NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
2019, VOL. 40, NO. 10, 1113–1135
https://doi.org/10.1080/01630563.2018.1552965

No Gap Second-Order Optimality Conditions for Circular Conic Programs

Yue Lu[a], Jein-Shan Chen[b], and Ning Zhang[c]

[a]School of Mathematical Sciences, Tianjin Normal University, Tianjin, China; [b]Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan; [c]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong

ABSTRACT
In this article, we study the second-order optimality conditions for a class of circular conic optimization problem. First, the explicit expressions of the tangent cone and the second-order tangent set for a given circular cone are derived. Then, we establish the closed-form formulation of critical cone and calculate the "sigma" term of the aforementioned optimization problem. At last, in light of tools of variational analysis, we present the associated no gap second-order optimality conditions. Compared to analogous results in the literature, our approach is intuitive and straightforward, which can be manipulated and verified. An example is illustrated to this end.

ARTICLE HISTORY
Received 29 May 2018
Revised 23 November 2018
Accepted 23 November 2018

KEYWORDS
Circular cone; no gap second-order optimality conditions; second-order tangent set; "sigma" term; tangent cone

1. Introduction

Consider the following general circular conic optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & h(x)=0, \\ & g(x)\le 0, \\ & \big(G_1^i(x),G_2^i(x)\big)\in\mathcal{L}_{\theta_i},\ \ i=1,2,\dots,J, \end{array} \tag{1.1}$$

where $f:\mathcal{R}^n\to\mathcal{R},\ \ h:\mathcal{R}^n\to\mathcal{R}^l,\ g:\mathcal{R}^n\to\mathcal{R}^m,\ \ G_1^i:\mathcal{R}^n\to\mathcal{R},\ G_2^i:\mathcal{R}^n\to\mathcal{R}^{s_i-1}\ \ (i=1,2,\dots,J)$ are assumed to be twice continuously differentiable. Here $\mathcal{L}_{\theta_i}$ denotes a circular cone in $\mathcal{R}^{s_i}$ given by

$$\mathcal{L}_{\theta_i}:=\left\{(x_1,x_2)\in\mathcal{R}\times\mathcal{R}^{s_i-1}\,|\,\|x_2\|\le x_1\tan\theta_i\right\} \tag{1.2}$$

with θ_i being its half-aperture angle and $\theta_i\in(0,\frac{\pi}{2})$. From definition, it is clear that $\mathcal{L}_{\frac{\pi}{4}}$ is the set of second-order cone $\mathcal{K}^{s_i}$.

During the past decade, optimization problems associated with circular conic constraints have become an important type of conic programing problems, which is used to modelize engineering problems. In particular, when dealing with the optimal grasping manipulation problems for

CONTACT Jein-Shan Chen jschen@math.ntnu.edu.tw Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan.

multifingered robots [1], the normal force of the ith finger u_{i1} and the associated another forces u_{i2}, u_{i3} satisfy the following condition

$$\|(u_{i2}, u_{i3})\| \leq \mu u_{i1},$$

where $||\cdot||$ represents the Euclidean norm defined in $\mathcal{R}^n$ and μ denotes the friction that depends on the angle θ. If $\mu = \tan\theta$ and $\theta \neq \frac{\pi}{4}$, then the above problem is a typical circular cone constrained problem. At the same time, many researchers have paid attention to theoretical analysis and algorithm design for circular conic programs. Recently, some fundamental results including the spectral factorization and the metric projection onto a given circular cone $\mathcal{L}_\theta$ are established in [2–4]. On the other hand, due to the nonself-duality of circular cones, there exist very few algorithms for dealing with circular conic programs. More specifically, some algorithms including prime-dual interior-point algorithms and smoothing Newton algorithm have been proposed for circular conic programing problems, see [5–7]. In addition, for circular conic complementarity problems, some merit functions are constructed in [8].

From theoretical aspect of optimization, variational geometries including contingent cone, inner tangent cone, outer second-order tangent set and inner second-order tangent set are crucial to establishing optimality conditions [9–11]. Generally speaking, there have been two technical ways to obtain the aforementioned variational geometries regrading circular cone $\mathcal{L}_\theta$. The first one follows from the methodology proposed by Zhou and Chen in their article [2], which depends on the relationship between the circular cone $\mathcal{L}_\theta$ and the second-order cone $\mathcal{K}^s$, that is,

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{L}_\theta \quad \Longleftrightarrow \quad \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{K}^s. \tag{1.3}$$

The other approach is through differential properties of vector-valued functions associated with circular cones [12–16], in which the following circular cone function

$$f^{\mathcal{L}_\theta}(x) := f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)},$$

is employed. Here $f : \mathcal{R} \to \mathcal{R}$ is a given real-valued function and $x = (x_1, x_2) \in \mathcal{R} \times \mathcal{R}^{s-1}$ has the spectral decomposition given by

$$x := \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)},$$

where

$$\lambda_1(x) := x_1 - ||x_2||\cot\theta, \quad \lambda_2(x) := x_1 + ||x_2||\tan\theta$$

and

$$u_x^{(1)} := \frac{1}{1+\cot^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \cot\theta \cdot I \end{bmatrix} \begin{bmatrix} 1 \\ -\bar{x}_2 \end{bmatrix}$$

$$u_x^{(2)} := \frac{1}{1+\tan^2\theta}\begin{bmatrix}1 & 0\\ 0 & \tan\theta\cdot I\end{bmatrix}\begin{bmatrix}1\\ \bar{x}_2\end{bmatrix}$$

with $\bar{x}_2 := x_2/||x_2||$ if $x_2 \neq 0$ and $\bar{x}_2$ being any vector $w \in \mathcal{R}^{s-1}$ satisfying $||w|| = 1$ if $x_2 = 0$. The tangent cone and the second-order tangent set of $\mathcal{L}_\theta$ can be characterized by the directional derivatives of circular cone functions, see [16, Section 4] for more details. Compared to the above two methods, in this article, we present an alternative way to obtain the explicit forms of the tangent cone and the second-order tangent set of $\mathcal{L}_\theta$, which only relies on basic definitions of its variational geometries and an useful lemma about how to calculate these results under the case for the level set of a class of Lipschitz continuous convex functions (see Lemma 2.2 below). In other words, our approach is intuitive and straightforward, which can be manipulated and verified. An example is illustrated to this end.

With the development of modern optimization, second-order optimality theory plays an important role in perturbation analysis [17–20], stability analysis [21–24] and numerical algorithm design [25]. Among these topics, the characterization of no gap second-order optimality condition is a very important issue, which is closely related to the quadratic growth condition. It was shown by Drusvyatskiy and Lewis [26] recently that the quadratic growth condition has a strongly impact on establishing the metric subregularity and calmness of set-valued mappings, the existence of error bounds and convergence rates of numerical algorithms. From different views, the metric subregularity and the calmness of set-valued mappings are the core concepts in nonsmooth calculus and perturbation analysis of variational problems. We refer the readers to the monographs by Dontchev and Rockafellar [27], Bonnans and Shapiro [19] and references therein for a comprehensive study on both theory and applications of related subjects [28–32]. However, to our best knowledge, no results about the no gap second-order optimality conditions for the general circular conic optimization problem (1.1) have been reported.[1] Hence, the purpose of this article aims to fill this gap and the contributions of our research can be summarized as follows.

[1]While finalizing a first version of this work, the authors became aware of an important observation made in Bonnans et al. [5], mainly focus on perturbation analysis on second-order cone programming. One possible way to obtain the results discussed in this article is to transform the circular conic constraints to the second-order cone constraints via the relation (1.3) and then adapt the conclusions based on the framework of second-order cone programming [5]. However, in this article we adopt a constructive way to deal with our mentioned issues. We have the following two reasons: (a) Through these qualitative analysis, we can learn more details on the structure of circular cone, which plays a crucial role on developing optimization theory for nonsymmetric cones. (b) The parameters in our discussion have an important effect on establishing the associated error bound analysis as Drusvyatskiy and Lewis [8] and consequently analyzing convergence rate of numerical algorithms such as proximal point method and its variants.

a. We propose an alternative way to derive the variational geometries of a given circular cone $\mathcal{L}_\theta$.
b. We present explicit forms of the critical cone and the "sigma" term for the given circular conic program (1.1).
c. We establish the equivalent relationship between the no gap second-order optimality conditions and the quadratic growth condition of (1.1).

The rest of this article is organized as follows. In Section 2, we recall some frequently used concepts from variational analysis [9, 11] and explore the variational geometries (including the tangent cone and the second-order tangent set) of a given circular cone. In Section 3, we first present the closed-form of the critical cone and then calculate the "sigma" term of (1.1) directly. After these preparations, we state the no gap second-order optimality conditions for the given circular conic optimization problem. Moreover, we illustrate an example to verify these results in Section 4. Finally, some concluding remarks are drawn in Section 5.

1.1. Notation and terminology

In what follows, we use $\mathrm{dist}(x,\Omega)$ to denote the distance between the vector x and the given set $\Omega \subseteq \mathcal{R}^n$, that is, $\mathrm{dist}(x,\Omega) := \inf_{z\in\Omega} ||x-z||$. $\mathcal{L}_\theta^*$ is the dual cone of a given circular cone $\mathcal{L}_\theta$, which is defined by $\mathcal{L}_\theta^* := \{v \in \mathcal{R}^s \,|\, v^T x \geq 0, \forall x \in \mathcal{L}_\theta\}$. From [2, Theorem 2.1], the structure of $\mathcal{L}_\theta^*$ can be described as

$$\mathcal{L}_\theta^* = \left\{(x_1, x_2) \in \mathcal{R} \times \mathcal{R}^{s-1} \,|\, ||x_2|| \leq x_1 \cot\theta\right\} = \mathcal{L}_{\frac{\pi}{2}-\theta}.$$

The interior and the boundary of $\mathcal{L}_\theta$ are denoted by $\mathrm{int}\,\mathcal{L}_\theta$ and $\mathrm{bd}\,\mathcal{L}_\theta$, respectively. In addition, we let $\ker(A)$ and $\mathrm{range}(A)$ denote the kernel and the range of A, respectively, that is,

$$\ker(A) := \{x \,|\, Ax = 0\}, \quad \mathrm{range}(A) := \{y \,|\, \exists\, x \text{ such that } y = Ax\}.$$

For a lower semicontinuous function $\psi : \mathcal{R}^n \to \mathcal{R}$, the directional derivative of ψ at x along the direction h is denoted by $\psi'(x;h)$, which is given by

$$\psi'(x;h) := \lim_{t\downarrow 0} \frac{\psi(x+th) - \psi(x)}{t}.$$

If ψ is directionally differentiable at x at every direction h, we say that ψ is directionally differentiable at x. Moreover, the parabolic second-order directional derivative of ψ at x is defined by

$$\psi''(x;h,w) := \lim_{t\downarrow 0} \frac{\psi\left(x + th + \frac{1}{2}t^2 w\right) - \psi(x) - \psi'(x,h)}{\frac{1}{2}t^2}.$$

2. Basic tools for the circular cone

As mentioned, we recall some concepts from variational analysis that will be used for subsequent analysis. First, we review the definitions of the tangent cone and the second-order tangent set for a given closed set $\Omega \subseteq \mathcal{R}^n$, which come from Bonnans and Shaprio's monograph [19, Definition 2.54 and Definition 3.28].

Definition 2.1. Let $\Omega \subseteq \mathcal{R}^n$ be a given closed set and $x \in \Omega$. The (Bouligand-Severi) tangent/contingent cone to Ω at $x \in \Omega$ is defined by

$$\mathcal{T}_\Omega(x) := \left\{ h \in \mathcal{R}^n \,|\, \exists\, t_n \downarrow 0, \mathrm{dist}(x + t_n h, \Omega) = o(t_n) \right\}.$$

Similarly, the inner tangent cone to Ω at $x \in \Omega$ is given in the form of

$$\mathcal{T}^i_\Omega(x) := \left\{ h \in \mathcal{R}^n \,|\, \mathrm{dist}(x + th, \Omega) = o(t), t \geq 0 \right\}.$$

In addition, if $h \in \mathcal{T}_\Omega(x)$, the outer second-order tangent set to Ω at x along the direction h is defined as

$$\mathcal{T}^2_\Omega(x,h) := \left\{ w \in \mathcal{R}^n \,|\, \exists\, t_n \downarrow 0, \mathrm{dist}\left(x + t_n h + \frac{1}{2} t_n^2 w, \Omega \right) = o\left(t_n^2\right) \right\}.$$

Similarly, if $h \in \mathcal{T}^i_\Omega(x)$, the inner second-order tangent set to Ω at x along the direction h is given by

$$\mathcal{T}^{i,2}_\Omega(x,h) := \left\{ w \in \mathcal{R}^n \,|\, \mathrm{dist}\left(x + th + \frac{1}{2} t^2 w, \Omega \right) = o(t^2), t \geq 0 \right\}.$$

Let $\Omega \subseteq \mathcal{R}^n$ be a closed convex set and $x \in \Omega$. It follows from [19, Section 2.2.4] that the contingent cone $\mathcal{T}_\Omega(x)$ coincides with the inner tangent cone $\mathcal{T}^i_\Omega(x)$, that is, $\mathcal{T}_\Omega(x) = \mathcal{T}^i_\Omega(x)$. In addition, if the set Ω is second-order regular at x (see [19, Definition 3.85] for details), the following conditions hold at x:

(i) $\mathcal{T}^2_\Omega(x,h) = \mathcal{T}^{i,2}_\Omega(x,h)$ for all $h \in \mathcal{T}_\Omega(x)$.

(ii) For any $h \in \mathcal{T}_\Omega(x)$ and for any sequence $x + t_n h + \frac{1}{2} t_n^2 h \in \Omega$ such that $t_n r_n \longrightarrow 0$ and

$$\lim_{n \to \infty} \mathrm{dist}\left(r_n, \mathcal{T}^2_\Omega(x,h)\right) = 0.$$

Moreover, from [2, Theorem 2.8], we know that the circular cone $\mathcal{L}_\theta$ is closed and second-order regular. Hence, in the sequel we only need to figure out the explicit forms for the contingent cone $\mathcal{T}_{\mathcal{L}_\theta}(x)$ and the outer second-order tangent set $\mathcal{T}^2_{\mathcal{L}_\theta}(x,h)$. To this end, we need a technical lemma, which describes the tangent cone and the second-order tangent set for a level set of a given convex function. We only state it without presenting its proof because it can be found in [19, Proposition 2.61 and Proposition 3.30].

Lemma 2.2 *Let $\psi : \mathcal{R}^n \to \mathcal{R}$ be a lower semicontinuous convex function. Consider the associated level set $\Omega := \{x \in \mathcal{R}^n \mid \psi(x) \leq 0\}$. Suppose that ψ is Lipschitz continuous at x and $\psi(x) = 0$. In addition, there exists $\bar{x} \in \mathcal{R}^n$ such that $\psi(\bar{x})<0$ (Slater condition). Then,*

$$\mathcal{T}_\Omega(x) = \{h \in \mathcal{R}^n \mid \psi'(x;h) \leq 0\}. \tag{2.1.}$$

Moreover, for a given $h \in \mathcal{R}^n$ satisfying $\psi'(x;h) = 0$, the outer second-order tangent set to Ω at x along the direction h can be described as

$$\mathcal{T}^2_\Omega(x,h) = \{w \in \mathcal{R}^n \mid \psi''(x;h,w) \leq 0\}. \tag{2.2}$$

With Lemma 2.2, we are ready to express the explicit form of the tangent cone $\mathcal{T}_{\mathcal{L}_\theta}(x)$ at any given $x \in \mathcal{R}^s$.

Theorem 2.3. *Let $x = (x_1, x_2) \in \mathcal{R} \times \mathcal{R}^{s-1}$. Then, the tangent cone to $\mathcal{L}_\theta$ at x can be written as*

$$\mathcal{T}_{\mathcal{L}_\theta}(x) = \begin{cases} \mathcal{R}^s, & \text{if } x \in \operatorname{int} \mathcal{L}_\theta, \\ \mathcal{L}_\theta, & \text{if } x = 0, \\ \{(h_1,h_2) \in \mathcal{R} \times \mathcal{R}^{s-1} \mid h_2^T x_2 - h_1 x_1 \tan^2\theta \leq 0\}, & \text{if } x \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0\}. \end{cases}$$

Proof. The explicit form of $\mathcal{T}_{\mathcal{L}_\theta}(x)$ is deduced by discussing two cases.

(a) If $x \in \operatorname{int} \mathcal{L}_\theta$ or $x=0$, from Definition 2.1, we immediately obtain

$$\mathcal{T}_{\mathcal{L}_\theta}(x) = \begin{cases} \mathcal{R}^s, & \text{if } x \in \text{int } \mathcal{L}_\theta, \\ \mathcal{L}_\theta, & \text{if } x = 0. \end{cases}$$

(b) If $x \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0\}$, then $x_1 \tan\theta = ||x_2|| \neq 0$. Using the definition of $\mathcal{L}_\theta$ as in (1.2), $\mathcal{L}_\theta$ can be rewritten as

$$\mathcal{L}_\theta = \left\{(x_1, x_2) \in \mathcal{R} \times \mathcal{R}^{s-1} \mid \phi(x) \leq 0\right\},$$

where $\phi : \mathcal{R}^s \to \mathcal{R}$ is given by $\phi(x) := ||x_2|| - x_1 \tan\theta$. It is easy to verify that ϕ is continuously differentiable, Lipschitz continuous at x and the corresponding Slater condition holds under this case. Hence, it follows from Lemma 2.2 that $\mathcal{T}_{\mathcal{L}_\theta}(x)$ can be described as

$$\mathcal{T}_{\mathcal{L}_\theta}(x) = \{h \in \mathcal{R}^s \mid \phi'(x;h) \leq 0\}. \tag{2.3}$$

Note that

$$\phi'(x;h) = \nabla\phi(x)^T h = \begin{bmatrix} -\tan\theta & \dfrac{x_2}{||x_2||} \end{bmatrix}^T \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{h_2^T x_2}{||x_2||} - h_1 \tan\theta.$$

Applying the relation $x_1 \tan\theta = ||x_2||$ and (2.3) yield that

$$\mathcal{T}_{\mathcal{L}_\theta}(x) = \{(h_1,h_2) \in \mathcal{R} \times \mathcal{R}^{s-1} \mid h_2^T x_2 - h_1 x_1 \tan^2\theta \leq 0\}.$$

Thus, the proof is complete. □

Next theorem describes the outer second-order tangent set $\mathcal{T}^2_{\mathcal{L}_\theta}(x,h)$ at any $x \in \mathcal{R}^s$ and $h \in \mathcal{T}_{\mathcal{L}_\theta}(x)$.

Theorem 2.4. *Let $x=(x_1,x_2) \in \mathcal{R} \times \mathcal{R}^{s-1}$ and $h=(h_1,h_2) \in \mathcal{T}_{\mathcal{L}_\theta}(x)$. The outer second-order tangent set to $\mathcal{L}_\theta$ at x along the direction h can be described as*

$$\mathcal{T}^2_{\mathcal{L}_\theta}(x,h) = \begin{cases} \mathcal{R}^s, & \text{if } h \in \operatorname{int} \mathcal{T}_{\mathcal{L}_\theta}(x), \\ \mathcal{T}_{\mathcal{L}_\theta}(h), & \text{if } x=0, \\ \Xi, & \text{if } x \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0\}, h \in \operatorname{bd} \mathcal{T}_{\mathcal{L}_\theta}(x), \end{cases}$$

where the set Ξ is defined by

$$\Xi := \left\{(w_1,w_2) \in \mathcal{R} \times \mathcal{R}^{s-1} \mid w_2^T x_2 - w_1 x_1 \tan^2\theta \le h_1^2 \tan^2\theta - ||h_2||^2\right\}.$$

Proof. Again, we derive the explicit form of $\mathcal{T}^2_{\mathcal{L}_\theta}(x,h)$ by discussing two cases.

(a) If $h \in \operatorname{int} \mathcal{T}_{\mathcal{L}_\theta}$ or $x=0$, from Definition 2.1, we have

$$\mathcal{T}^2_{\mathcal{L}_\theta}(x,h) = \begin{cases} \mathcal{R}^s, & \text{if } h \in \operatorname{int} \mathcal{T}_{\mathcal{L}_\theta}, \\ \mathcal{T}_{\mathcal{L}_\theta}(h), & \text{if } x=0. \end{cases}$$

(b) If $x \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0\}$ and $h \in \operatorname{bd} \mathcal{T}_{\mathcal{L}_\theta}(x)$, we have

$$0 \ne ||x_2|| = x_1 \tan\theta, \quad h_2^T x_2 - h_1 x_1 \tan^2\theta = 0. \tag{2.4}$$

Then, it follows from Lemma 2.2 that the second-order tangent set $\mathcal{T}^2_{\mathcal{L}_\theta}(x,h)$ has the form of

$$\mathcal{T}^2_{\mathcal{L}_\theta}(x,h) = \left\{w \in \mathcal{R}^s \mid \phi''(x;h,w) \le 0\right\}, \tag{2.5}$$

where $\phi(x) := ||x_2|| - x_1 \tan\theta$. Note that

$$\begin{aligned}
&\phi''(x;h,w) \\
&= \nabla\phi(x)^T w + h^T \nabla^2 \phi(x) h \\
&= \left[-\tan\theta \ \tfrac{x_2}{||x_2||}\right]^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + [h_1 h_2]^T \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{||x_2||} I_{s-1} - \frac{1}{||x_2||^3} x_2 x_2^T \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\
&= \frac{w_2^T x_2}{||x_2||} - w_1 \tan\theta + \frac{||h_2||^2}{||x_2||} - \frac{(h_2^T x_2)^2}{||x_2||^3} \\
&= \frac{1}{||x_2||}\left(w_2^T x_2 - w_1 x_1 \tan^2\theta\right) + \frac{||h_2||^2}{||x_2||} - \frac{(h_1 x_1 \tan^2\theta)^2}{(x_1 \tan\theta)^3} \\
&= \frac{1}{||x_2||}\left(w_2^T x_2 - w_1 x_1 \tan^2\theta\right) + \frac{||h_2||^2}{||x_2||} - \frac{h_1^2 \tan^2\theta}{x_1 \tan\theta} \\
&= \frac{1}{||x_2||}\left(w_2^T x_2 - w_1 x_1 \tan^2\theta + ||h_2||^2 - h_1^2 \tan^2\theta\right),
\end{aligned}$$

where the last two equalities are due to (2.4). Hence, under this case, the above expression together with (2.5) imply that

$$\mathcal{T}^2_{\mathcal{L}_\theta}(x,h) = \left\{(w_1,w_2) \in \mathcal{R}\times\mathcal{R}^{s-1} \,|\, w_2^T x_2 - w_1 x_1 \tan^2\theta \le h_1^2 \tan^2\theta - ||h_2||^2\right\}.$$

Thus, the proof is complete. □

To end this section, we introduce an useful complementarity property of the circular cone $\mathcal{L}_\theta$, which plays a major role in the analysis of the Karush-Kuhn-Tucker (KKT) condition for (1.1).

Theorem 2.5. *For any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathcal{R}\times\mathcal{R}^{s-1}$. The system*

$$x \in \mathcal{L}_\theta^*, \;\; y \in \mathcal{L}_\theta, \;\; x^T y = 0$$

has at least one solution if and only if one of the following cases holds.

(a) $x = 0_s$, $y \in \mathcal{L}_\theta$.
(b) $x \in \operatorname{int} \mathcal{L}_\theta^*$, $y = 0_s$.
(c) $x \in \operatorname{bd} \mathcal{L}_\theta^* \setminus \{0_s\}$, $y = 0_s$.
(d) $x \in \operatorname{bd} \mathcal{L}_\theta^* \setminus \{0_s\}$, $y \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0_s\}$, *and there exists* $\sigma > 0$ *such that* $x = \sigma(\mathcal{H}y)$, *where*

$$\mathcal{H} := \begin{bmatrix} \tan^2\theta & 0 \\ 0 & -I_{s-1} \end{bmatrix}.$$

Proof. The "sufficiency" direction is obvious from the definitions of $\mathcal{L}_\theta$ and $\mathcal{L}_\theta^*$. To prove the "necessity" direction, suppose that $x \in \mathcal{L}_\theta^*, \;\; y \in \mathcal{L}_\theta, \;\; x^T y = 0$. Then, from definitions, the cases (a)-(c) are trivial and we only need to verify the case (d). Taking $x \in \operatorname{bd} \mathcal{L}_\theta^* \setminus \{0_s\}$, $y \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0_s\}$, we have $x_2 \ne 0_{s-1}, \;\; y_2 \ne 0_{s-1}, \;\; ||x_2|| = x_1 \cot\theta$ and $||y_2|| = y_1 \tan\theta$. In addition, the relation $x^T y = 0$ yields that $x_1 y_1 + x_2^T y_2 = 0$, which implies $-x_2^T y_2 = ||x_2|| \cdot ||y_2||$ and there exists $\sigma > 0$ such that $x_2 = -\sigma y_2, \;\; x_1 y_1 = \sigma ||y_2||^2, \;\; y_1 \ne 0$ and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sigma \begin{bmatrix} \frac{||y_2||^2}{y_1} \\ -y_2 \end{bmatrix} = \sigma \begin{bmatrix} y_1 \tan^2\theta \\ -y_2 \end{bmatrix} = \sigma \begin{bmatrix} \tan^2\theta & 0 \\ 0 & -I_{s-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \sigma(\mathcal{H}y),$$

where the third equality is due to $||y_2|| = y_1 \tan\theta$. Thus, the proof is complete. □

3. Optimality conditions

This section aims to establish optimality conditions for the circular conic optimization problem (1.1). First of all, the Lagrangian function of (1.1) is defined as

$$L\left(x;\mu,\eta,\Gamma^1,\Gamma^2,...,\Gamma^J\right) := f(x) + h(x)^T\mu + g(x)^T\eta - \sum_{i=1}^{J} G^i(x)^T\Gamma^i, \tag{3.1}$$

where $\mu \in \mathcal{R}^l$, $\eta \in \mathcal{R}^m$. For simplicity, we write the vectors $G^i(x)$ and $\Gamma^i \in \mathcal{R}^{s_i}$ $(i = 1, 2\cdots, J)$ into the following form, respectively,

$$G^i(x) := \begin{bmatrix} G_1^i(x) \\ G_2^i(x) \end{bmatrix}, \quad \Gamma^i(x) := \begin{bmatrix} \Gamma_1^i(x) \\ \Gamma_2^i(x) \end{bmatrix}.$$

Let $\bar{x} \in \mathcal{R}^n$ be a local minimizer of (1.1) and Robinson's constraint qualification (RCQ) holds at $\bar{x}$, that is,

$$0 \in \operatorname{int}\left\{ \begin{bmatrix} h(\bar{x}) \\ g(\bar{x}) \\ G^1(\bar{x}) \\ \vdots \\ G^J(\bar{x}) \end{bmatrix} + \begin{bmatrix} \mathcal{J}h(\bar{x}) \\ \mathcal{J}g(\bar{x}) \\ \mathcal{J}G^1(\bar{x}) \\ \vdots \\ \mathcal{J}G^J(\bar{x}) \end{bmatrix} \mathcal{R}^n - \begin{bmatrix} 0_l \\ \mathcal{R}_-^m \\ \mathcal{L}_{\theta_1} \\ \vdots \\ \mathcal{L}_{\theta_J} \end{bmatrix} \right\},$$

where $\mathcal{J}h(\bar{x}), \mathcal{J}g(\bar{x})$ and $\mathcal{J}G^i(\bar{x})$ denote the derivatives of $h(x), g(x)$ and $\mathcal{J}G^i(x)$ at $\bar{x}$, respectively. Then, there exist $\bar{\mu} \in \mathcal{R}^l$, $\bar{\eta} \in \mathcal{R}^m$, $\bar{\Gamma}^i \in \mathcal{R}^{s_i}$ $(i = 1, 2, ..., J)$ satisfying the KKT condition

$$\begin{cases} \nabla_x L\left(\bar{x};\bar{\mu},\bar{\eta},\bar{\Gamma}^1,\bar{\Gamma}^2,...,\bar{\Gamma}^J\right) = 0, h(\bar{x}) = 0_l, \ \mathcal{R}_+^m \ni \bar{\eta} \perp g(\bar{x}) \in \mathcal{R}_-^m, \\ \mathcal{L}_{\theta_i}^* \ni \bar{\Gamma}^i \perp G^i(\bar{x}) \in \mathcal{L}_{\theta_i}, i = 1, 2, ..., J, \end{cases} \tag{3.2}$$

where "$a \perp b$" means that $a^Tb = 0$.

It is easy to see that the condition (3.2) is a special form of mathematical programing with equilibrium constraints (MPEC in brief). During the past two decades, MPECs have been drawn much attention not only in multiple applications such as engineering design and economics but also in the theoretical analysis themselves, we refer to the monographs [33, 34] and the references therein for more details.

In the sequel, if $(\bar{x};\bar{\mu},\bar{\eta},\bar{\Gamma}^1,\bar{\Gamma}^2,...,\bar{\Gamma}^J)$ satisfies the above system (3.2), we call $\bar{x}$ a stationary point of (1.1). In addition, the set of the associated Lagrangian multipliers $\Lambda(\bar{x})$ is defined by

$$\Lambda(\bar{x}) := \left\{ \left(\bar{\mu},\bar{\eta},\bar{\Gamma}^1,\bar{\Gamma}^2,...,\bar{\Gamma}^J\right) \middle| \begin{array}{c} \left(\bar{x};\bar{\mu},\bar{\eta},\bar{\Gamma}^1,\bar{\Gamma}^2,...,\bar{\Gamma}^J\right) \text{ satisfies} \\ \text{the KKT condition (10)} \end{array} \right\}.$$

For convenience, let us denote

$$\Omega := \begin{bmatrix} 0_l \\ \mathcal{R}^m_- \\ \mathcal{L}_{\theta_1} \\ \vdots \\ \mathcal{L}_{\theta_J} \end{bmatrix}, \quad \mathcal{G}(\bar{x}) := \begin{bmatrix} h(\bar{x}) \\ g(\bar{x}) \\ G^1(\bar{x}) \\ \vdots \\ G^J(\bar{x}) \end{bmatrix}, \quad \mathcal{Y} := \begin{bmatrix} \mathcal{R}^l \\ \mathcal{R}^m \\ \mathcal{R}^{s_1} \\ \vdots \\ \mathcal{R}^{s_J} \end{bmatrix}. \tag{3.3}$$

Then, the above RCQ can be rewritten as

$$\mathcal{J}\mathcal{G}(\bar{x})\mathcal{R}^n + \mathcal{T}_\Omega(\mathcal{G}(\bar{x})) = \mathcal{Y}. \tag{3.4}$$

Analogous to [19, Definition 4.70], the constraint nondegeneracy condition of (1.1) at $\bar{x}$ is defined by

$$\mathcal{J}\mathcal{G}(\bar{x})\mathcal{R}^n + \mathrm{lin}\{\mathcal{T}_\Omega(\mathcal{G}(\bar{x}))\} = \mathcal{Y}, \tag{3.5}$$

where $\mathrm{lin}\{\mathcal{T}_\Omega(\mathcal{G}(\bar{x}))\}$ denotes the linearity space of $\mathcal{T}_\Omega(\mathcal{G}(\bar{x}))$, which is the largest linear space contained in $\mathcal{T}_\Omega(\mathcal{G}(\bar{x}))$.

To understand the constraint nondegeneracy condition intuitively, we define the following index sets:

$$\begin{aligned}
I_+(\bar{x}) &:= \{i \,|\, g_i(\bar{x}) = 0,\ \ \bar{\eta}_i > 0,\ \ i = 1,2,\dots,m\},\\
I_0(\bar{x}) &:= \{i \,|\, g_i(\bar{x}) = 0,\ \ \bar{\eta}_i = 0,\ \ i = 1,2,\dots,m\},\\
I_-(\bar{x}) &:= \{i \,|\, g_i(\bar{x}) < 0,\ \ \bar{\eta}_i = 0,\ \ i = 1,2,\dots,m\},\\
I_G(\bar{x}) &:= \{i \,|\, G^i(\bar{x}) \in \mathrm{int}\,\mathcal{L}_{\theta_i},\ \ i = 1,2,\dots,J\},\\
Z_G(\bar{x}) &:= \{i \,|\, G^i(\bar{x}) = 0_{s_i},\ \ i = 1,2,\dots,J\},\\
B_G(\bar{x}) &:= \left\{i \,|\, G^i(\bar{x}) \in \mathrm{bd}\,\mathcal{L}_{\theta_j} \setminus \{0_{s_i}\},\ \ i = 1,2,\dots,J\right\}.
\end{aligned}$$

Theorem 3.1. *Let $\bar{x}$ be a stationary point of (1.1). Then, the following conditions are equivalent:*

(a) *The constraint nondegeneracy condition holds at $\bar{x}$.*

(b) *The vectors*

$$\begin{aligned}
&\mathcal{J}h^1(\bar{x})^T, \dots, \mathcal{J}h^l(\bar{x})^T,\\
&\mathcal{J}g^i(\bar{x})^T,\ \ i \in I_+(\bar{x}) \cup I_0(\bar{x}),\\
&\mathcal{J}G^i(\bar{x})^T\mathcal{H}_{\theta_i}G^i(\bar{x}),\ \ i \in B_G(\bar{x}),\\
&\mathcal{J}G^i(\bar{x})^T e^j_{s_i},\ \ j = 1,2,\dots,s_i,\ \ i \in Z_G(\bar{x})
\end{aligned}$$

are linearly independent, where $e^j_{s_i}$ denotes the jth column vector of the identity matrix I_{s_i} and $\mathcal{H}_{\theta_i}$ is defined by

$$\mathcal{H}_{\theta_i} := \begin{bmatrix} \tan^2\theta_i & 0 \\ 0 & -I_{s_i-1} \end{bmatrix}.$$

Proof. Without loss of generality, we assume that

$$I_G(\bar{x}) := \{1, 2, \ldots, J_1\}, \ Z_G(\bar{x}) := \{J_1+1, J_1+2, \ldots, J_2\},$$

$$B_G(\bar{x}) := \{J_2+1, J_2+2, \ldots, J\}.$$

It follows from Theorem 2.3 and (3.3) that the constraint nondegeneracy condition (3.5) can be described as

$$\begin{bmatrix} \mathcal{J}h(\bar{x}) \\ \mathcal{J}g(\bar{x}) \\ \Pi_{i\in I_G(\bar{x})}\mathcal{J}G^i(\bar{x}) \\ \Pi_{i\in Z_G(\bar{x})}\mathcal{J}G^i(\bar{x}) \\ \Pi_{i\in B_G(\bar{x})}\mathcal{J}G^i(\bar{x}) \end{bmatrix} \mathcal{R}^n + \mathrm{lin} \begin{bmatrix} \{0_l\} \\ \mathcal{T}_{\mathcal{R}^m_-}(g(\bar{x})) \\ \Pi_{i\in I_G(\bar{x})}\mathcal{R}^{s_i} \\ \Pi_{i\in Z_G(\bar{x})}\mathcal{L}_{\theta_i} \\ \Pi_{i\in B_G(\bar{x})}\mathcal{T}_{\mathcal{L}_{\theta_i}}(G^i(\bar{x})) \end{bmatrix} = \begin{bmatrix} \mathcal{R}^l \\ \mathcal{R}^m \\ \Pi_{i\in I_G(\bar{x})}\mathcal{R}^{s_i} \\ \Pi_{i\in Z_G(\bar{x})}\mathcal{R}^{s_i} \\ \Pi_{i\in B_G(\bar{x})}\mathcal{R}^{s_i} \end{bmatrix}, \tag{3.6}$$

where

$$\Pi_{i\in I_G(\bar{x})}\mathcal{J}G^i(\bar{x}) := \begin{bmatrix} \mathcal{J}G^1(\bar{x}) \\ \mathcal{J}G^2(\bar{x}) \\ \vdots \\ \mathcal{J}G^{J_1}(\bar{x}) \end{bmatrix}, \quad \Pi_{i\in I_G(\bar{x})}\mathcal{R}^{s_i} := \begin{bmatrix} \mathcal{R}^{s_1} \\ \mathcal{R}^{s_2} \\ \vdots \\ \mathcal{R}^{s_{J_1}} \end{bmatrix},$$

$$\Pi_{i\in Z_G(\bar{x})}\mathcal{J}G^i(\bar{x}) := \begin{bmatrix} \mathcal{J}G^{J_1+1}(\bar{x}) \\ \mathcal{J}G^{J_1+2}(\bar{x}) \\ \vdots \\ \mathcal{J}G^{J_2}(\bar{x}) \end{bmatrix}, \quad \Pi_{i\in Z_G(\bar{x})}\mathcal{R}^{s_i} := \begin{bmatrix} \mathcal{R}^{s_{J_1+1}} \\ \mathcal{R}^{s_{J_1+1}} \\ \vdots \\ \mathcal{R}^{s_{J_2}} \end{bmatrix},$$

$$\Pi_{i\in B_G(\bar{x})}\mathcal{J}G^i(\bar{x}) := \begin{bmatrix} \mathcal{J}G^{J_2+1}(\bar{x}) \\ \mathcal{J}G^{J_2+2}(\bar{x}) \\ \vdots \\ \mathcal{J}G^{J}(\bar{x}) \end{bmatrix}, \quad \Pi_{i\in B_G(\bar{x})}\mathcal{R}^{s_i} := \begin{bmatrix} \mathcal{R}^{s_{J_2+1}} \\ \mathcal{R}^{s_{J_2+1}} \\ \vdots \\ \mathcal{R}^{s_J} \end{bmatrix},$$

$$\Pi_{i\in Z_G(\bar{x})}\mathcal{L}_{\theta_i} := \begin{bmatrix} \mathcal{L}_{\theta_{J_1+1}} \\ \mathcal{L}_{\theta_{J_1+2}} \\ \vdots \\ \mathcal{L}_{\theta_{J_2}} \end{bmatrix}, \quad \Pi_{i\in B_G(\bar{x})}\mathcal{T}_{\mathcal{L}_{\theta_i}}(G^i(\bar{x})) := \begin{bmatrix} \mathcal{T}_{\mathcal{L}_{\theta_{J_2+1}}}(G^{J_2+1}(\bar{x})) \\ \mathcal{T}_{\mathcal{L}_{\theta_{J_2+2}}}(G^{J_2+2}(\bar{x})) \\ \vdots \\ \mathcal{T}_{\mathcal{L}_{\theta_J}}(G^J(\bar{x})) \end{bmatrix}.$$

Notice that

$$\mathrm{lin}\left\{\mathcal{T}_{\mathcal{R}^m_-}\big(g(\bar{x})\big)\right\} := \left\{\eta \in \mathcal{R}^m \mid \eta_i = 0, i \in I_+(\bar{x}) \cup I_0(\bar{x})\right\},$$

$$\mathrm{lin}\{\mathcal{L}_{\theta_i}\} := \{0_{s_i}\}, \;\; i \in Z_G(\bar{x}).$$

Taking $i \in B_G(\bar{x})$, the explicit description of $\mathcal{T}_{\mathcal{L}_{\theta_i}}(G^i(\bar{x}))$ implies that

$$\begin{aligned}\mathrm{lin}\left\{\mathcal{T}_{\mathcal{L}_{\theta_i}}\big(G^i(\bar{x})\big)\right\} &= \left\{\big(\Gamma_1^i,\Gamma_2^i\big) \mid \Gamma_1^i(\bar{x})G_1^i(\bar{x})\tan^2\theta_i - \big(\Gamma_2^i\big)^T G_2^i(\bar{x}) = 0\right\}\\ &= \left\{\big(\Gamma_1^i,\Gamma_2^i\big) \;\middle|\; \begin{bmatrix} G_1^i(\bar{x}) \\ G_2^i(\bar{x}) \end{bmatrix}^T \begin{bmatrix} \tan^2\theta_i & 0 \\ 0 & -I_{s_i-1} \end{bmatrix} \begin{bmatrix} \Gamma_1^i \\ \Gamma_2^i \end{bmatrix} = 0\right\}\\ &= \ker\Big(G^i(\bar{x})^T\mathcal{H}_{\theta_i}\Big).\end{aligned}$$

Hence, the equality (3.6) is equivalent to

$$\begin{bmatrix} \mathcal{J}h(\bar{x}) \\ \mathcal{J}g(\bar{x}) \\ \Pi_{i\in Z_G(\bar{x})}\mathcal{J}G^i(\bar{x}) \\ \Pi_{i\in B_G(\bar{x})}\mathcal{J}G^i(\bar{x}) \end{bmatrix}\mathcal{R}^n + \begin{bmatrix} \{0_l\} \\ \mathrm{lin}\mathcal{T}_{\mathcal{R}^m_-}\big(g(\bar{x})\big) \\ \Pi_{i\in Z_G(\bar{x})}0_{s_i} \\ \Pi_{i\in B_G(\bar{x})}\ker\Big(G^i(\bar{x})^T\mathcal{H}_{\theta_i}\Big) \end{bmatrix} = \begin{bmatrix} \mathcal{R}^l \\ \mathcal{R}^m \\ \Pi_{i\in Z_G(\bar{x})}\mathcal{R}^{s_i} \\ \Pi_{i\in B_G(\bar{x})}\mathcal{R}^{s_i} \end{bmatrix}.$$

By taking the orthogonal complements for both sides of the above equality, we obtain

$$\begin{aligned}&\ker\Big[\mathcal{J}h(\bar{x})^T \;\; \mathcal{J}g(\bar{x})^T \;\; \mathcal{J}G^{J_1+1}(\bar{x})^T \;\; \cdots \;\; \mathcal{J}G^{J_2}(\bar{x})^T \;\; \mathcal{J}G^{J_2+1}(\bar{x})^T \;\; \cdots \;\; \mathcal{J}G^{J}(\bar{x})^T\Big]\\ &\cap\;\; \mathcal{R}^l \times \left\{\eta \in \mathcal{R}^m \mid \eta_i = 0, i \in I_-(\bar{x})\right\} \times \mathcal{R}^{s_{J_1+1}} \times \cdots \times \mathcal{R}^{s_{J_2}}\\ &\qquad \times \mathrm{range}\Big(\mathcal{H}^T_{\theta_{J_2+1}}G^{J_2+1}(\bar{x})\Big) \times \cdots \times \mathrm{range}\Big(\mathcal{H}^T_{\theta_J}G^J(\bar{x})\Big)\\ &= 0_l \times 0_m \times 0_{s_{J_1+1}} \times \cdots \times 0_{s_{J_2}} \times 0_{s_{J_2+1}} \times \cdots 0_{s_J}.\end{aligned} \tag{3.7}$$

Let $\mu = (\mu_1, \ldots, \mu_l)^T$, $\eta_i \in \mathcal{R}, i \in I_+(\bar{x}) \cup I_0(\bar{x})$, $\Gamma^i \in \mathcal{R}^{s_i}, i \in Z_G(\bar{x})$, $p_i \in \mathcal{R}, i \in B_G(\bar{x})$ satisfying

$$\mathcal{J}h(\bar{x})^T\mu + \sum_{i\in I_+(\bar{x})\cup I_0(\bar{x})}\mathcal{J}g^i(\bar{x})^T\eta_i + \sum_{i\in Z_G(\bar{x})}\mathcal{J}G^i(\bar{x})^T\Gamma^i + \sum_{i\in B_G(\bar{x})}\mathcal{J}G^i(\bar{x})^T\mathcal{H}^T_{\theta_i}G^i(\bar{x})p_i = 0.$$

This together with (3.7) yields

$$\mu = 0_l, \quad \eta_i = 0, \; i \in I_+(\bar{x}) \cup I_0(\bar{x}),$$
$$\Gamma^i = 0_{s_i}, \; i \in Z_G(\bar{x}), \quad p_i = 0, \; i \in B_G(\bar{x}),$$

which means that the constraint nondegeneracy condition holds at $\bar{x}$ if and only if the vectors

$$\mathcal{J}h^1(\bar{x})^T, \dots, \mathcal{J}h^l(\bar{x})^T, \quad \mathcal{J}g^i(\bar{x})^T, \; i \in I_+(\bar{x}) \cup I_0(\bar{x}),$$
$$\mathcal{J}G^i(\bar{x})^T e^j_{s_i}, \; j = 1, 2, \dots, s_i, \; i \in Z_G(\bar{x}), \quad \mathcal{J}G^i(\bar{x})^T \mathcal{H}_{\theta_i} G^i(\bar{x}), \; i \in B_G(\bar{x})$$

are linearly independent. Thus, the proof is complete. □

Similar to [19, Theorem 3.9], we establish the first-order optimality condition of (1.1) in the following theorem.

Theorem 3.2. *Let $\bar{x}$ be a local minimizer of (1.1) and RCQ (3.4) holds at $\bar{x}$. Then the set $\Lambda(\bar{x})$ is nonempty, convex and compact. Furthermore, if the constraint nondegeneracy condition (3.5) holds at $\bar{x}$, the set $\Lambda(\bar{x})$ is a singleton.*

Let $\bar{x}$ be a stationary point of (1.1), the corresponding critical cone at $\bar{x}$ is defined by

$$\mathcal{C}(\bar{x}) := \left\{ d \in \mathcal{R}^n \;\middle|\; \begin{array}{c} \mathcal{J}h(\bar{x})d = 0_l, \;\; \nabla f(\bar{x})^T d = 0, \\ \mathcal{J}g(\bar{x})d \in \mathcal{T}_{\mathcal{R}^m_-}(g(\bar{x})), \\ \mathcal{J}G^i(\bar{x})d \in \mathcal{T}_{\mathcal{L}_{\theta_i}}(G^i(\bar{x})), \;\; i = 1, 2, \dots, J \end{array} \right\}.$$

If $\Lambda(\bar{x})$ is nonempty, then there exist $\bar{\mu} \in \mathcal{R}^l$, $\bar{\eta} \in \mathcal{R}^m_+$ and $\bar{\Gamma}^i \in \mathcal{L}^*_{\theta_i}$ $(i = 1, 2, \dots, J)$ such that $\mathcal{C}(\bar{x})$ can be rewritten as

$$\mathcal{C}(\bar{x}) = \left\{ d \in \mathcal{R}^n \;\middle|\; \begin{bmatrix} \mathcal{J}h(\bar{x})d \\ \mathcal{J}g(\bar{x})d \\ \mathcal{J}G^1(\bar{x})d \\ \vdots \\ \mathcal{J}G^J(\bar{x})d \end{bmatrix} \in \begin{bmatrix} \{0_l\} \\ \mathcal{T}_{\mathcal{R}^m_-}(g(\bar{x})) \\ \mathcal{T}_{\mathcal{L}_{\theta_1}}(G^1(\bar{x})) \\ \vdots \\ \mathcal{T}_{\mathcal{L}_{\theta_J}}(G^J(\bar{x})) \end{bmatrix} \cap \begin{bmatrix} -\bar{\mu} \\ -\bar{\eta} \\ \bar{\Gamma}^1 \\ \vdots \\ \bar{\Gamma}^J \end{bmatrix}^\perp \right\}. \tag{3.8}$$

With Theorem 2.5, the following theorem shows the explicit expression of $\mathcal{C}(\bar{x})$.

Theorem 3.3. *Let $\bar{x}$ be a stationary point of (1.1), $\bar{w} := (\bar{\mu}, \bar{\eta}, \bar{\Gamma}^1, \dots, \bar{\Gamma}^J) \in \mathcal{R}^l \times \mathcal{R}^m \times \mathcal{R}^{s_1} \times \cdots \times \mathcal{R}^{s_J}$ and $\bar{w} \in \Lambda(\bar{x})$. Then, the critical cone $\mathcal{C}(\bar{x})$ can be described as*

$$\mathcal{C}(\bar{x})=\left\{d\in\mathcal{R}^n \;\middle|\; \begin{array}{ll} (\mathcal{J}h(\bar{x})d)_k=0, & k=1,2,\dots,l,\\ \left(\mathcal{J}g(\bar{x})d\right)_i=0, & i\in I_+(\bar{x}),\\ \left(\mathcal{J}g(\bar{x})d\right)_i\le 0, & i\in I_0(\bar{x}),\\ \mathcal{J}G^j(\bar{x})d\in\mathcal{T}_{\mathcal{L}_{\theta_j}}\left(G^j(\bar{x})\right), & \bar{\Gamma}^j=0_{s_j},\\ \mathcal{J}G^j(\bar{x})d=0, & \bar{\Gamma}^j\in\operatorname{int}\mathcal{L}^*_{\theta_j},\\ \mathcal{J}G^j(\bar{x})d\in\mathcal{R}_+\left(\mathcal{H}_{\theta_j}\bar{\Gamma}^j\right), & \bar{\Gamma}^j\in\operatorname{bd}\mathcal{L}^*_{\theta_j}\setminus\left\{0_{s_j}\right\}, G^j(\bar{x})=0,\\ \left(\mathcal{J}G^j(\bar{x})d\right)^T\bar{\Gamma}^j=0, & \bar{\Gamma}^j\in\operatorname{bd}\mathcal{L}^*_{\theta_j}\setminus\left\{0_{s_j}\right\}, G^j(\bar{x})\in\operatorname{bd}\mathcal{L}_{\theta_j}\setminus\left\{0_{s_j}\right\}. \end{array}\right\}, \tag{3.9}$$

where the set $\mathcal{R}_+(\mathcal{H}_{\theta_j}\bar{\Gamma}^j)$ is defined by

$$\mathcal{R}_+\left(\mathcal{H}_{\theta_j}\bar{\Gamma}^j\right):=\left\{\sigma\mathcal{H}_{\theta_j}\bar{\Gamma}^j \mid \sigma\ge 0\right\}.$$

Proof. From the equality (3.8), we have

$$\mathcal{C}(\bar{x})=\left\{d\in\mathcal{R}^n \;\middle|\; \begin{array}{ll} (\mathcal{J}h(\bar{x})d)_k=0, & k=1,2,\dots,l,\\ \left(\mathcal{J}g(\bar{x})d\right)_i\le 0, \left(\mathcal{J}g(\bar{x})d\right)_i\bar{\eta}_i=0, & i\in I_+(\bar{x})\cup I_0(\bar{x}),\\ \mathcal{J}G^j(\bar{x})d\in\mathcal{T}_{\mathcal{L}_{\theta_j}}\left(G^j(\bar{x})\right), \left(\mathcal{J}G^j(\bar{x})d\right)^T\bar{\Gamma}^j=0, & j=1,2,\dots,J. \end{array}\right\}. \tag{3.10}$$

By the definitions of $I_+(\bar{x})$ and $I_0(\bar{x})$, we notice that the equalities in the second row of (2.15) are equivalent to

$$\begin{array}{ll} \left(\mathcal{J}g(\bar{x})d\right)_i=0, & i\in I_+(\bar{x}),\\ \left(\mathcal{J}g(\bar{x})d\right)_i\le 0, & i\in I_0(\bar{x}). \end{array}$$

To proceed, we analyze the remain part of the theorem by discussing four cases:

Case (1): If $\bar{\Gamma}^j=0_{s_j}$, then the third row of (3.10) becomes $\mathcal{J}G^i(\bar{x})d\in\mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))$.

Case (2): If $\bar{\Gamma}^j\in\operatorname{int}\mathcal{L}^*_{\theta_j}$, from the KKT condition (3.2), then $G^j(\bar{x})=0_{s_j}$. The explicit form of $\mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))$ defined in Theorem 2.3 implies that $\mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))=\mathcal{L}_{\theta_j}$. From the last row of (3.10), we obtain $\mathcal{J}G^j(\bar{x})d\in\mathcal{L}_{\theta_j}$. It follows from Theorem 2.5 and $(\mathcal{J}G^j(\bar{x})d)^T\bar{\Gamma}^j=0$ that $\mathcal{J}G^j(\bar{x})d=0_{s_j}$.

Case (3): If $\bar{\Gamma}^j\in\operatorname{bd}\mathcal{L}^*_{\theta_j}\setminus\{0_{s_j}\}$ and $G^j(\bar{x})=0_{s_j}$, then $\mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))=\mathcal{L}_{\theta_j}$ and $\mathcal{J}G^j(\bar{x})d\in\mathcal{L}_{\theta_j}\cap(\bar{\Gamma}^j)^{\perp}$. It follows from Theorem 2.5 that $\mathcal{J}G^j(\bar{x})d=0_{s_j}$ or there exists $\sigma>0$ such that $\mathcal{J}G^j(\bar{x})d=\sigma\mathcal{H}_{\theta_j}\bar{\Gamma}^j$. Hence, we have $\mathcal{J}G^j(\bar{x})d\in\mathcal{R}_+(\mathcal{H}_{\theta_j}\bar{\Gamma}^j)$.

Case (4): If $\bar{\Gamma}^j\in\operatorname{bd}\mathcal{L}^*_{\theta_j}\setminus\{0_{s_j}\}$, $G^j(\bar{x})\in\operatorname{bd}\mathcal{L}_{\theta_j}\setminus\{0_{s_j}\}$, we have

$$\mathcal{T}_{\mathcal{L}_{\theta_j}}\left(G^j(\bar{x})\right)=\left\{(h_1,h_2)\mid h_2^TG_2^j(\bar{x})-h_1G_1^j(\bar{x})\tan^2\theta_j\le 0\right\}.$$

Combining the above equality with the fact $\mathcal{J}G^j(\bar{x})d \in \mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x})) \cap (\bar{\Gamma}^j)^{\perp}$ as in (3.10), we obtain

$$\begin{aligned} &\left(\mathcal{J}G_2^j(\bar{x})d\right)^T G_2^j(\bar{x}) - \left(\mathcal{J}G_1^j(\bar{x})d\right) G_1^j(\bar{x}) \tan^2\theta_j \le 0, \\ &\left(\mathcal{J}G_2^j(\bar{x})d\right)^T \bar{\Gamma}_2^j + \left(\mathcal{J}G_1^j(\bar{x})d\right) \bar{\Gamma}_1^j = 0. \end{aligned} \tag{3.11}$$

From the KKT condition (3.2), we know $(\bar{\Gamma}^j)^T G^j(\bar{x}) = 0$. Because $\bar{\Gamma}^j \in \operatorname{bd} \mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\}$, $G^j(\bar{x}) \in \operatorname{bd} \mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}$, by the case (d) in Theorem 2.5, there exists $\sigma > 0$ such that $\bar{\Gamma}^j = \sigma \mathcal{H}_{\theta_j} G^j(\bar{x})$ and $(\mathcal{J}G_2^j(\bar{x})d)^T G_2^j(\bar{x}) - (\mathcal{J}G_1^j(\bar{x})d) G_1^j(\bar{x}) \tan^2\theta_j = 0$. Under this case, the equality (3.11) reduces to $(\mathcal{J}G^j(\bar{x})d)^T \bar{\Gamma}^j = 0$.

From the above discussions, the conclusion holds at the given stationary point $\bar{x}$. Thus, the proof is complete. □

Next, we calculate the "sigma" term of the optimization problem (1.1) in the below lemma, which plays an important role in describing the second-order optimality conditions for (1.1).

Lemma 3.4. *Let $\bar{x}$ be a stationary point of (1.1), $\bar{w} := (\bar{\mu}, \bar{\eta}, \bar{\Gamma}^1, \ldots, \bar{\Gamma}^J) \in \Lambda(\bar{x}) \subseteq \mathcal{R}^l \times \mathcal{R}^m \times \mathcal{R}^{s_1} \times \cdots \times \mathcal{R}^{s_J}$, $d \in \mathcal{C}(\bar{x})$, and RCQ (3.4) holds at $\bar{x}$. Denote the sigma term of (1.1) by*

$$\Upsilon\left(\left(\bar{\mu}, \bar{\eta}, -\bar{\Gamma}^1, \ldots, -\bar{\Gamma}^J\right), \mathcal{T}_\Omega^2(\mathcal{G}(\bar{x}), \mathcal{J}\mathcal{G}(\bar{x})d)\right),$$

where $\Upsilon(\cdot, \mathcal{T}_\Omega^2(\cdot,\cdot))$ means the support function of the second-order tangent set $\mathcal{T}_\Omega^2(\cdot,\cdot)$. Then, we have

$$\Upsilon\left(\left(\bar{\mu}, \bar{\eta}, -\bar{\Gamma}^1, \ldots, -\bar{\Gamma}^J\right), \mathcal{T}_\Omega^2(\mathcal{G}(\bar{x}), \mathcal{J}\mathcal{G}(\bar{x})d)\right) = d^T \left(\sum_{j=1}^J A^j\left(\bar{x}; \bar{\mu}, \bar{\eta}, \bar{\Gamma}^j\right)\right) d,$$

where the matrix $A^j(\bar{x}; \bar{\mu}, \bar{\eta}, \bar{\Gamma}^j)$ is defined by

$$A^j\left(\bar{x}; \bar{\mu}, \bar{\eta}, \bar{\Gamma}^j\right) := \begin{cases} \dfrac{\bar{\Gamma}_1^j}{G_1^j(\bar{x})} \cot^2\theta_j \left(\mathcal{J}G^j(\bar{x})\right)^T \mathcal{H}_{\theta_j} \mathcal{J}G^j(\bar{x}), & \text{if } G^j(\bar{x}) \in \operatorname{bd} \mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. From the definitions of Ω and $\mathcal{G}(x)$, we have

$$\begin{aligned} &\Upsilon\left(\left(\bar{\mu}, \bar{\eta}, -\bar{\Gamma}^1, \ldots, -\bar{\Gamma}^J\right), \mathcal{T}_\Omega^2(\mathcal{G}(\bar{x}), \mathcal{J}\mathcal{G}(\bar{x})d)\right) \\ = \ &\Upsilon\left(\bar{\mu}, \mathcal{T}_{\{0_l\}}^2(h(\bar{x}), \mathcal{J}h(\bar{x})d)\right) + \Upsilon\left(\bar{\eta}, \mathcal{T}_{\mathcal{R}_-^m}^2\left(g(\bar{x}), \mathcal{J}g(\bar{x})d\right)\right) \\ &+ \sum_{j=1}^J \Upsilon\left(-\bar{\Gamma}^j, \mathcal{T}_{\mathcal{L}_{\theta_j}}^2\left(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d\right)\right). \end{aligned} \tag{3.12}$$

Because $d \in \mathcal{C}(\bar{x})$, we know that $h(\bar{x}) = 0_l$ and $\mathcal{J}h(\bar{x})d = 0_l$. In addition, the definition of $\mathcal{T}^2_{\{0_l\}}(h(\bar{x}), \mathcal{J}h(\bar{x})d)$ implies that $\Upsilon(\bar{\mu}, \mathcal{T}^2_{\{0_l\}}(h(\bar{x}), \mathcal{J}h(\bar{x})d)) = 0$. For the second part of the right-hand side of (3.12), it follows from [19, Remark 3.47] that $\Upsilon(\bar{\eta}, \mathcal{T}^2_{\mathcal{R}^m_-}(g(\bar{x}), \mathcal{J}g(\bar{x})d)) = 0$. To proceed, we focus on discussing the last part of the above explicit formulas for the "sigma" term. From Theorem 2.4, the second-order tangent set $\mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d)$ has the following form:

(a) If $\mathcal{J}G^j(\bar{x})d \in \operatorname{int} \mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))$, then $\mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d) = \mathcal{R}^{s_j}$.
(b) If $G^j(\bar{x}) = 0_{s_j}$, then $\mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d) = \mathcal{T}_{\mathcal{L}_{\theta_j}}(\mathcal{J}G^j(\bar{x})d)$.
(c) If $G^j(\bar{x}) \in \operatorname{bd} \mathcal{L}_{\theta_j} \setminus \{0\}, \mathcal{J}G^j(\bar{x})d \in \operatorname{bd} \mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))$, then $\mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d) = \Xi^j$, where the set Ξ^j is defined by

$$\Xi^j := \left\{ \left(w_1^j, w_2^j\right) \in \mathcal{R} \times \mathcal{R}^{s_j - 1} \;\middle|\; \begin{array}{l} \left(w_2^j\right)^T G_2^j(\bar{x}) - w_1^j G_1^j(\bar{x}) \tan^2\theta_j \\ \leq \left(\mathcal{J}G_1^j(\bar{x})d\right)^2 \tan^2\theta_j - ||\mathcal{J}G_2^j(\bar{x})d||^2 \end{array} \right\}.$$

Because $d \in \mathcal{C}(\bar{x})$, we have $\mathcal{J}G^j(\bar{x})d \in \mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x})) \cap (\bar{\Gamma}^j)^\perp$. It follows from the KKT condition (3.2) that $-\bar{\Gamma}^j \in \mathcal{N}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))$, where $G^j(\bar{x}) \in \mathcal{L}_{\theta_j}$ and $\mathcal{N}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))$ denotes the normal cone of $\mathcal{L}_{\theta_j}$ at $G^j(\bar{x})$ in the sense of convex analysis [10], that is,

$$\left(-\bar{\Gamma}^j\right)^T \left(\tilde{G}^j - G^j(\bar{x})\right) \leq 0, \;\; \forall \tilde{G}^j \in \mathcal{L}_{\theta_j}. \tag{3.13}$$

For any given $w^j = (w_1^j, w_2^j) \in \mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d)$, there exist $\{t_n\} \downarrow 0$ and $(w^j)^n \to w^j$ such that $G^j(\bar{x}) + t_n \mathcal{J}G^j(\bar{x})d + \frac{1}{2}t_n^2 (w^j)^n \in \mathcal{L}_{\theta_j}$. From (3.13), we have $(-\bar{\Gamma}^j)^T(t_n \mathcal{J}G^j(\bar{x})d + \frac{1}{2}t_n^2 (w^j)^n) \leq 0$. Furthermore, due to the fact $\mathcal{J}G^j(\bar{x})d \in (\bar{\Gamma}^j)^\perp$, one can obtain that $(-\bar{\Gamma}^j)^T (w^j)^n \leq 0$. Taking $n \to +\infty$, we deduce $(-\bar{\Gamma}^j)^T w^j \leq 0$. Hence, $\Upsilon(-\bar{\Gamma}^j, \mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d)) \leq 0$. From the definition of $\mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d)$, if $\mathcal{J}G^j(\bar{x})d \in \operatorname{int} \mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))$, $G^j(\bar{x}) = 0_{s_j}$ or $\mathcal{J}G^j(\bar{x})d = 0$, then $0_{s_j} \in \mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d)$. In these cases,

$$\Upsilon\left(-\bar{\Gamma}^j, \mathcal{T}^2_{\mathcal{L}_{\theta_j}}\left(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d\right)\right) = 0.$$

Next, we consider the case $G^j(\bar{x}) \in \operatorname{bd}\mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}$, $\mathcal{J}G^j(\bar{x})d \in \operatorname{bd} \mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}))$. For simplicity, we denote $\mathcal{T}_j^2 := \mathcal{T}^2_{\mathcal{L}_{\theta_j}}(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d)$. Then, we have

$$\begin{aligned} & \Upsilon\left(-\bar{\Gamma}^j, \mathcal{T}_j^2\right) \\ = \;\; & \sup_{(w_1^j, w_2^j) \in \mathcal{T}_j^2} \left\{ -\left(\bar{\Gamma}_1^j w_1^j + \left(\bar{\Gamma}_2^j\right)^T w_2^j\right) \;\middle|\; \begin{array}{l} \left(w_2^j\right)^T G_2^j(\bar{x}) - w_1^j G_1^j(\bar{x}) \tan^2\theta_j \\ \leq \left(\mathcal{J}G_1^j(\bar{x})d\right)^2 \tan^2\theta_j - ||\mathcal{J}G_2^j(\bar{x})d||^2 \end{array} \right\}. \end{aligned}$$

Using the KKT condition (3.2), we know $\mathcal{L}^*_{\theta_j} \ni \bar{\Gamma}^j \perp G^j(\bar{x}) \in \mathcal{L}_{\theta_j}$. Hence, applying Theorem 2.5 yields that $\bar{\Gamma}^j = 0_{s_j}$ or $\bar{\Gamma}^j \in \mathrm{bd}\,\mathcal{L}^*_{\theta_j} \setminus \{0_{s_j}\}$. If the first case occurs, then

$$\Upsilon\left(-\bar{\Gamma}^j, \mathcal{T}^2_{\mathcal{L}_{\theta_j}}\left(G^j(\bar{x}), \mathcal{J}G^j(\bar{x})d\right)\right) = 0.$$

In other case, we use the fact (d) in Theorem 2.5, there exists $\sigma>0$ such that $\bar{\Gamma}^j = \sigma \mathcal{H}_{\theta_j} G^j(\bar{x})$. The following facts

$$\bar{\Gamma}^j \in \mathrm{bd}\mathcal{L}^*_{\theta_j} \setminus \{0_{s_j}\},\ \ G^j(\bar{x}) \in \mathrm{bd}\mathcal{L}_{\theta_j} \setminus \{0\},\ \ \left(\bar{\Gamma}^j\right)^T G^j(\bar{x}) = 0$$

imply that $\sigma = \frac{\bar{\Gamma}^j_1}{G^j_1(\bar{x})} \cot^2\theta_j$. In addition, we have

$$\begin{aligned}
-\left(\bar{\Gamma}^j_1 w^j_1 + \left(\bar{\Gamma}^j_2\right)^T w^j_2\right) &= -\frac{\bar{\Gamma}^j_1}{G^j_1(\bar{x})} \cot^2\theta_j \left(\mathcal{H}_{\theta_j} G^j(\bar{x})\right)^T w^j \\
&= -\frac{\bar{\Gamma}^j_1}{G^j_1(\bar{x})} \cot^2\theta_j \begin{bmatrix} G^j_1(\bar{x}) \\ G^j_2(\bar{x}) \end{bmatrix}^T \begin{bmatrix} \tan^2\theta_j & 0 \\ 0 & -I_{s_j-1} \end{bmatrix} \begin{bmatrix} w^j_1 \\ w^j_2 \end{bmatrix} \\
&= \frac{\bar{\Gamma}^j_1}{G^j_1(\bar{x})} \cot^2\theta_j \left(\left(w^j_2\right)^T G^j_2(\bar{x}) - w^j_1 G^j_1(\bar{x}) \tan^2\theta_j\right).
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
\Upsilon\left(-\bar{\Gamma}^j, \mathcal{T}^2_j\right) &= \frac{\bar{\Gamma}^j_1}{G^j_1(\bar{x})} \cot^2\theta_j \left(\left(\mathcal{J}G^j_1(\bar{x})d\right)^2 \tan^2\theta_j - ||\mathcal{J}G^j_2(\bar{x})d||^2\right) \\
&= \frac{\bar{\Gamma}^j_1}{G^j_1(\bar{x})} \cot^2\theta_j \begin{bmatrix} \mathcal{J}G^j_1(\bar{x})d \\ \mathcal{J}G^j_2(\bar{x})d \end{bmatrix}^T \begin{bmatrix} \tan^2\theta_j & 0 \\ 0 & -I_{s_j-1} \end{bmatrix} \begin{bmatrix} \mathcal{J}G^j_1(\bar{x})d \\ \mathcal{J}G^j_2(\bar{x})d \end{bmatrix}^T \\
&= \frac{\bar{\Gamma}^j_1}{G^j_1(\bar{x})} \cot^2\theta_j d^T \left(\mathcal{J}G^j(\bar{x})\right)^T \mathcal{H}_{\theta_j} \mathcal{J}G^j(\bar{x})d,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\Upsilon\left(\left(\bar{\mu}, \bar{\eta}, -\bar{\Gamma}^1, \ldots, -\bar{\Gamma}^J\right), \mathcal{T}^2_\Omega(\mathcal{G}(\bar{x}), \mathcal{J}\mathcal{G}(\bar{x})d)\right) \\
&= \sum_{j=1}^J \Upsilon\left(-\bar{\Gamma}^j, \mathcal{T}^2_j\right) = d^T \left(\sum_{j=1}^J A^j\left(\bar{x}; \bar{\mu}, \bar{\eta}, \bar{\Gamma}^j\right)\right) d,
\end{aligned}$$

where

$$A^j\left(\bar{x}; \bar{\mu}, \bar{\eta}, \bar{\Gamma}^j\right) := \begin{cases} \frac{\bar{\Gamma}^j_1}{G^j_1(\bar{x})} \cot^2\theta_j \left(\mathcal{J}G^j(\bar{x})\right)^T \mathcal{H}_{\theta_j} \mathcal{J}G^j(\bar{x}), & \text{if } G^j(\bar{x}) \in \mathrm{bd}\ \mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the proof is complete. □

Because both sets $\mathcal{R}^m_-$ and $\mathcal{L}_{\theta_i}(i = 1, 2, ..., J))$ are second-order regular, similar to [19, Theorem 3.86], we state in the following theorem that there is no gap between the second-order necessary and second-order sufficient conditions for the general circular conic optimization problem (1.1), in which we also establish the equivalent relationship between the no gap second-order optimality condition and the quadratic growth condition.

Theorem 3.5. *Suppose that $\bar{x}$ is a local minimizer of (1.1) and RCQ (3.4) holds at $\bar{x}$. Then, the following inequality holds at any given $d \in \mathcal{C}(\bar{x})$,*

$$\sup_{(\bar{\mu},\bar{\eta},\bar{\Gamma}^1,...,\bar{\Gamma}^J)\in\Lambda(\bar{x})} d^T\left(\nabla^2_{xx}L\left(\bar{x};\bar{\mu},\bar{\eta},\bar{\Gamma}^1,...,\bar{\Gamma}^J\right) - \sum_{j=1}^{J} A^j\left(\bar{x};\bar{\mu},\bar{\eta},\bar{\Gamma}^j\right)\right)d \geq 0.$$

Conversely, let $\bar{x}$ be a feasible solution of (1.1) satisfying the first-order optimality conditions (3.2). Suppose that RCQ (3.4) holds at $\bar{x}$. Then, for any given $d \in \mathcal{C}(\bar{x}) \setminus \{0_n\}$, the condition

$$\sup_{(\bar{\mu},\bar{\eta},\bar{\Gamma}^1,...,\bar{\Gamma}^J)\in\Lambda(\bar{x})} d^T\left(\nabla^2_{xx}L\left(\bar{x};\bar{\mu},\bar{\eta},\bar{\Gamma}^1,...,\bar{\Gamma}^J\right) - \sum_{j=1}^{J} A^j\left(\bar{x};\bar{\mu},\bar{\eta},\bar{\Gamma}^j\right)\right)d>0$$

is necessary and sufficient for the quadratic growth condition at the point $\bar{x}$:

$$f(x) \geq f(\bar{x}) + c||x-\bar{x}||^2, \quad \forall x \in \mathcal{N} \cap \mathcal{F}$$

for some constant $c > 0$ and a neighborhood $\mathcal{N}$ of $\bar{x}$, where $\mathcal{F}$ denotes the feasible set of (1.1), that is,

$$\mathcal{F} := \left\{x \in \mathcal{R}^n \,|\, h(x) = 0, g(x) \leq 0, \left(G^i_1(x), G^i_2(x)\right) \in \mathcal{L}_{\theta_i}(i = 1, 2, ..., J)\right\}.$$

4. Example

In this section, we present an example to illustrate these results established in this article.

Example 4.1. Consider the following circular conic optimization problem

$$\begin{array}{ll} \min & x_3 \\ \text{s.t.} & 1-x_2^2 = 0, \\ & 2x_2-x_1^2 \leq 0, \\ & \left(\sqrt{3}x_3, x_1^2\right) \in \mathcal{L}_{\frac{\pi}{6}} \subset \mathcal{R}^2 \end{array}$$

at the reference point $x^* = (0, -1, 0)^T \in \mathcal{R}^3$.

It is not hard to see that

$$\begin{array}{l} f(x) := x_3,\ \ h(x) := 1-x_2^2,\ \ g(x) := 2x_2-x_1^2, \\ G(x) := (G_1(x), G_2(x)),\ \ G_1(x) := \sqrt{3}x_3,\ \ G_2(x) := x_1^2 \end{array}$$

and the Lagrangian function is given by

$$\begin{aligned} L(x;\mu,\eta,\Gamma) &:= f(x) + h(x)\mu + g(x)\eta - (G_1(x)\Gamma_1 + G_2(x)\Gamma_2) \\ &= x_3 + \left(1-x_2^2\right)\mu + \left(2x_2-x_1^2\right)\eta - \left(\sqrt{3}x_3\Gamma_1 + x_1^2\Gamma_2\right), \end{aligned}$$

where $\mu \in \mathcal{R}$, $\eta \in \mathcal{R}_+$, $\Gamma = (\Gamma_1, \Gamma_2) \in \mathcal{R}^2$ are the associated multipliers. In addition, the KKT condition can be characterized as

$$\begin{array}{l} -2x_1\eta - 2x_1\Gamma_2 = 0,\ \ -2x_2\mu + 2\eta = 0, \\ 1-\sqrt{3}\Gamma_1 = 0,\ \ 1-x_2^2 = 0, \\ 0 \le \eta \perp \left(2x_2-x_1^2\right) \le 0, \\ \mathcal{L}_{\frac{\pi}{6}}^* \ni \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \perp \begin{bmatrix} \sqrt{3}x_3 \\ x_1^2 \end{bmatrix} \in \mathcal{L}_{\frac{\pi}{6}}. \end{array}$$

By direct calculation, the corresponding multipliers are obtained:

$$\mu = 0,\ \ \eta = 0,\ \ \Gamma_1 = \frac{1}{\sqrt{3}},\ \ |\Gamma_2| \le 1.$$

Next, we will verify the corresponding constraint qualifications at x^*. Notice that

$$\begin{array}{r} h(x^*) = 0,\ \ \mathcal{J}h(x^*) = (0,2,0), \\ g(x^*) = -2<0,\ \ \mathcal{J}g(x^*) = (0,2,0), \\ \left(G_1(x^*), G_2(x^*)\right) = (0,0),\ \ \mathcal{J}G_1(x^*) = \left(0,0,\sqrt{3}\right),\ \ \mathcal{J}G_2(x^*) = (0,0,0). \end{array}$$

It follows from [19, Corollary 2.101] that RCQ holds at x^* if there exists at least one vector $w = (w_1, w_2, w_3)^T \in \mathcal{R}^3$ satisfying the following system

$$\begin{array}{l} h(x^*) + \mathcal{J}h(x^*)w = 0, \\ g(x^*) + \mathcal{J}g(x^*)w<0, \\ \begin{bmatrix} G_1(x^*) \\ G_2(x^*) \end{bmatrix} + \begin{bmatrix} \mathcal{J}G_1(x^*) \\ \mathcal{J}G_2(x^*) \end{bmatrix} w \in \mathrm{int}\mathcal{L}_{\frac{\pi}{6}} \end{array} \iff \begin{array}{l} w_2 = 0, \\ -1 + w_2<0, \\ w_3>0. \end{array}$$

It is obvious that the right-hand side of system holds at $\bar{w} = (0,0,1)^T$, which says that RCQ holds at x^*. Note that

$$\mathcal{J}h(x^*)^T = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},\ \ \mathcal{J}G(x^*)^T e_1 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{3} \end{bmatrix},\ \ \mathcal{J}G(x^*)^T e_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are linearly dependent. From Theorem 3.1, the constraint nondegeneracy condition is false at x^*. In addition, we can verify that Theorem 3.2 holds at x^*, which uses the fact that the multiplier set

$$\Lambda(x^*) = \left\{(\mu, \eta, \Gamma) \mid \mu = 0,\ \eta = 0,\ \Gamma_1 = \frac{1}{\sqrt{3}},\ \ |\Gamma_2| \le 1\right\}.$$

is a nonempty, convex compact set.

Finally, we analyze the corresponding critical cone and no gap second-order optimality condition at the reference point x^*. In this case, we have

$$g(x^*) < 0,\ \ G(x^*) = (0,0),\ \ \Gamma^* = \left(\frac{1}{\sqrt{3}}, \Gamma_2^*\right),$$

where $\Gamma_2^* \in \mathcal{R}$ satisfies the relation $|\Gamma_2^*| \le 1$. It follows from Theorem 3.3 that

$$\mathcal{C}(x^*) = \begin{cases} \left\{d \in \mathcal{R}^3 \mid \mathcal{J}h(x^*)d = 0, \mathcal{J}G(x^*)d \in \mathcal{R}_+\left(\mathcal{H}_{\frac{\pi}{3}}\Gamma^*\right)\right\} & \text{if } \Gamma_2^* = \pm 1, \\ \left\{d \in \mathcal{R}^3 \mid \mathcal{J}h(x^*)d = 0, \mathcal{J}G(x^*)d = 0\right\} & \text{otherwise.} \end{cases}$$

From the above definition, we obtain

$$\mathcal{C}(x^*) = \left\{d = (d_1, d_2, d_3)^T \in \mathcal{R}^3 \mid d_1 \in \mathcal{R},\ \ d_2 = 0,\ \ d_3 = 0\right\}.$$

Furthermore, the Hessian matrix $\nabla^2_{xx}L(x; \mu, \eta, \Gamma)$ and the matrix $A(x; \mu, \eta, \Gamma)$ in the "sigma" term are given by

$$\nabla^2_{xx}L(x^*; \mu^*, \eta^*, \Gamma^*) = \begin{bmatrix} -2\Gamma_2^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A(x^*; \mu^*, \eta^*, \Gamma^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, for any given

$$d \in \mathcal{C}(x^*) := \left\{d = (d_1, d_2, d_3)^T \in \mathcal{R}^3 \mid d_1 \in \mathcal{R},\ \ d_2 = 0,\ \ d_3 = 0\right\},$$

we have

$$\sup_{(\mu^*, \eta^*, \Gamma^*) \in \Lambda(x^*)} d^T\left(\nabla^2_{xx}L(x^*; \mu^*, \eta^*, \Gamma^*) - A(x^*; \mu^*, \eta^*, \Gamma^*)\right)d = \sup_{|\Gamma_2^*| \le 1} -2d_1^2\Gamma_2^* = 2d_1^2 \ge 0,$$

which implies that the second-order necessary condition holds at x^*. Moreover, the no gap second-order optimality condition at x^* is equivalent to the conclusion that there exist a positive constant c and a feasible neighborhood $\mathcal{N}^*$ around x^* such that

$$x_3 \ge c\left(x_1^2 + (x_2 + 1)^2 + x_3^2\right),\ \ \forall x = (x_1, x_2, x_3)^T \in \mathcal{N}^*.$$

From this, it is not hard to find that the above inequality is true if we set $c = 1$ and

$$\mathcal{N}^* = \left\{ x = (x_1, x_2, x_3)^T \in \mathcal{R}^3 \,|\, x_1 = 0,\ \ x_2 = -1,\ \ x_3 \in [0, 1] \right\}.$$

5. Concluding remarks

In this article, we characterize the no gap second-order optimality conditions for a class of circular conic optimization problems. As byproducts, we present the explicit descriptions for the critical cone and the "sigma" term of the given programs as well. Meanwhile, we establish the equivalent form of the quadratic growth condition, which fills the gap in the optimality theory of circular cone programing.

It should be emphasized that in this article we develop a primal approach to deriving optimality conditions for circular cone programing problems by using tangential approximations. In contrast, there is a dual approach to these and related issues based on employing normal cones. For example, Zhou, Chen and Mordukhovich [35] recently present some calculations of normal cones and related coderivatives to the circular cone mapping (i.e., the dual second-order constructions), in which these results were employed to deriving second-order characterizations of crucial stability issues of variational analysis in circular cone programing. How these results can be extended to the general case such as the circular cone programing problem (1.1)? We believe that it is possible to follow the scheme of [9] and the results in [35] to answer this question. On the other hand, our theoretical results are obtained under some assumptions such as the Robinson constraint qualification or the constraint nondegeneracy condition. However, in the recent development of nonlinear programing, some weaker CQs are proposed to achieve the task of stability issues such as complete characterizations of tilt stability [21, 36]. How to construct the weaker CQs for (1.1) maybe another interesting topic for our study. As mentioned above, the no gap second-order optimality conditions also play a crucial impact on some issues in numerical design such as error bound and complexity analysis. Would it be possible to establish these results for the given problem (1.1)? We leave these further discussions as our future work.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The first author is supported by the Doctoral Foundation of Tianjin Normal University [grant number 52XB1513], the National Natural Science Foundation of China [grant number 11601389, 11871372]. The second author is supported by Ministry of Science and Technology, Taiwan.

ORCID

Jein-Shan Chen http://orcid.org/0000-0002-4596-9419

References

[1] Ko, C.-H., Chen, J.-S. (2013). Optimal grasping manipulation for multifingered robots using semismooth Newton method. *Math. Probl. Eng.* 681710–681719. pages. doi:10.1155/2013/681710.

[2] Zhou, J. C., Chen, J.-S. (2013). Properties of circular cone and spectral factorization associated with circular cone. *J. Nonlinear Convex Anal.* 14(4):807–816.

[3] Zhou, J. C., Chen, J.-S. (2014). The vector-valued functions associated with circular cones. *Abstr. Appl. Anal.* 2014:603542. doi:10.1155/2014/603542.

[4] Zhou, J. C., Chen, J.-S., Hung, H.-F. (2013). Circular cone convexity and some inequalities associated with circular cones. *J. Inequal. Appl.* 2013(1):571–517. doi:10.1186/1029-242X-2013-571.

[5] Bai, Y. Q., Gao, X. R., Wang, G. Q. (2015). Primal-dual interior-point algorithms for convex quadratic circular cone optimization. *Numer. Algebra Control Optim.* 5(2):211–231. doi:10.3934/naco.2015.5.211.

[6] Bai, Y. Q., Ma, P. F., Zhang, J. (2015). A polynomial-time interior-point method for circular coneprogramming based on kernel functions. *JIMO* 12(2):739–756. doi:10.3934/jimo.2016.12.739.

[7] Chi, X., Wan, Z., Zhu, Z., Yuan, L. (2016). A nonMonotone smoothing Newton method for circular cone programming. *Optimization* 65(12):2227–2250. doi:10.1080/02331934.2016.1217861.

[8] Miao, X. H., Guo, S. J., Qi, N., Chen, J.-S. (2016). Constructions of complementarity functions and merit functions for circular cone complementarity problem. *Comput. Optim. Appl.* 63(2):495–522. doi:10.1007/s10589-015-9781-1.

[9] Mordukhovich, B. S. (2006). *Variational Analysis and Generalized Differentiation*, Vols. I and II. Berlin: Springer.

[10] Rockafellar, R. T. (1996). *Convex Analysis*. Princeton: Princeton University Press.

[11] Rockafellar, R. T., Wets, R.J.B. (1998). *Variational Analysis*. Berlin: Springer.

[12] Ma, P. F., Bai, Y. Q., Chen, J.-S. (2016). A self-concordant interior point algorithm for nonsymmetric circular cone programming. *J. Nonlinear Convex Anal.* 17(2):225–241.

[13] Zhou, J. C., Chen, J.-S. (2017). Monotonicity and circular cone monotonicity associated with circular cones. *Set-Valued Var. Anal.* 25(2):211–232. doi:10.1007/s11228-016-0374-7.

[14] Zhou, J. C., Chang, Y.-L., Chen, J.-S. (2015). The H-differentiability and calmness of circular cone functions. *J. Glob. Optim.* 63(4):811–833. doi:10.1007/s10898-015-0312-5.

[15] Zhou, J. C., Tang, J. Y., Chen, J.-S. (2016). Further relationship between second-order cone and positive semidefinite cone. *Optimization* 65(12):2115–2133. doi:10.1080/02331934.2016.1220553.

[16] Zhou, J. C., Tang, J. Y., Chen, J.-S. (2017). Parabolic second-order directional differentiability in the hadamard sense of the vector-valued functions associated with circular cones. *J. Optim. Theory Appl.* 172(3):802–823. doi:10.1007/s10957-016-0935-9.

[17] Bonnans, J. F., Cominetti, R., Shapiro, A. (1999). Second order optimality conditions based on parabolic second order tangent sets. *SIAM J. Optim.* 9(2):466–492. doi:10.1137/S1052623496306760.

[18] Bonnans, J. F. H., Ramirez, C. (2005). Perturbation analysis of second-order cone programming problems. *Math. Prog.* 104(2/3):205–227. doi:10.1007/s10107-005-0613-4.

[19] Bonnans, J. F., Shapiro, A. (2000). *Perturbation Analysis of Optimization Problems*. New York, NY: Springer.

[20] Shapiro, A. (1997). First and second order analysis of nonlinear semidefinite programs. *Math. Prog.* 77(1):301–320. doi:10.1007/BF02614439.

[21] Mordukhovich, B. S., Nghia, T. T. A. (2015). Second-order characterizations of tilt stability with applications to nonlinear programming. *Math. Prog.* 149(1/2):83–104. doi:10.1007/s10107-013-0739-8.

[22] Mordukhovich, B. S., Outrata, J. V., Ramirez, H. (2015). Second-order variational analysis in conic programming with applications to optimality conditions and stability. *SIAM J. Optim.* 25(1):76–101. doi:10.1137/120903221.

[23] Mordukhovich, B. S., Outrata, J. V., Sarabi, M. E. (2014). Full stability of locally optimal solutions in second-order cone programs. *SIAM J. Optim.* 24(4):1581–1613. doi:10.1137/130928637.

[24] Mordukhovich, B. S., Rockafellar, R. T. (2012). Second-order subdifferential calculus with applications to tilt stability in optimization. *SIAM J. Optim.* 22(3):953–986. doi:10.1137/110852528.

[25] Bonnans, J. F., Gilbert, J. C., Lemarechal, C., Sagastizabal, C. A. (2006). *Numerical Optimization: Theoretical and Practical Aspects*. Berlin: Springer.

[26] Drusvyatskiy, D., Lewis, A. S. (2018). Error Bounds, quadratic Growth, and Linear Convergence of Proximal Methods. Math. Oper. Res. 43(3):919–948. doi:10.1287/moor.2017.0889.

[27] Dontchev, A. L., Rockafellar, R. T. (2009). *Implicit Functions and Solution Mappings*. New York, NY: Springer.

[28] Fabian, M., Henrion, R., Kruger, A. Y., Outrata, J. V. (2010). Error bounds: Necessary and sufficient conditions. *Set-Valued Variat. Anal.* 18(2):121–149. doi:10.1007/s11228-010-0133-0.

[29] Ioffe, A. D., Outrata, J. V. (2008). On metric and calmness qualification conditions in subdifferential calculus. *Set-Valued Anal.* 16(2/3):199–227. doi:10.1007/s11228-008-0076-x.

[30] Gfrerer, H., Outrata, J. V. (2016). On Lipschitzian properties of implicit multifunctions. *SIAM J. Optim.* 26(4):2160–2189. doi:10.1137/15M1052299.

[31] Li, G. Y., Mordukhovich, B. S. (2012). Hölder metric subregularity with applications to proximal point method. *SIAM J. Optim.* 22:1655–1684. doi:10.1137/120864660.

[32] Mordukhovich, B. S., Sarabi, M. E. (2018). Critical multipliers in variational systems via second-order generalized differentiation. *Math. Prog.* 169(2):605–648. doi:10.1007/s10107-017-1155-2.

[33] Luo, Z., Pang, J., Ralph, D. (1996). *Mathematical Programs with Equilibrium Constraints*. Cambridge, UK: Cambridge University Press.

[34] J., Outrata, M. K. [*ssmile*]ocvara and J. Zowe (1998). *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*. Boston, MA: Kluwer Academic Publishers.

[35] Zhou, J. C., Chen, J.-S., Mordukhovich, B. S. (2015). Variational analysis of circular cone programs. *Optimization* 64(1):113–147. doi:10.1080/02331934.2014.951043.

[36] Gfrerer, H., Mordukhovich, B. S. (2015). Complete characterizations of tilt stability in nonlinear programming under weakest qualification conditions. *SIAM J. Optim.* 25(4):2081–2119. doi:10.1137/15M1012608.