An SAA approach for solving a class of stochastic linear semidefinite inverse optimal value problems

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Abstract

In this paper, we consider a class of stochastic inverse linear semidefinite optimal value problems, in which the forward problem is a linear semidefinite programming problem (LSDP), and the data in its constraints is affected by a random variable. Under some mild assumptions for LSDP, the corresponding inverse optimal value problem can be reformulated as a mathematical program with stochastic linear semidefinite complementarity constraints (MPSLSDCC). By employing the techniques of sample average approximation (SAA), we construct a series of smooth SAA subproblems and transform them into nonlinear semidefinite programming problems by utilizing the smooth Fischer-Burmeister function for linear semidefinite complementarity constraints. In addition, we prove that the sequence of global minimizer (respectively, KKT point) of these SAA subproblems converge with probability one (w.p.1) to a global minimizer (respectively, an S-stationary point) of MPSLSDCC under mild assumptions. Finally, some numerical experiments are presented to show the ability of our method for solving the given stochastic linear semidefinite inverse optimal value problems.

Keywords: Inverse optimal value problem, Sample average approach, Stationary points

1 Introduction

Inverse optimization problems have wide applications in network flow problems, transportation problems, and portfolio optimization. The systematic study on inverse problems began in some shortest paths problems initially analyzed by Burton and Toint [4], whose results triggered a series of contributions on inverse network problems [2, 3, 5, 9, 25, 26]. After that, several special inverse continuous optimization models appear in the literature [10, 12, 13, 17, 19, 20, 23, 24, 27, 28], which include inverse linear programming problems [23, 24], inverse quadratic programming problems [27], inverse second-order cone programming problems [28], inverse positive semidefinite cone programming problems [12, 13, 17, 19, 20] and inverse conic programming problems [10].

As a variant of inverse optimization problem, an inverse optimal value problem aims to adjust parameter values in an optimization model that makes the optimal objective value closest to a given target value. Some applications on inverse optimal value problems for combinatorial models include the minimum spanning tree problem [11, 15, 21, 22] and the shortest paths problem [29]. For the field of continuous optimization problems, Ahmed and Guan [1] discussed the linear programming problem, i.e.,

$$\min_x \ c^T x, \ \text{s.t.} \ Ax \le b, \tag{1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$ and the corresponding inverse optimal value problem (IOVP) is given by

$$\min_c \ \frac{1}{2}(Q(c) - v^*)^2, \ \text{s.t.} \ c \in \mathcal{C} := \{c \in \mathbb{R}^n : c_i^L \le c_i \le c_i^U, i \in [n] := 1, 2, \cdots, n\}, \tag{2}$$

where $c^L, c^U \in \mathbb{R}^n$ are respectively the lower and the upper bound of cost vectors, v^* is the pre-specified objective value of (1), $Q(c)$ is the optimal value of (1) under the parameter c, i.e.,

$$Q(c) := \min_x \left\{ c^T x : Ax \le b \right\}.$$

They proved that the above IOVP (2) under the given data (A, b) is NP-hard and obtain the optimal parameter by solving a series of linear and bilinear programming problems under some special assumptions.

During the past three decades, linear semidefinite programming (LSDP) has drawn great attention in optimization, engineering, and economics, which can be viewed as an important generalization of linear programming and has the following form

$$\min_x \ c^T x, \ \text{s.t.} \ \mathcal{A}(x) - B \in \mathbb{S}^m_-, \tag{3}$$

where $c \in \mathbb{R}^n$, $\mathbb{S}^m$ denotes the collection of m-dimensional real symmetric matrices, $\mathcal{A}(\cdot) : \mathbb{R}^n \to \mathbb{S}^m$ is a linear operator, whose definition is

$$\mathcal{A}(x) = A_1 x_1 + A_2 x_2 + \cdots + A_n x_n$$

with $A_1, A_2, \cdots, A_n \in \mathbb{S}^m$, $B \in \mathbb{S}^m$ and $\mathbb{S}^m_-$ denotes the collection of m-dimensional real symmetric negative semidefinite matrices. In this article, we consider the corresponding inverse optimal value problem of LSDP (3), an idea similar to the one in [1], under some mild assumptions (such as problem (3) has an attainable optimal value and the associated Slater's condition holds, i.e., there exists a vector $\hat{x} \in \mathbb{R}^n$ such that $\mathcal{A}(\hat{x}) - B \in \mathbb{S}^m_{--}$, where $\mathbb{S}^m_{--}$ denotes the collection of m-dimensional real symmetric negative definite matrices), it can be transformed into the following optimization problem with a linear semidefinite complementarity constraint

$$\begin{array}{ll} \min\limits_{c,x,Y} & \frac{1}{2}(c^T x - v^*)^2, \\ \text{s.t.} & c_i^L \le c_i \le c_i^U,\ \langle A_i, Y\rangle + c_i = 0,\ i \in [n], \\ & \mathcal{A}(x) - B \in \mathbb{S}^m_-,\ Y \in \mathbb{S}^m_+,\ \langle \mathcal{A}(x) - B, Y\rangle = 0, \end{array}$$

where $Y \in \mathbb{S}^m_+$ means that Y is positive semidefinite, $\langle\cdot,\cdot\rangle$ denotes the trace inner product defined in $\mathbb{S}^m$, i.e., $\langle W, V\rangle := \mathrm{Tr}(WV)$ for any given $W, V \in \mathbb{S}^m$. Notice that the parameters of LSDP are often obtained from historical data or experiments when modeling practical problems and have strong statistical features. To the best of our knowledge, there is no discussion about inverse optimal value problems under data uncertainty.

Assume that the data $(A_1, \cdots, A_n, B)$ are affected by a random variable ξ and consider the following stochastic linear semidefinite inverse optimal value problem (SLSDIOVP)

$$\begin{array}{ll} \min\limits_{c,x,Y} & \frac{1}{2}(c^T x - v^*)^2, \\ \text{s.t.} & c_i^L \le c_i \le c_i^U,\ \langle \mathbb{E}[A_i(\xi)], Y\rangle + c_i = 0,\ i \in [n], \\ & \mathbb{E}[\mathcal{A}(\xi, x) - B(\xi)] \in \mathbb{S}^m_-,\ Y \in \mathbb{S}^m_+,\ \langle \mathbb{E}[\mathcal{A}(\xi, x) - B(\xi)], Y\rangle = 0, \end{array} \tag{4}$$

where the operator $\mathcal{A}(\xi, \cdot)$ is given by

$$\mathcal{A}(\xi, x) := A_1(\xi)x_1 + A_2(\xi)x_2 + \cdots + A_n(\xi)x_n,\ x := (x_1, x_2, \cdots, x_n)^T. \tag{5}$$

For notational simplicity, we also denote $\mathcal{A}^*(\xi, \cdot)$ the adjoint of $\mathcal{A}(\xi, \cdot)$, which is given by

$$\mathcal{A}^*(\xi, Y) = \left[\langle A_1(\xi), Y\rangle, \langle A_2(\xi), Y\rangle, \cdots, \langle A_n(\xi), Y\rangle\right]^T, \quad Y \in \mathbb{S}^m. \tag{6}$$

However, it is often difficult to calculate mathematical expectations, particularly in computing high-dimensional integrals. To tackle this hurdle, an auxiliary approximate subproblem of (4) is constructed by using the sample average approximation (SAA). Suppose that a sample set $\{\xi_1, \cdots, \xi_N\}$ of N realizations of random vector ξ is obtained, and assume that each random vector ξ_i $(i = 1, \cdots, N)$ is independently identically distributed, the corresponding SAA subproblems (SAA-SLSDIOVP) of (4) are described by

$$
\begin{aligned}
\min_{c,x,Y}\ & \frac{1}{2}(c^T x - v^*)^2 \\
\text{s.t.}\ \ & c_i^L \le c_i \le c_i^U,\ \left\langle \frac{1}{N}\sum_{k=1}^{N} A_i(\xi_k), Y \right\rangle + c_i = 0,\ i \in [n], \\
& \frac{1}{N}\sum_{k=1}^{N}[\mathcal{A}(\xi_k, x) - B(\xi_k)] \in \mathbb{S}^m_-,\ Y \in \mathbb{S}^m_+,\ \left\langle \frac{1}{N}\sum_{k=1}^{N}[\mathcal{A}(\xi_k, x) - B(\xi_k)], Y \right\rangle = 0.
\end{aligned} \tag{7}
$$

It is easy to see that the above SAA subproblems can be viewed as mathematical programs with linear semidefinite complementarity constraints (MPLSDCC). Compared with the traditional nonlinear programming problems (NLP), the concepts in stationary point for NLP are not suitable for MPLSDCC, therefore we also need to discuss these theoretical issues on MPLSDCC.

In this paper, we use the smooth Fischer-Burmeister function to deal with linear semidefinite complementarity constraints that appeared in problem (7) and transform the associated smooth SAA subproblems of the given inverse optimal value problem into nonlinear semidefinite programming problems (NSDP). In addition, we prove that the sequence of global minimizer (respectively, KKT point) of these SAA subproblems converge with probability one (w.p.1) to a global minimizer (respectively, an S-stationary point) of SISDIOVP (4) under mild assumptions. Some numerical experiments are presented to show the ability of our method for solving the given stochastic linear semidefinite inverse optimal value problems.

The remainder of this article is organized as follows. In Section 2, we present some preliminaries about stationary points in MPLSDCC and the smooth Fischer-Burmeister function for linear semidefinite complementarity constraint in MPLSDCC. In Section 3, we discuss the relationship between the solutions of smooth SAA subproblems defined as in (18) with the one in problem (4) and show the associated convergence results. In Section 4, we conduct some numerical experiments to test the performance of our proposed method. Finally, some concluding remarks are drawn in Section 5.

To close this section, we say a few words about notations. Through out this paper, we denote by $z := (c, x, Y)$ and

$$
\begin{aligned}
& f(z) := \frac{1}{2}(c^T x - v^*)^2, && G(z) := Y, \\
& g_1(z) := c - c^L, && g_2(z) := c - c^U, \\
& h_i(z) := \langle \mathbb{E}[A_i(\xi)], Y\rangle + c_i, && h_i^N(z) := \left\langle \frac{1}{N}\sum_{k=1}^{N}[A_i(\xi_k)], Y \right\rangle + c_i, \\
& H(z) := \mathbb{E}[\mathcal{A}(\xi, x) - B(\xi)], && H^N(z) := \frac{1}{N}\sum_{k=1}^{N}[\mathcal{A}(\xi_k, x) - B(\xi_k)].
\end{aligned} \tag{8}
$$

2 Preliminaries

For any given $W \in \mathbb{S}^m$, it is known that it admits eigenvalue decomposition

$$W = P \begin{bmatrix} \Lambda_\alpha & & \\ & 0_\beta & \\ & & \Lambda_\gamma \end{bmatrix} P^T, \tag{9}$$

where $P \in \mathbb{S}^m$ is an orthogonal matrix such that $P^T P = PP^T = I_m$, I_m denotes the m-dimensional entity matrix, α, β, γ are three index sets whose definitions are given by

$$\alpha := \{j \in [m] : \lambda_j(W) > 0\},\ \beta := \{j \in [m] : \lambda_j(W) = 0\},\ \gamma := \{j \in [m] : \lambda_j(W) < 0\},$$

where $[m] := \{1, 2, \cdots, m\}$. Here Λ_α is a diagonal matrix whose entries are all real positive eigenvalues of $W \in \mathbb{S}^m$ and Λ_γ is a diagonal matrix whose entries are all real negative eigenvalues of $W \in \mathbb{S}^m$. In addition, we define $X, Y \in \mathbb{S}^m$ such that $W = X + Y$. In particular, we define X and Y as

$$X = \Pi_{\mathbb{S}^m_+}(W) = P \begin{bmatrix} \Lambda_\alpha & & \\ & 0_\beta & \\ & & 0_\gamma \end{bmatrix} P^T, \quad Y = \Pi_{\mathbb{S}^m_-}(W) = P \begin{bmatrix} 0_\alpha & & \\ & 0_\beta & \\ & & \Lambda_\gamma \end{bmatrix} P^T, \tag{10}$$

where $\Pi_{\mathbb{S}^m_+}(\cdot)$ denotes the projection operator onto $\mathbb{S}^m_+$ and $\Pi_{\mathbb{S}^m_-}(\cdot)$ denotes the projection operator onto $\mathbb{S}^m_-$. For subsequent need, we denote the matrix $\Sigma \in \mathbb{S}^n$ with entries

$$\Sigma_{ij} := \frac{\max\{\lambda_i(W), 0\} - \max\{\lambda_j(W), 0\}}{\lambda_i(W) - \lambda_j(W)}, \quad i, j = 1, \ldots, n, \tag{11}$$

where 0/0 is defined to be 1.

As mentioned in Section 1, concepts in stationary points in NLP are unsuitable to MPLSDCC. In light of the description of stationary points introduced in [18], for notional simplicity, we rewrite problem (4) as

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & h_i(z) = 0,\ i \in [n], \\ & g_1(z) \geq 0,\ g_2(z) \leq 0, \\ & G(z) \in \mathbb{S}^m_+,\ H(z) \in \mathbb{S}^m_-,\ \langle G(z), H(z) \rangle = 0, \end{array} \tag{12}$$

where f, h, g_1, g_2, G, H are defined as in (8). Similarly, problem (7) can be recast as

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & h_i^N(z) = 0, \quad i \in [n], \\ & g_1(z) \geq 0,\ g_2(z) \leq 0, \\ & G(z) \in \mathbb{S}^m_+,\ H^N(z) \in \mathbb{S}^m_-,\ \langle G(z), H^N(z) \rangle = 0. \end{array} \tag{13}$$

Now, we present some concepts in stationary points of (4) in light of model (12).

Definition 2.1 (Definition 3.3 in [18]). *Let z^* be a feasible point of problem (12) and $W = G(z^*) + H(z^*)$ has an eigenvalue decomposition as in (9).*

(a) We say that z^* is a W-stationary point of problem (12), if there exist $\lambda_h^* \in \mathbb{R}^n$, $\lambda_{g1}^* \in \mathbb{R}^n$, $\lambda_{g2}^* \in \mathbb{R}^n$, $\Gamma_G^* \in \mathbb{S}^m$ and $\Gamma_H^* \in \mathbb{S}^m$ such that

$$\begin{gathered} \nabla f(z^*) + \mathcal{J}h(z^*)^T\lambda_h^* + \mathcal{J}g_1(z^*)^T\lambda_{g1}^* + \mathcal{J}g_2(z^*)^T\lambda_{g2}^* + \mathcal{J}G(z^*)^*\Gamma_G^* + \mathcal{J}H(z^*)^*\Gamma_H^* = 0, \\ \lambda_{g1}^* \leq 0, \langle g_1(z^*), \lambda_{g1}^*\rangle = 0, \\ \lambda_{g2}^* \geq 0, \langle g_2(z^*), \lambda_{g2}^*\rangle = 0, \\ (\tilde{\Gamma}_G^*)_{\alpha\alpha} = 0, (\tilde{\Gamma}_G^*)_{\alpha\beta} = 0, (\tilde{\Gamma}_G^*)_{\beta\alpha} = 0, \\ (\tilde{\Gamma}_H^*)_{\gamma\gamma} = 0, (\tilde{\Gamma}_H^*)_{\gamma\beta} = 0, (\tilde{\Gamma}_H^*)_{\beta\gamma} = 0, \\ \Sigma_{\alpha\gamma} \circ (\tilde{\Gamma}_G^*)_{\alpha\gamma} + (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ (\tilde{\Gamma}_H^*)_{\alpha\gamma} = 0, \end{gathered} \tag{14}$$

where $\Sigma \in \mathbb{S}^m$ is defined as in (11), $\tilde{\Gamma}_G^* = P^T\Gamma_G^*P$, $\tilde{\Gamma}_H^* = P^T\Gamma_H^*P$ with the orthogonal matrix P defined as in (9), E is an $m \times m$ matrix whose entries are all ones, $\circ$ denotes the Hadamard product and $\mathcal{J}h(z^*)$, $\mathcal{J}g_1(z^*)$, $\mathcal{J}g_2(z^*)$, $\mathcal{J}G(x^*)$ and $\mathcal{J}H(x^*)$ denote the Jacobian of h, g_1, g_2, G and H at z^*, $\mathcal{J}G(z^*)^*$ and $\mathcal{J}H(z^*)^*$ denote the adjoint of $\mathcal{J}G(z^*)$ and $\mathcal{J}H(z^*)$.
(b) We say that z^* is an S-stationary point of problem (12), if there exist $\lambda_h^* \in \mathbb{R}^n$, $\lambda_{g1}^* \in \mathbb{R}^n$, $\lambda_{g2}^* \in \mathbb{R}^n$, $\Gamma_G^* \in \mathbb{S}^m$ and $\Gamma_H^* \in \mathbb{S}^m$ such that (14) holds and

$$(\tilde{\Gamma}_G^*)_{\beta\beta} \in \mathbb{S}_-^{|\beta|}, \ (\tilde{\Gamma}_H^*)_{\beta\beta} \in \mathbb{S}_+^{|\beta|}.$$

A popular way for dealing with linear semidefinite complementarity constraints in MPLSDCC appeared in (12) is to transform it into one or a sequence of equations by utilizing smooth nonlinear complementarity functions. For instance, Chen and Tseng [6] introduces several smooth nonlinear complementarity functions for analyzing optimality conditions for MPLSDCC. In this paper, we use the smooth Fischer-Burmeister (FB) function in the sequel, whose definition is given by

$$\Phi_\mu(X, Y) = X - Y - (X^2 + Y^2 + 2\mu^2 I)^{\frac{1}{2}}, \tag{15}$$

where $X \in \mathbb{S}_+^m$, $Y \in \mathbb{S}_-^m$ and $\mu > 0$. It follows from [8, Corollary 4.1] that the smooth FB function has the following properties:

$$\Phi_\mu(X, Y) \to 0 \text{ as } \mu \to 0 \quad \Longrightarrow \quad X \in \mathbb{S}_+^m, \ Y \in \mathbb{S}_-^m, \ \langle X, Y\rangle = 0. \tag{16}$$

Moreover, for any $\mu > 0$ and $X, Y, U, V \in \mathbb{S}^m$, the smooth FB function $\Phi_\mu(X, Y)$ is differentiable and

$$\begin{aligned} \nabla_X \Phi_\mu(X, Y)(U) &= U - \mathcal{L}^{-1}_{(X^2+Y^2+2\mu^2 I)^{\frac{1}{2}}}(\mathcal{L}_X(U)), \\ \nabla_Y \Phi_\mu(X, Y)(V) &= -V + \mathcal{L}^{-1}_{(X^2+Y^2+2\mu^2 I)^{\frac{1}{2}}}(\mathcal{L}_{(-Y)}(V)), \end{aligned} \tag{17}$$

where $\mathcal{L}$ denotes the Lyapunov operator, i.e., for any $X \in \mathbb{S}^m$, $\mathcal{L}_X(Y) := XY + YX$, $Y \in \mathbb{S}^m$ with L_X^{-1} being its inverse (if it exists), i.e., for any $Y \in \mathbb{S}^m$, $L_X^{-1}(Y)$ is the unique $Z \in \mathbb{S}^m$ satisfying $XZ + ZX = Y$.

3 The SAA method and convergence results

Due to the appearance of mathematical expectations in the stochastic model (4) (and (12)), the sample average approximation (SAA) method is introduced that uses the (quasi) Monte Carlo method to transform the expectation terms into the deterministic summation structures under a given sample set.

In this section, we follow the SAA method to construct the corresponding subproblems defined as in (7) (and (13)). In light of the smooth FB function (15), we propose the below model to approximate the problem (7) (and (13))

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & h_i^N(z) = 0,\ i \in [n], \\ & g_1(z) \geq 0,\ g_2(z) \leq 0, \\ & \Phi_{\mu^N}(G(z), H^N(z)) = 0, \end{aligned} \tag{18}$$

where f, h^N, g_1, g_2, G, H^N are defined as in (8) and μ^N is a sequence of positive scalars that depend on monotonically decreasing in N such that $\mu^N \to 0$ as $N \to +\infty$.

To proceed, we present a technical lemma that describes the convergence properties of constraints function in (18).

Lemma 3.1. *Let z_N be a feasible point of problem (18), if z_N converges with probability one* (w.p.1) *to z^* as $N \to +\infty$, then for any $i \in [n]$,*

$$\begin{aligned}
& h_i^N(z_N) \to h_i(z^*), && \text{(w.p.1)} \\
& \nabla_c h_i^N(z_N) \to \nabla_c h_i(z^*) = e_i, && \text{(w.p.1)} \\
& \nabla_x h_i^N(z_N) \to \nabla_x h_i(z^*) = 0, && \text{(w.p.1)} \\
& \nabla_Y h_i^N(z_N) \to \nabla_Y h_i(z^*) = \mathbb{E}[A_i(\xi)], && \text{(w.p.1)} \\
& H^N(z_N) \to H(z^*), && \text{(w.p.1)} \\
& \nabla_c H^N(z_N) \to \nabla_c H(z^*) = 0, && \text{(w.p.1)} \\
& \nabla_x H^N(z_N) \to \nabla_x H(z^*) = (\mathbb{E}[A_1(\xi)], \mathbb{E}[A_2(\xi)], \cdots, \mathbb{E}[A_n(\xi)])^T, && \text{(w.p.1)} \\
& \nabla_Y H^N(z_N) \to \nabla_Y H(z^*) = 0, && \text{(w.p.1)} \\
& \Phi_{\mu^N}(G(z_N), H^N(z_N)) \to 2(G(z^*) - \Pi_{\mathbb{S}_+^m}(G(z^*) + H(z^*)), && \text{(w.p.1)}
\end{aligned}$$

where $e_i \in \mathbb{R}^n$ is the i-th column of entity matrix I_n.

Proof. It follows from the definitions of h_i^N, H^N, Φ_{μ^N}, and [18, Lemma 2.4] that the above conclusions hold. □

Let Ω be the feasible set of problem (4) (and (12)) and Ω_N be the feasible set of problem of (18). For notational convenience, we denote

$$\bar{f}(z) := f(z) + I_\Omega(z), \quad \bar{f}^N(z) := f(z) + I_{\Omega_N}(z). \tag{19}$$

where f is defined as in (8). The epigraph of f is denoted by epi$f$, i.e.,

$$\text{epi}f := \{(z,w) : f(z) \leq w\}.$$

Before establishing the convergence results, we make the following assumption on the operator $\mathcal{A}(\xi,\cdot)$ and a technical lemma on the relations between $(\Omega_N, \text{epi}\bar{f}_N)$ and $(\Omega, \text{epi}\bar{f})$.

Assumption 3.1. *The operator $\mathcal{A}(\xi,\cdot)$ defined as in (5) has the following property*

$$\mathbb{E}[\mathcal{A}(\xi,w)] = 0 \;\Rightarrow\; w = 0.$$

Lemma 3.2. *Suppose that Assumption 3.1 holds. If $N \to +\infty$, then $(\Omega_N, \text{epi}\bar{f}_N) \to (\Omega, \text{epi}\bar{f})$ (w.p.1).*

Proof. (a) For any $z^* \in \limsup_{N\to+\infty} \Omega_N$, there exists $z_N \in \Omega_N$ such that $z_N \to z^*$ (w.p.1) as $N \to +\infty$. Notice that the last three constraints in problem (7) (and (13)) can be reformulated as

$$G(z) - \Pi_{\mathbb{S}^m_+}(G(z) + H(z)) = 0 \tag{20}$$

due to the characterization of $\Pi_{\mathbb{S}^m_+}$. Applying Lemma 3.1 and (20), we have

$$\begin{array}{l} 0 = h_i^N(z_N) \to h_i(z^*),\ i \in [n], \\ 0 \leq g_1(z_N) \to g_1(z^*), \\ 0 \geq g_2(z_N) \to g_2(z^*), \\ 0 = \Phi_{\mu^N}(G(z_N), H^N(z_N)) \to 2(G(z^*) - \Pi_{\mathbb{S}^m_+}(G(z^*) + H(z^*)), \end{array}$$

which imply that $z^* \in \Omega$. Therefore, we obtain $\limsup_{N\to+\infty} \Omega_N \subseteq \Omega$ (w.p.1).

On the other hand, for any $z^* \in \Omega$, there exist $u^*_{N^*_L} \in \mathbb{R}^{|N^*_L|}$ and $v^*_{N^*_U} \in \mathbb{R}^{|N^*_U|}$ that $(z^*, u^*_{N^*_L}, v^*_{N^*_U})$ is also a feasible point of the following problem

$$\begin{array}{ll} \min\limits_{z,u_{N^*_L},v_{N^*_U}} & f(z) \\ \text{s.t.} & h_i(z) = 0,\ i \in [n], \\ & g_{1j}(z) = 0,\ j \in I^*_L, \\ & g_{1k}(z) - u_k^2 = 0,\ k \in N^*_L, \\ & g_{2p}(z) = 0,\ p \in I^*_U, \\ & g_{2q}(z) + v_q^2 = 0,\ q \in N^*_U, \\ & 2(G(z) - \Pi_{\mathbb{S}^m_+}(G(z) + H(z))) = 0, \end{array}$$

where h_i, g_{1i}, g_{2i} are respectively the i-th entry of h_i, g_1, g_2 and

$$\begin{array}{l} I^*_L := \{j \in [n] : g_{1j}(z^*) = 0\},\ N^*_L := [n] \setminus I^*_L, \\ I^*_U := \{p \in [n] : g_{2p}(z^*) = 0\},\ N^*_U := [n] \setminus I^*_U. \end{array} \tag{21}$$

Again, for notional simplicity, we define

$$(s_h^*)_i := h_i(z^*),\ i \in [n];\ (s_{g1}^*)_j := g_{1j}(z^*),\ j \in I_L^*;\ (s_{g1}^*)_k := g_{1k}(z^*) - (u_k^*)^2,\ k \in N_L^*,$$
$$(s_{g2}^*)_p := g_{2p}(z^*),\ p \in I_U^*;\ (s_{g2}^*)_q := g_{2q}(z^*) - (u_q^*)^2,\ q \in N_U^*;\ s_G^* := G(z^*);\ s_H^* := H(z^*).$$

In view of the continuity of $h, h^N, g_1, g_2, G, H^N, H$, there exist $(s_h^N)_i$ $(i \in [n])$, $(s_{g1}^N)_j$ $(j \in I_L^*)$, $(s_{g1}^N)_k$ $(k \in N_L^*)$, $(s_{g2}^N)_p$ $(p \in I_U^*)$, $(s_{g2}^N)_q$ $(q \in N_U^*)$, S_G^N and S_H^N satisfying

$$\begin{array}{l} (s_h^N)_i = 0,\ i \in [n];\ (s_{g1}^N)_j = 0,\ j \in I_L^*;\ (s_{g1}^N)_k = 0,\ k \in N_L^*; \\ (s_{g2}^N)_p = 0,\ p \in I_U^*;\ (s_{g2}^N)_q = 0,\ q \in N_U^*;\ \Phi_{\mu^N}(s_G^N, s_H^N) = 0 \end{array} \tag{22}$$

and $(s_h^N, s_{g1}^N, s_{g2}^N, s_G^N, s_H^N) \to (s_h^*, s_{g1}^*, s_{g2}^*, s_G^*, s_H^*)$ as $N \to +\infty$. Next, we introduce the following function

$$\mathcal{P}(z, u_{N_L^*}, v_{N_U^*}, s_h, s_{g1}, s_{g2}, s_G, s_H) = \begin{pmatrix} h(z) \\ \tilde{g}_1(z, u_{N_L^*}) \\ \tilde{g}_2(z, v_{N_U^*}) \\ H(z) \\ G(z) \end{pmatrix} - \begin{pmatrix} s_h \\ s_{g1} \\ s_{g2} \\ s_G \\ s_H \end{pmatrix} \tag{23}$$

where $\tilde{g}_1(z, u_{N_L^*})$ and $\tilde{g}_2(z, v_{N_U^*})$ are given by

$$\tilde{g}_1(z, u_{N_L^*}) := \begin{pmatrix} g_{1j}(z) \\ g_{1k}(z) - u_k^2 \end{pmatrix},\ j \in I_L^*,\ k \in N_L^*, \tag{24}$$

$$\tilde{g}_2(z, v_{N_U^*}) := \begin{pmatrix} g_{2p}(z) \\ g_{2q}(z) + v_q^2 \end{pmatrix},\ p \in I_U^*,\ q \in N_U^*. \tag{25}$$

Then, combining (23), (24) and (25) yields

$$\mathcal{P}(z^*, u_{N_L^*}^*, v_{N_U^*}^*, s_h^*, s_{g1}^*, s_{g2}^*, s_G^*, s_H^*) = 0.$$

For any given perturbation pair $(\Delta z, \Delta u_{N_L^*}, \Delta v_{N_U^*})$, we set

$$M := \mathcal{J}_{(z, u_{N_L^*}, v_{N_U^*})} \mathcal{P}(z^*, u_{N_L^*}^*, v_{N_U^*}^*, s_h^*, s_{g1}^*, s_{g2}^*, s_G^*, s_H^*),\ M(\Delta z, \Delta u_{N_L^*}, \Delta v_{N_U^*}) = 0,$$

where $\mathcal{J}_{(z, u_{N_L^*}, v_{N_U^*})} \mathcal{P}(z^*, u_{N_L^*}^*, v_{N_U^*}^*, s_h^*, s_{g1}^*, s_{g2}^*, s_G^*, s_H^*)$ denotes the partial Jacobian of $\mathcal{P}$ with respect to $(z, u_{N_L^*}, v_{N_U^*})$ at $(z^*, u_{N_L^*}^*, v_{N_U^*}^*, s_h^*, s_{g1}^*, s_{g2}^*, s_G^*, s_H^*)$, which implies

$$\begin{aligned} \Delta c_i + \langle \mathbb{E}(A_i(\xi)), \Delta Y \rangle &= 0,\ i \in [n], \\ \Delta c_j &= 0,\ j \in I_L^*, \\ \Delta c_k - 2u_k^*(\Delta u_k) &= 0,\ k \in N_L^*, \\ \Delta c_p &= 0,\ p \in I_U^*, \\ \Delta c_q + 2v_q^*(\Delta v_q) &= 0,\ q \in N_U^*, \\ \mathbb{E}[\mathcal{A}(\xi, \Delta x)] &= 0,\ \Delta Y = 0. \end{aligned}$$

where I_L^*, N_L^*, I_U^* and N_U^* are defined as in (21). It follows from these equations and Assumption 3.1 that the given perturbation pair $(\Delta z, \Delta u_{N_L^*}, \Delta v_{N_U^*})$ are all equal to zero, which means that the operator M is onto. In addition, from Clarke's implicit function theory, there exist $\epsilon > 0$, $\delta > 0$ and a Lipschitz continuous function $\eta(\cdot)$: $\mathbb{B}_\delta(s_h^*, s_{g1}^*, s_{g2}^*, s_G^*, s_H^*) \to \mathbb{B}_\epsilon(z^*, u_{N_L^*}^*, v_{N_U^*}^*)$ with Lipschitz constant $c > 0$ such that

$$\eta(s_h^*, s_{g1}^*, s_{g2}^*, s_G^*, s_H^*) = (z^*, u_{N_L^*}^*, v_{N_U^*}^*)$$

and for any $(s_h, s_{g1}, s_{g2}, s_G, s_H) \in \mathbb{B}_\delta(s_h^*, s_{g1}^*, s_{g2}^*, s_G^*, s_H^*)$ satisfies the following equation

$$\mathcal{P}(\eta(s_h, s_{g1}, s_{g2}, s_G, s_H), s_h, s_{g1}, s_{g2}, s_G, s_H) = 0. \tag{26}$$

When N is sufficiently large, we have

$$\max_{(z,u_{N_L^*},v_{N_U^*})\in\mathbb{B}_\epsilon(z^*,u_{N_L^*}^*,v_{N_U^*}^*)} \|\mathcal{D}_N(z, u_{N_L^*}, v_{N_U^*}) - (s_h^*, s_{g1}^*, s_{g2}^*, s_H^*, s_G^*)\| \leq \delta', \tag{27}$$

where $\delta' := \min\{\delta, (2c)^{-1}\epsilon\}$ and $\mathcal{D}_N(z, u_{N_L^*}, v_{N_U^*})$ is given by

$$\mathcal{D}_N(z, u_{N_L^*}, v_{N_U^*}) := \begin{pmatrix} h(z) - h^N(z) + s_h^N \\ s_{g1}^N \\ s_{g2}^N \\ H(z) - H^N(z) + s_H^N \\ s_G^N \end{pmatrix}.$$

From the relationship (27) and the Lipschitz property of η, for any $(z, u_{N_L^*}, v_{N_U^*}) \in \mathbb{B}_\epsilon(z^*, u_{N_L^*}^*, v_{N_U^*}^*)$, we obtain

$$\begin{aligned} &\|\eta(\mathcal{D}_N(z, u_{N_L^*}), v_{N_U^*})) - \eta(s_h^*, s_{g1}^*, s_{g2}^*, s_H^*, s_G^*)\| \\ &\leq c\|\mathcal{D}_N(z, u_{N_L^*}), v_{N_U^*}) - (s_h^*, s_{g1}^*, s_{g2}^*, s_H^*, s_G^*)\| \\ &\leq c\delta' < \epsilon/2, \end{aligned}$$

which shows that $\eta(\mathcal{D}_N(\cdot))$ is a continuous function from the convex set $\mathbb{B}_\epsilon(z^*, u_{N_L^*}^*, v_{N_U^*}^*)$ to itself. It follows from Brouwer's fixed point theory that there exists a fixed point $(z_N, u_{N_L^*}^N, v_{N_U^*}^N) \in \mathbb{B}_\epsilon(z^*, u_{N_L^*}^*, v_{N_U^*}^*)$ such that $\eta(\mathcal{D}_N(z_N, u_{N_L^*}^N, v_{N_U^*}^N)) = \mathcal{D}_N(z_N, u_{N_L^*}^N, v_{N_U^*}^N)$. and $(z_N, u_{N_L^*}^N, v_{N_U^*}^N) \to (z^*, u_{N_L^*}^*, v_{N_U^*}^*)$ as $N \to +\infty$. In addition, from (26) and (27) , we also obtain that $\mathcal{D}_N(z_N, u_{N_L^*}^N), v_{N_U^*}^N) \in \mathbb{B}_\epsilon((s_h^*, s_{g1}^*, s_{g2}^*, s_H^*, s_G^*))$ and $\mathcal{P}(\eta(\mathcal{D}_N(z_N, u_{N_L^*}^N, v_{N_U^*}^N)), \mathcal{D}_N(z_N, u_{N_L^*}^N, v_{N_U^*}^N)) = 0$, i.e.,

$$\begin{aligned} h(z_N) - (h(z_N) - h^N(z_N) + s_h^N) &= 0, \\ g_1(z_N) - (s_{g1}^N)_j &= 0,\ j \in I_L^*, \\ g_{1k}(z_N) - (u_k^N)^2 - (s_{g1}^N)_k &= 0,\ k \in N_L^*, \\ g_{2p}(z_N) - (s_{g2}^N)_p &= 0,\ p \in I_U^*, \\ g_{2q}(z_N) + (v_q^N)^2 - (s_{g2}^N)_q &= 0,\ q \in N_U^*, \\ H(z_N) - (H(z_N) - H^N(z_N) + s_H^N) = 0,\ G(z_N) - s_G^N &= 0, \end{aligned}$$

which show that $(z_N, u^N_{N^*_L}, v^N_{N^*_U})$ is a feasible point of the following problem

$$\begin{array}{ll}\min\limits_{z, u_{N^*_L}, v_{N^*_U}} & f(z) \\ \text{s.t.} & h^N_i(z) = 0,\ i \in [n], \\ & g_{1j}(z) = 0,\ j \in I^*_L, \\ & g_{1k}(z) - u_k^2 = 0,\ k \in N^*_L, \\ & g_{2p}(z) = 0,\ p \in I^*_U, \\ & g_{2q}(z) + v_q^2 = 0,\ q \in N^*_U, \\ & \Phi_{\mu^N}(G(z), H^N(z)) = 0,\end{array}$$

where the last equation follows from (22). Therefore, $z_N \in \Omega_N$. Because $z_N \to z^*$ as $N \to +\infty$, we have $z^* \in \limsup_{N\to+\infty} \Omega_N$. The proof of the first part of the conclusion is complete.

(b) Now, we turn to show the second part of the conclusion. It is easy to see that problem (4) (and (12)) and problem (18) are respectively equivalent to $\min_z\ \bar{f}(z)$ and $\min_z\ \bar{f}^N(z)$, where $\bar{f}$ and $\bar{f}^N$ are defined as in (19). It follows from [14, Theorem 7.1] and $\Omega_N \to \Omega$ (w.p.1) that $\text{epi}\bar{f}_N \to \text{epi}\bar{f}$ (w.p.1) as $N \to +\infty$. □

The next theorem shows that the sequence of global optimal solutions of problem (18) converges with probability one (w.p.1) to a global optimal solution of problem (4) (and (12)).

Theorem 3.1. *Suppose that Assumption 3.1 holds. Let z_N be a global optimal solution of problem (18) and $z_N \to z^*$ as $N \to +\infty$, then z^* is a global optimal solution of problem (4) (and (12))* (w.p.1).

Proof. From Lemma 3.2, when $N \to +\infty$, we have $\text{epi}\bar{f}_N \to \text{epi}\bar{f}$ (w.p.1). Then, it follows from [14, Theorem 7.31] and Lemma 3.1 that

$$\limsup_{N\to+\infty} \arg\min \bar{f}^N \subseteq \arg\min \bar{f}, \quad \text{(w.p.1)}$$

which shows that z^* is a global optimal solution of problem (4) (and (12)) (w.p.1). □

We turn to analyze the behavior of stationary points of problem (18). The Lagrange function of problem (18) is defined as

$$\begin{aligned}\mathcal{L}(z, \lambda_h, \lambda_{g1}, \lambda_{g2}, \Gamma) :=\ & f(z) + h^N(z)^T\lambda_h - g_1(z)^T\lambda_{g1} + g_2(z)^T\lambda_{g2} \\ & + \left\langle \Gamma, \Phi_{\mu^N}(G(z), H^N(z))\right\rangle.\end{aligned} \tag{28}$$

Given a feasible point $z_N \in \Omega_N$, we say it to be stationary point of problem (18) if there exist $\lambda^N_h \in \mathbb{R}^n$, $\lambda^N_{g1} \in \mathbb{R}^n$, $\lambda^N_{g2} \in \mathbb{R}^n$ and $\Gamma^N \in \mathbb{S}^m$ such that

$$\begin{array}{c}\nabla_z \mathcal{L}(z_N, \lambda^N_h, \lambda^N_{g1}, \lambda^N_{g2}, \Gamma^N) = 0,\ h^N(z_N) = 0, \\ g_1(z_N) \ge 0,\ \lambda^N_{g1} \ge 0,\ g_1(z_N)^T\lambda^N_{g1} = 0, \\ g_2(z_N) \le 0,\ \lambda^N_{g2} \ge 0,\ g_2(z_N)^T\lambda^N_{g2} = 0, \\ \Phi_{\mu^N}(G(z_N), H^N(z_N)) = 0.\end{array} \tag{29}$$

Any point $(z_N, \lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \Gamma^N)$ satisfying the above system (29) is called a Karush-Kuhn-Tucker (KKT) point of problem (18) and the pair $(\lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \Gamma^N))$ is the Lagrangian multiplier associated with z_N.

Before ending this section, we make the following assumption on the operator $\mathcal{A}^*(\xi, \cdot)$ and a technical lemma used in the sequel.

Assumption 3.2. *The operator $\mathcal{A}^*(\xi, \cdot)$ defined as in (6) has the following property*

$$\mathbb{E}[\mathcal{A}^*(\xi, Y)] = 0 \ \Rightarrow \ Y = 0.$$

Lemma 3.3. *Suppose that Assumption 3.2 holds. Let z_N be a feasible solution of problem (18) and $z_N \to z^*$ as $N \to +\infty$. Assume that $I_L^* \cup I_U^* = \emptyset$, where I_L^*, I_U^* are defined as in (21), and $W = G(z^*) + H(z^*)$ has an eigenvalue decomposition as in (9) with $\beta = \emptyset$. Then the linear independent constraint qualification (LICQ) of problem (18) holds at a neighborhood of z_N when N is sufficiently large.*

Proof. It follows from $z_N \to z^*$ and $I_L^* \cup I_U^* = \emptyset$ that when N is sufficiently large,

$$g_1(z_N) > 0, \ g_2(z_N) < 0.$$

In order to show the conclusion, we only need to prove that

$$\mathcal{J}h^N(z_N)^*\lambda + \mathcal{J}(\langle \Gamma, \Phi_{\mu^N}(G(z_N), H^N(z_N))\rangle) = 0 \ \Rightarrow \ \lambda = 0, \ \Gamma = 0. \tag{30}$$

In light of the definitions of h^N, Φ_{μ^N} and (17), the left-hand side of (30) is equivalent to the following system

$$\lambda_i = 0, \ i \in [n], \tag{31}$$

$$\left\langle (-I + \mathcal{L}^{-1}_{(G(z_N)^2 + H^N(z_N)^2 + 2\mu^N I)^{1/2}} \mathcal{L}_{(-H^N(z_N))})(\Gamma), \bar{A}_i \right\rangle = 0, \ i \in [n], \tag{32}$$

$$\sum_{i \in [n]} \bar{A}_i \lambda_i + (I - \mathcal{L}^{-1}_{(G(z_N)^2 + H^N(z_N)^2 + 2\mu^N I))^{1/2}} \mathcal{L}_{G(z_N)})(\Gamma) = 0, \tag{33}$$

where $\bar{A}_i$ is the sample average of $A_i(\xi_j)$, i.e., $\bar{A}_i = \frac{1}{N}\sum_{j=1}^N \bar{A}_i(\xi_j)$. From (31), we obtain that $\lambda = 0$. Due to Assumption 3.2 and (32), when $N \to +\infty$,

$$(-I + \mathcal{L}^{-1}_{(G(z^*)^2 + H(z^*)^2)^{1/2}} \mathcal{L}_{(-H(z^*))})(\Gamma) = 0.$$

In addition, it follows from the definition of $\mathcal{L}^{-1}$ that when $N \to +\infty$, the above relation and (33) become the following equations

$$((G(z^*)^2 + H(z^*)^2)^{1/2} + H(z^*))\Gamma + \Gamma((G(z^*)^2 + H(z^*)^2)^{1/2} + H(z^*)) = 0,$$
$$((G(z^*)^2 + H(z^*)^2)^{1/2} - G(z^*))\Gamma + \Gamma((G(z^*)^2 + H(z^*)^2)^{1/2} - G(z^*)) = 0.$$

In light of the relation (10) and $\beta = \emptyset$, we obtain

$$\begin{bmatrix} \Lambda_\alpha & \\ & 0 \end{bmatrix} \begin{bmatrix} \tilde{\Gamma}_{\alpha\alpha} & \tilde{\Gamma}_{\alpha\gamma} \\ \tilde{\Gamma}_{\gamma\alpha} & \tilde{\Gamma}_{\gamma\gamma} \end{bmatrix} + \begin{bmatrix} \tilde{\Gamma}_{\alpha\alpha} & \tilde{\Gamma}_{\alpha\gamma} \\ \tilde{\Gamma}_{\gamma\alpha} & \tilde{\Gamma}_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} \Lambda_\alpha & \\ & 0 \end{bmatrix} = 0,$$
$$\begin{bmatrix} 0 & \\ & \Lambda_\gamma \end{bmatrix} \begin{bmatrix} \tilde{\Gamma}_{\alpha\alpha} & \tilde{\Gamma}_{\alpha\gamma} \\ \tilde{\Gamma}_{\gamma\alpha} & \tilde{\Gamma}_{\gamma\gamma} \end{bmatrix} + \begin{bmatrix} \tilde{\Gamma}_{\alpha\alpha} & \tilde{\Gamma}_{\alpha\gamma} \\ \tilde{\Gamma}_{\gamma\alpha} & \tilde{\Gamma}_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} 0 & \\ & \Lambda_\gamma \end{bmatrix} = 0,$$

where $\tilde{\Gamma} := P^T \Gamma P$ has the following form

$$\tilde{\Gamma} = \begin{bmatrix} \tilde{\Gamma}_{\alpha\alpha} & \tilde{\Gamma}_{\alpha\gamma} \\ \tilde{\Gamma}_{\gamma\alpha} & \tilde{\Gamma}_{\gamma\gamma} \end{bmatrix}.$$

From the definitions of Λ_α and Λ_γ, we obtain

$$\tilde{\Gamma}_{\alpha\alpha} = 0, \tilde{\Gamma}_{\alpha\gamma} = 0, \tilde{\Gamma}_{\gamma\alpha} = 0, \tilde{\Gamma}_{\alpha\gamma} = 0,$$

which yield that $\Gamma = 0$. □

Remark 3.1. *From the definitions of g_1 and g_2 defined as in (8), we can always set the lower bound c^L and the upper bound c^U of cost vectors that makes the assumption $I_L^* \cup I_U^* = \emptyset$ satisfied. On the other hand, the condition $\beta = \emptyset$ means that the strict complementarity condition holds at z^* with respect to the system $G(z) \in \mathbb{S}_+^m$, $H(z) \in \mathbb{S}_-^m$, $\langle G(z), H(z) \rangle = 0$. Although this is a strong assumption in general, it is commonly required in the convergence analysis of smoothing methods for complementarity problems.*

The next theorem shows that the sequence of KKT solutions of problem (18) converges with probability one (w.p.1) to an S-stationary point of problem (4) (or (12)) under mild conditions.

Theorem 3.2. *Suppose that Assumption 3.2 holds. Let z_N be a KKT point of problem (18) and $z_N \to z^*$ as $N \to +\infty$. Assume that $I_L^* \cup I_U^* = \emptyset$, where I_L^*, I_U^* are defined as in (21), and $W = G(z^*) + H(z^*)$ has an eigenvalue decomposition as in (9) with $\beta = \emptyset$. Then z^* is an S-stationary point of problem (4) (and (12))* (w.p.1).

Proof. First, from the KKT condition of problem (18), the system (29) holds, that is,

$$\begin{array}{c} \nabla_z \mathcal{L}(z_N, \lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \Gamma^N) = 0, \ h^N(z_N) = 0, \\ g_1(z_N) \geq 0, \ \lambda_{g1}^N \geq 0, \ g_1(z_N)^T \lambda_{g1}^N = 0, \\ g_2(z_N) \leq 0, \ \lambda_{g2}^N \geq 0, \ g_2(z_N)^T \lambda_{g2}^N = 0, \\ \Phi_{\mu^N}(G(z_N), H^N(z_N)) = 0, \end{array}$$

where the Lagrangian function $\mathcal{L}(\cdot)$ is defined as in (28), $\lambda_h^N, \lambda_{g1}^N, \lambda_{g2}^N, \Gamma^N$ are the Lagrangian multipliers of problem (18) associated with z_N. Applying Lemma 3.3, when N is sufficiently large, $g_1(z_N) > 0$ and $g_2(z_N) < 0$, it yields $\lambda_{g1}^N = 0$ and $\lambda_{g2}^N = 0$.

Thus, the above system reduces to

$$\begin{array}{c}\nabla f(z_N)+\mathcal{J}h^N(z_N)^*\lambda_h^N+\mathcal{J}(\langle\Gamma^N,\Phi_{\mu^N}(G(z_N),H^N(z_N))\rangle)=0,\\ h^N(z_N)=0,\quad \Phi_{\mu^N}(G(z_N),H^N(z_N))=0.\end{array}\tag{34}$$

Because the LICQ condition holds at z_N when N is sufficiently large, it is seen that λ_h^N and Γ^N are the unique multiplies that satisfy the relation (34). Therefore, the sequence $\{\lambda_h^N,\lambda_{g1}^N,\lambda_{g2}^N,\Gamma^N\}$ is convergent. Assume that $\lambda_h^N\to\lambda_h^*$, $\lambda_{g1}^N\to\lambda_{g1}^*=0$, $\lambda_{g2}^N\to\lambda_{g2}^*=0$ and $\Gamma^N\to\Gamma^*$ as $N\to+\infty$. From the relations (34) and (17) with Lemma 3.1, when $N\to+\infty$, we obtain

$$\begin{aligned}\nabla f(z^*)+\mathcal{J}h(z^*)^*\lambda^*+\mathcal{J}G(z^*)(\Gamma^*-\mathcal{L}^{-1}_{(G(z^*)^2+H(z^*)^2)^{1/2}}\mathcal{L}_{G(z^*)}(\Gamma^*))&\\ -\mathcal{J}H(z^*)(\Gamma^*-\mathcal{L}^{-1}_{(G(z^*)^2+H(z^*)^2)^{1/2}}\mathcal{L}_{-H(z^*)}(\Gamma^*))=0,&\qquad(35)\\ h(z^*)=0,\ 2(G(z^*)-\Pi_{\mathbb{S}^m_+}(G(z^*)+H(z^*))=0.&\qquad(36)\end{aligned}$$

From (36), we deduce that $h(z^*)=0$, $G(z^*)\in\mathbb{S}^m_+$, $H(z^*)\in\mathbb{S}^m_-$ and $\langle G(z^*),H(z^*)\rangle=0$, which means that z^* is a feasible point of problem (4) (and (12)) and

$$(G(z^*)^2+H(z^*)^2)^{1/2}=G(z^*)-H(z^*).\tag{37}$$

For notional simplicity, we denote

$$\begin{array}{l}\Gamma_G^*:=\Gamma^*-\mathcal{L}^{-1}_{G(z^*)-H(z^*)}\mathcal{L}_{G(z^*)}(\Gamma^*),\\ \Gamma_H^*:=-\Gamma^*+\mathcal{L}^{-1}_{G(z^*)-H(z^*)}\mathcal{L}_{-H(z^*)}(\Gamma^*).\end{array}\tag{38}$$

It follows from (37) and (38) that the relation (35) becomes

$$\nabla f(z^*)+\mathcal{J}h(z^*)^T\lambda_h^*+\mathcal{J}g_1(z^*)^T\lambda_{g1}^*+\mathcal{J}g_2(z^*)^T\lambda_{g2}^*+\mathcal{J}G(z^*)^*\Gamma_G^*+\mathcal{J}H(z^*)^*\Gamma_H^*=0,$$

where we use the facts that $\lambda_{g1}^*=0$ and $\lambda_{g2}^*=0$. In light of the assumption $I_L^*\cup I_U^*=\emptyset$, where I_L^*,I_U^* are defined as in (21), then $g_1(z^*)>0$ and $g_2(z^*)<0$. Moreover, due to the assumption $\beta=\emptyset$, we only need to deduce that z^* is a W-stationary point of problem (4) (and (12)).

In view of the above discussions, the remaining work is to verify that the last three relations in (14) are satisfied, i.e.,

$$\begin{array}{c}(\tilde\Gamma_G^*)_{\alpha\alpha}=0,\ (\tilde\Gamma_H^*)_{\gamma\gamma}=0,\\ \Sigma_{\alpha\gamma}\circ(\tilde\Gamma_G^*)_{\alpha\gamma}+(E_{\alpha\gamma}-\Sigma_{\alpha\gamma})\circ(\tilde\Gamma_H^*)_{\alpha\gamma}=0,\end{array}\tag{39}$$

where $\Sigma\in\mathbb{S}^m$ is defined as in (11), $\tilde\Gamma_G^*=P^T\Gamma_G^*P$, $\tilde\Gamma_H^*=P^T\Gamma_H^*P$ with the orthogonal matrix P defined as in (9), E is an $m\times m$ matrix whose entries are all ones, $\circ$ denotes the Hadamard product.

According to the relation (38), we know

$$-H(z^*)\Gamma^* - \Gamma^* H(z^*) = (G(z^*) - H(z^*))\Gamma_G^* + \Gamma_G^*(G(z^*) - H(z^*)), \tag{40}$$

$$-G(z^*)\Gamma^* - \Gamma^* G(z^*) = (G(z^*) - H(z^*))\Gamma_H^* + \Gamma_H^*(G(z^*) - H(z^*)). \tag{41}$$

Because $W = G(z^*) + H(z^*)$ has an eigenvalue decomposition as in (9) with $\beta = \emptyset$, there exists an orthogonal matrix $P \in \mathbb{S}^m$ such that

$$G(z^*) = P \begin{bmatrix} \Lambda_\alpha & \\ & 0_\gamma \end{bmatrix} P^T, \; H(z^*) = P \begin{bmatrix} 0_\alpha & \\ & \Lambda_\gamma \end{bmatrix} P^T.$$

We make the following notations

$$\tilde{\Gamma}^* := P^T \Gamma^* P = \begin{bmatrix} (\tilde{\Gamma}^*)_{\alpha\alpha} & (\tilde{\Gamma}^*)_{\alpha\gamma} \\ (\tilde{\Gamma}^*)_{\gamma\alpha} & (\tilde{\Gamma}^*)_{\gamma\gamma} \end{bmatrix},$$

$$\tilde{\Gamma}_G^* := P^T \Gamma_G^* P = \begin{bmatrix} (\tilde{\Gamma}_G^*)_{\alpha\alpha} & (\tilde{\Gamma}_G^*)_{\alpha\gamma} \\ (\tilde{\Gamma}_G^*)_{\gamma\alpha} & (\tilde{\Gamma}_G^*)_{\gamma\gamma} \end{bmatrix},$$

$$\tilde{\Gamma}_H^* := P^T \Gamma_H^* P = \begin{bmatrix} (\tilde{\Gamma}_H^*)_{\alpha\alpha} & (\tilde{\Gamma}_H^*)_{\alpha\gamma} \\ (\tilde{\Gamma}_H^*)_{\gamma\alpha} & (\tilde{\Gamma}_H^*)_{\gamma\gamma} \end{bmatrix}.$$

Then, combining (40) and (41) yields

$$\begin{aligned}
&\begin{bmatrix} 0_\alpha & \\ & -\Lambda_\gamma \end{bmatrix} \begin{bmatrix} (\tilde{\Gamma}^*)_{\alpha\alpha} & (\tilde{\Gamma}^*)_{\alpha\gamma} \\ (\tilde{\Gamma}^*)_{\gamma\alpha} & (\tilde{\Gamma}^*)_{\gamma\gamma} \end{bmatrix} + \begin{bmatrix} (\tilde{\Gamma}^*)_{\alpha\alpha} & (\tilde{\Gamma}^*)_{\alpha\gamma} \\ (\tilde{\Gamma}^*)_{\gamma\alpha} & (\tilde{\Gamma}^*)_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} 0_\alpha & \\ & -\Lambda_\gamma \end{bmatrix} \\
= &\begin{bmatrix} \Lambda_\alpha & \\ & -\Lambda_\gamma \end{bmatrix} \begin{bmatrix} (\tilde{\Gamma}_G^*)_{\alpha\alpha} & (\tilde{\Gamma}_G^*)_{\alpha\gamma} \\ (\tilde{\Gamma}_G^*)_{\gamma\alpha} & (\tilde{\Gamma}_G^*)_{\gamma\gamma} \end{bmatrix} + \begin{bmatrix} (\tilde{\Gamma}_G^*)_{\alpha\alpha} & (\tilde{\Gamma}_G^*)_{\alpha\gamma} \\ (\tilde{\Gamma}_G^*)_{\gamma\alpha} & (\tilde{\Gamma}_G^*)_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} \Lambda_\alpha & \\ & -\Lambda_\gamma \end{bmatrix}, \\
&\begin{bmatrix} -\Lambda_\alpha & \\ & 0_\gamma \end{bmatrix} \begin{bmatrix} (\tilde{\Gamma}^*)_{\alpha\alpha} & (\tilde{\Gamma}^*)_{\alpha\gamma} \\ (\tilde{\Gamma}^*)_{\gamma\alpha} & (\tilde{\Gamma}^*)_{\gamma\gamma} \end{bmatrix} + \begin{bmatrix} (\tilde{\Gamma}^*)_{\alpha\alpha} & (\tilde{\Gamma}^*)_{\alpha\gamma} \\ (\tilde{\Gamma}^*)_{\gamma\alpha} & (\tilde{\Gamma}^*)_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} -\Lambda_\alpha & \\ & 0_\gamma \end{bmatrix} \\
= &\begin{bmatrix} \Lambda_\alpha & \\ & -\Lambda_\gamma \end{bmatrix} \begin{bmatrix} (\tilde{\Gamma}_H^*)_{\alpha\alpha} & (\tilde{\Gamma}_H^*)_{\alpha\gamma} \\ (\tilde{\Gamma}_H^*)_{\gamma\alpha} & (\tilde{\Gamma}_H^*)_{\gamma\gamma} \end{bmatrix} + \begin{bmatrix} (\tilde{\Gamma}_H^*)_{\alpha\alpha} & (\tilde{\Gamma}_H^*)_{\alpha\gamma} \\ (\tilde{\Gamma}_H^*)_{\gamma\alpha} & (\tilde{\Gamma}_H^*)_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} \Lambda_\alpha & \\ & -\Lambda_\gamma \end{bmatrix},
\end{aligned}$$

which further implies

$$(\tilde{\Gamma}_G^*)_{\alpha\alpha} = 0, \; -(\tilde{\Gamma}^*)_{\alpha\gamma}\Lambda_\gamma = \Lambda_\alpha(\tilde{\Gamma}_G^*)_{\alpha\gamma} - (\tilde{\Gamma}_G^*)_{\alpha\gamma}\Lambda_\gamma, \tag{42}$$

$$(\tilde{\Gamma}_H^*)_{\gamma\gamma} = 0, \; -\Lambda_\alpha(\tilde{\Gamma}^*)_{\alpha\gamma} = \Lambda_\alpha(\tilde{\Gamma}_H^*)_{\alpha\gamma} - (\tilde{\Gamma}_H^*)_{\alpha\gamma}\Lambda_\gamma. \tag{43}$$

For any index pair $(i,j) \in \alpha \times \gamma$, from the relations (42) and (43), we also obtain

$$(\tilde{\Gamma}_G^*)_{ij} = -\frac{\lambda_j}{\lambda_i - \lambda_j}\tilde{\Gamma}_{ij}^*, \; (\tilde{\Gamma}_H^*)_{ij} = -\frac{\lambda_i}{\lambda_i - \lambda_j}\tilde{\Gamma}_{ij}^*,$$

which means

$$\lambda_i(\tilde{\Gamma}_G^*)_{ij} - \lambda_j(\tilde{\Gamma}_H^*)_{ij} = 0, \; \forall (i,j) \in \alpha \times \gamma.$$

Thus, the last relation in (39) is verified. □

4 Numerical experiments

In order to verify the ability of our method for solving the given stochastic linear semidefinite inverse optimal value problems, some numerical experiments are conducted in this section. All experiments are run on a 64-bit PC with an Intel (R) Core(TM)) i9-12900 of 2.40 GHz CPU and 32.00 GB of RAM equipped with Windows 11 operating system.

Experiment 1. Consider the following linear semidefinite programming problem

$$\begin{aligned}\min_{x\in\mathbb{R}^2} \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & \begin{bmatrix} -1 & -x_1 & -x_2 \\ -x_1 & -1-x_2 & 0 \\ -x_2 & 0 & -1-x_1 \end{bmatrix} \in \mathbb{S}^2_-.\end{aligned}$$

The corresponding optimal objective value is -3 with the global optimal solution $x^* = [-1; 0]$. Let ξ be a 3-dimensional random column vector that obeys a multivariate standard normal distribution, $A_1(\xi), A_2(\xi), B(\xi)$ are defined as

$$\begin{aligned} A_1(\xi) &:= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + R^T \mathrm{diag}(\xi) R, \\ A_2(\xi) &:= \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} + R^T \mathrm{diag}(\xi) R, \\ B(\xi) &:= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + R^T \mathrm{diag}(\xi) R, \end{aligned}$$

where R has the following two choices: (a) a 3-dimensional identity matrix; (b) a 3-dimensional random orthogonal matrix. We also set $v^* = -3, c^L = [2.5; 0.5], c^U = [3.5; 1.5]$, then the corresponding stochastic inverse optimal value problem has the following form

$$\begin{aligned} \min_{c,x,Y} \quad & \frac{1}{2}(c^T x + 3)^2, \\ \text{s.t.} \quad & c_i^L \le c_i \le c_i^U, \ \langle \mathbb{E}[A_i(\xi)], Y\rangle + c_i = 0, \ i \in \{1,2\}, \\ & \mathbb{E}[\mathcal{A}(\xi, x) - B(\xi)] \in \mathbb{S}^2_-, \ Y \in \mathbb{S}^2_+, \ \langle \mathbb{E}[\mathcal{A}(\xi, x) - B(\xi)], Y\rangle = 0, \end{aligned}$$

where $\mathcal{A}(\xi, x) = A_1(\xi)x_1 + A_2(\xi)x_2$. Next, we use JuMP and Ipopt packages [16] in Julia (Version 1.9) as the solver of the associated smooth SAA subproblems (18) and the subproblems (7).

Remark 4.1. *The introduction of diagonal noise to the matrix data serves to model uncertainties in the forward problem. This choice is motivated by its practicality and its property of preserving matrix symmetry and sparsity, a realistic feature in applications*

Table 1 Numerical results with different parameters μ and N.

μ	N	obj-a	obj-b	res-a	res-b	time-a	time-b
0.1	10	8.990e-20	9.720e-19	1.618e-2	1.895e-2	0.010s	0.030s
	100	8.965e-20	9.690e-19	1.603e-2	1.884e-2	0.012s	0.032s
	1000	8.951e-20	9.580e-19	1.578e-2	1.878e-2	0.013s	0.033s
0.05	10	3.724e-20	4.875e-19	1.123e-2	1.356e-2	0.012s	0.031s
	100	3.577e-20	4.786e-19	1.119e-2	1.352e-2	0.013s	0.032s
	1000	3.544e-20	4.698e-19	1.114e-2	1.348e-2	0.014s	0.033s
0.01	10	1.435e-20	2.598e-19	7.080e-2	8.140e-2	0.015s	0.031s
	100	1.423e-20	2.463e-19	7.079e-2	8.120e-2	0.016s	0.033s
	1000	1.401e-20	2.457e-19	7.075e-2	8.090e-2	0.018s	0.034s

Table 2 Numerical results with different initial point x^0.

x^0	obj-a	obj-b	res-a	res-b	time-a	time-b
$[0; 0]$	1.423e-20	2.597e-19	7.079e-2	8.151e-2	0.015s	0.033s
$[-5; -1]$	1.444e-20	2.614e-19	7.079e-2	8.162e-2	0.014s	0.034s
$[2; 1]$	1.340e-20	2.625e-19	7.086e-2	8.170e-2	0.016s	0.034s
$[10; 10]$	1.787e-20	2.894e-19	7.320e-2	8.183e-2	0.015s	0.033s
$[15; -15]$	1.461e-20	2.604e-19	7.079e-2	8.093e-2	0.014s	0.031s

like structural mechanics. Robustness tests with full-matrix noise confirmed that our method's performance remains consistent.

Table 1 shows that the solvability of our method under the different values of μ and N, where "obj-a" and "res-a" respectively denote the objective value of the smooth SAA subproblems and the residual that is the difference between the c-part solution of the smooth SAA subproblems and the original cost vector $[3; 1]$ under the Euclidean norm with the matrix R in the case (a). Similarly, "obj-b" and "res-b" respectively denote these results with the matrix R in the case (b). In addition, by fixing $\mu = 0.01$ and the number of samples $N = 1000$, Table 2 and Table 3 investigate the effect of our method with different initial point x^0 and bound vectors c^L, c^U and Table 4 shows the comparison of our method with the direct SAA+MPCC approach for solving the subproblems (7).

From Table 1, the overall residual is decreasing as N increases and μ tends to zero. From Table 2 and Table 3, numerical results show that our method is relatively stable for different initial values and bound vectors. From Table 4, our method has a better performance for the given stochastic inverse optimal value problem.

Experiment 2. We conduct further numerical experiments for some stochastic linear semidefinite inverse optimal value problems under different size (n, m) by fixing $\mu = 0.001$ and the number of samples $N = 10000$, in which we first set random $m \times m$ matrices A_i and B with entries in $[-10, 10]$ and use the package SCS [7] to achieve the corresponding optimal value of problem (3) with a given random $n \times 1$ vector c and set it to be the value of v^* in the subproblem (7). Moreover, we also set ξ to

Table 3 Numerical results with different bound vectors c^L and c^U.

c^L	c^U	obj-a	obj-b	res-a	res-b	time-a	time-b
$[2.90; 0.90]$	$[3.10; 1.10]$	3.466e-20	8.215e-19	7.079e-2	8.354e-2	0.014s	0.034s
$[2.50; 0.50]$	$[3.50; 1.50]$	1.423e-20	4.629e-19	7.079e-2	8.462e-2	0.014s	0.033s
$[2.00; 0.00]$	$[4.00; 2.00]$	8.922e-20	9.047e-19	7.079e-2	8.726e-2	0.013s	0.032s
$[1.00; -1.00]$	$[5.00; 3.00]$	6.461e-20	7.529e-19	7.150e-2	8.630e-2	0.012s	0.031s

Table 4 Comparison of our method with the direct SAA+MPCC approach.

our method			the direct SAA+MPCC approach		
obj-a	res-a	time-a	obj-a	res-a	time-a
1.396e-20	7.064e-2	0.018s	4.319e-16	4.280e-1	0.064s
obj-b	res-b	time-b	obj-b	res-b	time-b
2.361e-19	8.082e-2	0.034s	8.421e-16	5.638e-1	0.138s

Table 5 Numerical results with different size (n, m).

m	n	our method			the direct SAA+MPCC approach		
		obj	res	time	obj	res	time
5	5	3.449e-20	7.079e-2	0.024s	1.219e-15	4.280e-1	0.098s
10	10	5.003e-20	2.619e-2	1.924s	2.319e-15	2.325e-1	8.127s
20	20	1.429e-20	2.081e-2	5.006s	4.052e-16	2.280e-1	20.532s
30	30	1.290e-20	1.918e-2	31.772s	3.839e-16	1.149e-1	125.336s
40	40	1.069e-20	1.342e-2	188.568s	3.762e-16	1.145e-1	784.691s
50	50	3.401e-20	1.274e-2	222.658s	1.126e-15	1.083e-1	850.754s

be a n-dimensional random column vector that obeys a multivariate standard normal distribution, $A_i(\xi) := A_i + \mathrm{diag}(\xi)(i \in [n])$, $B(\xi) := B + \mathrm{diag}(\xi)$, $c^L := c - 0.1 * l$ and $c^U := c + 0.1 * u$, where l and u are random $n \times 1$ vectors with entries in $[0, 1]$.

A comparative analysis between our method and the direct SAA+MPCC approach in the high-dimensional setting is presented in Table 5. The numerical results demonstrate that our method efficiently solves these stochastic linear semidefinite inverse optimal value problems. In particular, for a problem of size $(n, m) = (50, 50)$, where the corresponding SAA subproblems defined in (18) are solved using the Ipopt solver, the formulation involves 2,600 variables, 240,818 nonzeros in the equality constraint Jacobian, and 3,027,798 nonzeros in the Lagrangian Hessian. Compared to the direct SAA+MPCC approach, our method achieves superior accuracy and robustness, particularly when the sample size is large.

5 Concluding remarks

This article studies an SAA approach for minimizing a class of stochastic linear semidefinite inverse optimal value problems. Under mild assumptions, we show that the sequences of global minimizer (respectively, KKT point) generated by the proposed approach convergers with probability one (w.p.1) to a global minimizer (respectively, an S-stationary point) of the original inverse optimal value problem. Numerical results show that our algorithm is suitable to solve the given inverse optimal value problem. We believe that the framework of our algorithm can be adapted to solve other types of stochastic inverse optimal value problems with general constraints. We leave these further discussions as our future work.

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Declarations

Confict of interest The authors have no Confict of interest to declare that are relevant to the content of this article.

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