}^n$, let $\lambda_I^{(i)}(x), u_I^{(i)}(x)$ $(i = 1, 2)$ be defined as in (4.2). The following statements hold:*

- (a) *f is continuous at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{circ} is continuous at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*
- (b) *f is directionally differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{circ} is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*
- (c) *f is differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{circ} is differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*
- (d) *f is continuously differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{circ} is continuously differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

Example 4.5. Consider the elliptic cone

$$\mathcal{K}_M^n := \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|M\bar{x}_{n-1}\| \le x_n\right\}.$$

In this case, we know

$$Q = \begin{bmatrix} M^TM & 0 \\ 0 & -1 \end{bmatrix}, \ \bar{U}_{n-1} = \begin{bmatrix} \bar{U}_{n-1,n-1} \\ 0 \end{bmatrix}, \ u_n = e_n,$$
$$\bar{M} = (\bar{U}_{n-1,n-1}^T M^T M \bar{U}_{n-1,n-1})^{1/2}, \ \lambda_n = -1,$$

where M is any nonsingular matrix of order $n-1$ and $\bar{U}_{n-1,n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ is an orthogonal matrix satisfying the condition $\bar{U}_{n-1,n-1}\bar{M}^T\bar{M}\bar{U}_{n-1,n-1}^T = M^TM$. Therefore, we obtain

$$\begin{aligned}
(\bar{M}\bar{U}_{n-1}^T x)^T \bar{M}\bar{U}_{n-1}^T x &= x^T \bar{U}_{n-1}\bar{U}_{n-1,n-1}^T M^T M \bar{U}_{n-1,n-1}\bar{U}_{n-1}^T x \\
&= \bar{x}_{n-1}^T M^T M \bar{x}_{n-1}, \\
(\bar{M}^{-1}\bar{U}_{n-1}^T x)^T \bar{M}^{-1}\bar{U}_{n-1}^T x &= x^T \bar{U}_{n-1}\bar{U}_{n-1,n-1}^T M^T M \bar{U}_{n-1,n-1}\bar{U}_{n-1}^T x \\
&= \bar{x}_{n-1}^T M^T M \bar{x}_{n-1},
\end{aligned}$$

which show that $\|\bar{M}\bar{U}_{n-1}^T x\| = \|M\bar{x}_{n-1}\|$ and $\|\bar{M}^{-1}\bar{U}_{n-1}^T x\| = \|(M^{-1})^T\bar{x}_{n-1}\|$. If we set $w := \bar{U}_{n-1,n-1}^T\eta$, then $\eta \neq 0$ and $\bar{U}_{n-1}w = (\eta, 0) \in \mathbb{R}^{n-1}\times\mathbb{R}$, since $w \neq 0$ and the orthogonal property of $\bar{U}_{n-1.n-1}$. Moreover, by simple calculation, we also obtain

$$\begin{aligned}
(\bar{M}w)^T\bar{M}w &= w^T\bar{U}_{n-1,n-1}^T M^T M \bar{U}_{n-1,n-1} w \\
&= \eta^T M^T M \eta, \\
(\bar{M}^{-1}w)^T\bar{M}^{-1}w &= w^T\bar{U}_{n-1,n-1}^T M^{-1}(M^T)^{-1}\bar{U}_{n-1,n-1} w \\
&= \eta^T((M^{-1})^T)^T(M^{-1})^T\eta,
\end{aligned}$$

therefore we have $\|\bar{M}w\| = \|M\eta\|$ and $\|\bar{M}^{-1}w\| = \|(M^{-1})^T\eta\|$. With respect to the elliptic cone $\mathcal{K}_M^n$, we can decompose $x \in \mathbb{R}^n$ as

$$x = \lambda_I^{(1)}(x)\cdot u_I^{(1)}(x) + \lambda_I^{(2)}(x)\cdot u_I^{(2)}(x)$$

with $\lambda_I^{(i)}(x), u_I^{(i)}(x)$ $(i = 1,2)$ given by

$$\begin{aligned}
&\lambda_I^{(i)}(x) = x_n + (-1)^{i+1}\|M\bar{x}_{n-1}\|, \\
&u_I^{(i)}(x) = \begin{cases} \frac{1}{2}\begin{bmatrix} (-1)^{i+1}\dfrac{\bar{x}_{n-1}}{\|M\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} & \text{if } \bar{x}_{n-1} \neq 0, \\ \frac{1}{2}\begin{bmatrix} (-1)^{i+1}\dfrac{\eta}{\|M\eta\|} \\ 1 \end{bmatrix} & \text{if } \bar{x}_{n-1} = 0, \end{cases}
\end{aligned} \tag{4.3}$$

where η is any given nonzero vector in $\mathbb{R}^{n-1}$. Again, let $f : \mathbb{R} \to \mathbb{R}$ be a scalar function, the corresponding elliptic cone function is given by

$$f^{ell}(x) := f(\lambda_I^{(1)}(x))\cdot u_I^{(1)}(x) + f(\lambda_I^{(2)}(x))\cdot u_I^{(2)}(x),$$

where $\lambda_I^{(i)}(x)$, $u_I^{(i)}(x)$ $(i = 1,2)$ are defined as in (4.3). At the same time, the relations between f and f^{ell} are fulfilled in the following theorem.

Theorem 4.6 (Elliptic Cone Case)**.** *For any given function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}^n$, let $\lambda_I^{(i)}(x), u_I^{(i)}(x)$ $(i = 1, 2)$ be defined as in (4.3). The following statements hold:*

(a) *f is continuous at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{ell} is continuous at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

(b) *f is directionally differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{ell} is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

(c) *f is differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{ell} is differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

(d) *f is continuously differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f^{ell} is continuously differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.*

5. Concluding remarks

In this paper, we introduce a symmetric type of ellipsoidal cone function and have proved the underline results of this vector-valued function as follows:

(a) f is continuous at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f_I^{EC} is continuous at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.

(b) f is directionally differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f_I^{EC} is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.

(c) f is differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f_I^{EC} is differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.

(d) f is continuously differentiable at $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$ if and only if f_I^{EC} is continuously differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_I^{(1)}(x), \lambda_I^{(2)}(x)$.

We believe these results are useful for designing smooth numerical algorithms for solving ellipsoidal cone programming problems. A potential application is to analyze the following ellipsoidal cone complementarity problems

$$x \in \mathcal{K}_{\mathcal{E}},\ y \in (\mathcal{K}_{\mathcal{E}})^*_{\langle\cdot,\cdot\rangle},\ \langle x, y\rangle = 0,\ F(x, y, \zeta) = 0,$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l$ is a continuously differentiable mapping, which is similar to its special case under the second-order cone setting [11]. We leave further discussion on this topic as our future work.

References

[1] B. Alzalg and M. Pirhaji, *Elliptic cone optimization and prime-dual path-following algorithms*, Optimization. **66** (2017), 2245–2274.

[2] R. Bhattacharya, J. Fung and A. Tiwari, *Ellipsoidal cones and rendezvous of multiple agents*, 43rd IEEE Conference on Decision and Control, Nassau, Bahamas, 2004.

[3] Y.-L. Chang, C.-Y. Yang and J.-S. Chen, *Smooth and nonsmooth analysis of vector-valued functions associated with circular cones*, Nonlinear Anal. **85** (2013), 160–173.

[4] J.-S. Chen, *SOC Functions and Their Applications*, Springer Optimization and Its Applications 143, Springer, Singapore, 2019.

[5] J.-S. Chen, X. Chen and P. Tseng, *Analysis of nonsmooth vector-valued functions associated with second-order cone*, Math. Program. **101** (2004), 95–117.

[6] J.-S. Chen and S.-H. Pan, *A entropy-like proximal algorithm and the exponential multiplier method for symmetric cone programming*, Comput. Optim. Appl. **47** (2010), 477–499.

[7] J.-S. Chen, S.-H. Pan and T.-C. Lin, *A smoothing Newton method based on the generalized Fischer-Burmeister function for MCPs*, Nonlinear Anal. **72** (2010), 3739–3758.

[8] J.-S. Chen and P. Tseng, *An unconstrained smooth minimization reformulation of second-order cone complementarity problem*, Math. Program. **104** (2005), 293–327.

[9] F. Facchinei and J.-S. Pang, *Finite-dimensional variational inequalities and complementarity problems*, Vol. I, Springer, New York, 2003.

[10] J. Faraut and A. Kornyi, *Analysis on symmetric cones*, Oxford University Press, New York, 1994.

[11] M. Fukushima, Z.Q. Luo and P. Tseng, *Smoothing functions for second-order-cone complementarity problems*, SIAM J. Optim. **12** (2002), 436–460.

[12] Z.-H. Huang and T. Ni, *Smoothing algorithms for complementarity problems over symmetric cones*, Comput. Optim. Appl. **45** (2010), 557–579.

[13] L.-C. Kong, J. Sun and N.-H. Xiu, *A regularized smoothing Newton method for symmetric cone complementarity problems*, SIAM J. Optim. **19** (2008), 1028–1047.

[14] S.-H. Pan and J.-S. Chen, *A one-parametric class of merit functions for the symmetric cone complementarity problem*, J. Math. Anal. Appl. **355** (2009), 195–215.

[15] R.T. Rockafellar and R. Wets, *Variational analysis*, Springer-Verlag, Berlin, 2005.

[16] Y. Lu and J.-S. Chen, *On the self-duality and homogeneity of ellipsoidal cones*, J. Nonlinear Convex Anal. **19** (2018), 1331–1353.

[17] Y. Lu and J.-S. Chen, *The variational geometry, projection expression and decomposition associated with ellipsoidal cones*, J. Nonlinear Convex Anal. **20** (2019), 715–738.

[18] X.-H. Miao, Y. Lu and J.-S. Chen, *From symmteric cone optimization to nonsymmetric cone optimization: spectral decomposition, nonsmooth analysis, and projections onto nonsymmetric cones*, Pac. J. Optim. **14** (2018), 399–419.

[19] X.-H. Miao, N. Qi and J.-S. Chen, *Projection formula and one type of spectral factorization assciated with p-order cone*, J. Nonlinear Convex Anal. **18** (2017), 1699–1705.

[20] D.F. Sun and J. Sun, *Löwner's operator and spectral functions in Euclidean Jordan algebras*, Math. Oper. Res. **33** (2008), 421–445.

[21] A. Seeger and J. Vidal-Nuñez, *Measuring centrality and dispersion in directional datasets: the ellipsoidal cone covering approach*, J. Global Optim. **68** (2017), 279-306.

[22] R.J. Stern and H. Wolkowicz, *Invariant ellipsoidal cones*, Linear Algebra Appl. **150** (1991), 81–106.

[23] R.J. Stern and H. Wolkowicz, *Exponential nonnegativity on the ice cream cone*, SIAM J. Matrix Anal. Appl. **12** (1991), 160–165.

[24] J.-C. Zhou and J.-S. Chen, *Properties of circular cone and spectral factorization associated with circular cone*, J. Nonlinear Convex Anal. **14** (2013), 807–816.

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