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Augmented Lagrangian method for nonlinear circular conic programs: a local convergence analysis

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ABSTRACT

In this paper, we analyse a local convergence of augmented Lagrangian method (ALM) for a class of nonlinear circular conic optimization problems. In light of the singular value decomposition, the Debreu theorem and the implicit function theorem, we prove that the sequence generated by ALM converges to a local minimizer in the linear convergence rate under the constraint nondegeneracy condition and the strong second-order sufficient condition, in which the ratio constant is proportional to $1/\tau$, where τ is the associated penalty parameter with a given lower threshold. As a byproduct, we also derive explicit expressions of critical cone and its affine hull for the given nonlinear circular conic program.

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1. Introduction

Consider the following nonlinear circular conic program

$$
(\mathrm{P_{NCCP}})\qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & h_i(x) = 0,\ i \in \mathcal{I}_{\mathcal{E}} := \{1, 2, \dots, l\} \\ & (G_1^j(x), G_2^j(x)) \in \mathcal{L}_{\theta_j},\ j \in \mathcal{I}_{\mathcal{C}} := \{1, 2, \dots, J\}, \end{array}
$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h_i : \mathbb{R}^n \to \mathbb{R}$, $G_1^j : \mathbb{R}^n \to \mathbb{R}$, $G_2^j : \mathbb{R}^n \to \mathbb{R}^{s_j-1}$ are twice continuously differentiable, and $\mathcal{L}_{\theta_j}$ denotes a s_j-dimensional circular cone with $\theta_j \in (0, \frac{\pi}{2})$ being its half-aperture angle, i.e. $\mathcal{L}_{\theta_j} := \{(p_1^j, p_2^j) \in \mathbb{R} \times \mathbb{R}^{s_j-1} : \|p_2^j\| \le p_1^j \tan\theta_j\}$. The index sets $\mathcal{I}_{\mathcal{E}}$, $\mathcal{I}_{\mathcal{C}}$ correspond to the equality constraints and circular conic constraints, respectively.

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One can easy to see that the above model (P_{NCCP}) is a generalization of nonlinear second-order cone program

$$(P_{NSOCP}) \quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & h_i(x) = 0,\ i \in \mathcal{I}_{\mathcal{E}} := \{1, 2, \ldots, l\} \\ & (G_1^j(x), G_2^j(x)) \in \mathcal{K}^{s_j},\ j \in \mathcal{I}_{\mathcal{C}} := \{1, 2, \ldots, J\}, \end{array}$$

where $\mathcal{K}^{s_j}$ denotes a s_j-dimensional second-order cone, i.e. $\mathcal{K}^{s_j} := \{(p_1^j, p_2^j) \in \mathbb{R} \times \mathbb{R}^{s_j - 1} : \|p_2^j\| \leq p_1^j\}$, which corresponds to the case $\theta = \frac{\pi}{4}$ in (P_{NCCP}). The relations between the circular cone $\mathcal{L}_{\theta_j}$ and the second-order cone $\mathcal{K}^{s_j}$ follows from Zhou [1] that

$$x \in \mathcal{L}_{\theta_j} \Leftrightarrow S_j x \in \mathcal{K}^{s_j},\ S_j := \begin{bmatrix} \tan\theta_j & 0 \\ 0 & I_{s_j - 1} \end{bmatrix}, \tag{1}$$

where I_{s_j-1} is a $(s_j - 1)$-dimensional identity matrix. In addition, the circular cone $\mathcal{L}_{\theta_j}$ is not self-dual under the standard inner product [1, Theorem 2.1]. One may argue that the model (P_{NCCP}) can be transformed into (P_{NSOCP}) via the relation (1) and use the state-of-art algorithms for NSOCP to solve the corresponding transformed problem. However, this approach may not be acceptable from theoretical, numerical and modelling viewpoints: (a) The study of the vector-valued function induced by $\mathcal{L}_{\theta_j}$ can not be transformed into the one induced by $\mathcal{K}^{s_j}$ via the relation (1) directly [2]; (b) The scaling matrix S_j defined as in (1) may cause undesirable numerical performance due to round-off errors in computation process [3,4]; (c) Some circular conic programs (such as the models for solving support vector machines problems) may have smaller scales than the associated model using the second-order cone programs [5, Section 3]. Hence, it is necessary to study the above nonlinear circular conic programming (P_{NCCP}) deeply from theoretical analysis and numerical algorithms.

Recently, many researchers have paid attention to theoretical analysis on optimization problems with circular conic constraints [1,2,6–11]. However, due to the non-self duality of circular cones under the standard inner product and the standard Euclidean vector norm, there exist very few algorithms for dealing with circular conic programs. More specifically, some algorithms including prime-dual interior-point algorithms and smoothing Newton algorithm have been proposed for circular conic programming problems, one can refer to [3,4,12] for more details. As a classical method for solving constrained optimization problems, augmented Lagrangian method (ALM) was initially analysed by Hestenes [13] and Powell [14] in equality constrained problem. These results triggered off a series of contributions on the local convergence rate results for convex programming [15–17], nonlinear programming [18], nonlinear second-order cone programming [19] and nonlinear semidefinite programming [20]. However, to our best knowledge, no results about the local convergence analysis of ALM for nonlinear circular conic program (P_{NCCP}) have been reported. Hence, the purpose of this article aims to fill this gap and the contributions of our research can be summarized as follows:

(a) We present explicit expressions of the critical cone and its affine hull for nonlinear circular conic program ($\mathrm{P_{NCCP}}$).
(b) Under mild conditions, we prove that the sequence generated by the augmented Lagrangian method converges to a local minimizer in the linear convergence rate and the ratio constant is proportional to $1/\tau$, where τ is the associated penalty parameter with a given lower threshold.

The remainder of this paper is organized as follows. In Section 2, we recall some preliminary results on the geometric properties of circular cone. In Section 3, we present some analytic properties on the associated Lagrangian function of the given nonlinear conic program. After these preparations, a local convergence analysis of augmented Lagrangian method for solving ($\mathrm{P_{NCCP}}$) is established in Section 4. Finally, we draw some concluding remarks in Section 5.

1.1. Notations

To close this section, we introduce some notations that will be frequently used in the sequel. The Lagrangian function associated with problem ($\mathrm{P_{NCCP}}$) is defined as $\mathfrak{L}(x,\mu,\Gamma) := f(x) + \mu^T h(x) - \sum_{j\in\mathcal{I}_C}(\Gamma^j)^T G^j(x)$, where $\mu := (\mu_1,\dots,\mu_l)^T \in \mathbb{R}^l$, $h(x) := (h_1(x),\dots,h_l(x))^T \in \mathbb{R}^l$, $\Gamma := (\Gamma^1,\dots,\Gamma^J) \in \prod_{j\in\mathcal{I}_C}\mathbb{R}^{s_j}$, $\Gamma^j := (\Gamma^j_1,\Gamma^j_2) \in \mathbb{R}^{s_j}$ and $G^j(x) := (G^j_1(x), G^j_2(x)) \in \mathbb{R}^{s_j}$. Given a feasible point $x \in \mathbb{R}^n$, we say it to be a stationary point of ($\mathrm{P_{NCCP}}$) if there exist $\mu \in \mathbb{R}^l$ and $\Gamma \in \prod_{j\in\mathcal{I}_C}\mathbb{R}^{s_j}$ such that

$$\begin{aligned} &\nabla_x\mathfrak{L}(x,\mu,\Gamma) = 0_n, \quad h_i(x) = 0,\ i \in \mathcal{I}_\mathcal{E},\\ &\Gamma^j \in \mathcal{L}^*_{\theta_j}, \quad G^j(x) \in \mathcal{L}_{\theta_j},\ (\Gamma^j)^T G^j(x) = 0,\ j \in \mathcal{I}_C, \end{aligned} \tag{2}$$

where 0_n denotes the zero vector in $\mathbb{R}^n$ and $\mathcal{L}^*_{\theta_j}$ is the dual cone of $\mathcal{L}_{\theta_j}$, i.e. $\mathcal{L}^*_{\theta_j} := \{(y^j_1, y^j_2) \in \mathbb{R}\times\mathbb{R}^{s_j-1} : x^j_1 y^j_1 + (x^j_2)^T y^j_2 \geq 0,\ \forall (x^j_1, x^j_2) \in \mathcal{L}_{\theta_j}\}$. Any point $(x,\mu,\Gamma) \in \mathbb{R}^n \times \mathbb{R}^l \times \prod_{j\in\mathcal{I}_C}\mathbb{R}^{s_j}$ satisfying the above system (2) is called a Karush-Kuhn-Tucker (KKT) point of ($\mathrm{P_{NCCP}}$) and the pair (μ,Γ) is the Lagrange multiplier associated with $x \in \mathbb{R}^n$. In addition, we denote by $\Lambda(x)$ the collection of such multiplier vectors at x. As defined in [21, Section 2], the augmented Lagrangian function of ($\mathrm{P_{NCCP}}$) is

$$\begin{aligned} \mathfrak{L}_\tau(x,\mu,\Gamma) := &f(x) + \mu^T h(x) + \frac{\tau}{2}\|h(x)\|^2 \\ &+ \frac{1}{2\tau}\sum_{j\in\mathcal{I}_C}\left(\|\Pi_{\mathcal{L}^*_{\theta_j}}(\Gamma^j - \tau G^j(x))\|^2 - \|\Gamma^j\|^2\right), \end{aligned}$$

where $\tau > 0$ is the penalty parameter. For any given $w \in \mathbb{R}^{s_j}$, $\Pi_{\mathcal{L}^*_{\theta_j}}(\cdot)$ is the metric projection mapping onto $\mathcal{L}^*_{\theta_j}$, i.e.$\Pi_{\mathcal{L}^*_{\theta_j}}(w) := \operatorname{argmin}_{y\in\mathcal{L}^*_{\theta_j}} \|w - y\|^2$.

2. Preliminaries

This section recalls some topological properties and strongly semismoothness of Bouligand-subdifferential of projection operator regarding circular cones.

2.1. Topological properties

Let $\operatorname{int}\mathcal{L}_\theta$ and $\operatorname{bd}\mathcal{L}_\theta$ denote the interior and the boundary of $\mathcal{L}_\theta$, respectively. From [1, Theorem 2.1], we know

$$\operatorname{int}\mathcal{L}_\theta := \{(p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{s-1} : \|p_2\| < p_1 \tan\theta\},$$
$$\operatorname{bd}\mathcal{L}_\theta := \{(p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{s-1} : \|p_2\| = p_1 \tan\theta\}.$$

Moreover, the corresponding dual cone $\mathcal{L}^*_\theta$ and its dual have the following relations

$$\begin{aligned} &\mathcal{L}^*_\theta := \{(p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{s-1} : \|p_2\| \le p_1 \cot\theta\}, \\ &\mathcal{L}^*_\theta = \mathcal{L}_{\theta'}, \quad (\mathcal{L}^*_\theta)^* = \mathcal{L}_\theta, \quad \mathcal{L}^\circ_\theta := -\mathcal{L}^*_\theta = -\mathcal{L}_{\theta'}, \quad \theta' := \frac{\pi}{2} - \theta, \end{aligned} \tag{3}$$

where $\mathcal{L}^\circ_\theta$ is the polar cone of $\mathcal{L}_\theta$.

Remark 2.1: The interior and the boundary of $\mathcal{L}^*_\theta$ are respectively given by

$$\operatorname{int}\mathcal{L}^*_\theta = \{(p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{s-1} : \|p_2\| < p_1 \cot\theta\},$$
$$\operatorname{bd}\mathcal{L}^*_\theta = \{(p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{s-1} : \|p_2\| = p_1 \cot\theta\}.$$

As mentioned in [1, Section 3], for any given vector $p = (p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{s-1}$, its spectral decomposition with respect to $\mathcal{L}_\theta$ is given by

$$p = \lambda_1(p) \cdot v_p^{(1)} + \lambda_2(p) \cdot v_p^{(2)}, \tag{4}$$

where $\lambda_i(p)$, $v_p^{(i)}$ are the corresponding spectral values and spectral vectors, i.e.

$$\begin{aligned} \lambda_1(p) &:= p_1 - \|p_2\| \cot\theta, \\ v_p^{(1)} &:= \frac{1}{1+\cot^2\theta} \cdot \begin{pmatrix} 1 & 0^T_{s-1} \\ 0_{s-1} & \cot\theta \cdot I_{s-1} \end{pmatrix} \begin{pmatrix} 1 \\ -\bar{p}_2 \end{pmatrix}, \\ \lambda_2(p) &:= p_1 + \|p_2\| \tan\theta, \\ v_p^{(2)} &:= \frac{1}{1+\tan^2\theta} \cdot \begin{pmatrix} 1 & 0^T_{s-1} \\ 0_{s-1} & \tan\theta \cdot I_{s-1} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{p}_2 \end{pmatrix} \end{aligned} \tag{5}$$

with I_{s-1} being the $(s-1)$-dimensional identity matrix and $\bar{p}_2 := p_2/\|p_2\|$, if $p_2 \ne 0_{s-1}$; $\bar{p}_2$ is any unit vector defined in $\mathbb{R}^{s-1}$, otherwise. In light of [1, Theorem 3.1] and the spectral decomposition (4), the metric projection of p onto $\mathcal{L}_\theta$ takes the form of

$$\Pi_{\mathcal{L}_\theta}(p) = [\lambda_1(p)]_+ \cdot v_p^{(1)} + [\lambda_2(p)]_+ \cdot v_p^{(2)},$$
$$[\lambda_i(p)]_+ := \max\{0, \lambda_i(p)\}, \ (i = 1, 2).$$

It also follows from (5) that

$$\Pi_{\mathcal{L}_\theta}(p) = \begin{cases} p & \text{if } p \in \mathcal{L}_\theta, \\ 0_s & \text{if } p \in -\mathcal{L}_\theta^*, \\ u & \text{otherwise,} \end{cases} \qquad u := \begin{pmatrix} \dfrac{p_1 + \|p_2\| \tan\theta}{1 + \tan^2\theta} \\ \left(\dfrac{p_1 + \|p_2\| \tan\theta}{1 + \tan^2\theta} \tan\theta \right) \cdot \bar{p}_2 \end{pmatrix}. \tag{6}$$

Remark 2.2: These results together with the famous Moreau decomposition theorem [22] imply that

$$p = \Pi_{\mathcal{L}_\theta}(p) + \Pi_{\mathcal{L}_\theta^\circ}(p) = \Pi_{\mathcal{L}_\theta}(p)$$
$$- \Pi_{\mathcal{L}_\theta^*}(-p) = \Pi_{\mathcal{L}_\theta}(p) - \Pi_{\mathcal{L}_{\theta'}}(-p), \quad \theta' = \frac{\pi}{2} - \theta.$$

The augmented Lagrangian function of ($\mathrm{P_{NCCP}}$) can be recast as

$$\mathfrak{L}_\tau(x, \mu, \Gamma) = f(x) + \mu^T h(x) + \frac{\tau}{2}\|h(x)\|^2$$
$$+ \frac{1}{2\tau} \sum_{j \in \mathcal{I}_C} \left(\|\Pi_{\mathcal{L}_{\theta_j'}}(\Gamma^j - \tau G^j(x))\|^2 - \|\Gamma^j\|^2 \right) \tag{7}$$

with $\theta_j' := \frac{\pi}{2} - \theta_j, j \in \mathcal{I}_C$.

The next lemma shows an important observation on the following circular conic complementarity system

$$p = (p_1, p_2) \in \mathcal{L}_\theta^*, \quad q = (q_1, q_2) \in \mathcal{L}_\theta, \ p^T q = 0, \tag{8}$$

which plays an important role in the analysis of optimality conditions, see [6, Theorem 2.5] for more details.

Lemma 2.1 (Property of circular conic complementary system): *The circular conic complementarity system* (8) *has at least one solution if and only if one of the following holds:*

(a) $p = 0_s$, $q \in \mathcal{L}_\theta$;
(b) $p \in \operatorname{int} \mathcal{L}_\theta^*$, $q = 0_s$;
(c) $p \in \operatorname{bd} \mathcal{L}_\theta^* \setminus \{0_s\}$, $q = 0_s$;
(d) $p \in \operatorname{bd} \mathcal{L}_\theta^* \setminus \{0_s\}$, $q \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0_s\}$ *and there exist* $\sigma > 0$ *such that* $p = \sigma(\mathcal{H}q)$, *where* $\mathcal{H} := \begin{pmatrix} \tan^2\theta & 0_{s-1}^T \\ 0_{s-1} & -I_{s-1} \end{pmatrix}$.

2.2. Strongly semismoothness of bouligand-subdifferential of projection operator

In this subsection, we first recall some concepts on the Bouligand-subdifferential of a given mapping, which comes from [20, Definition 1] that adapts from Mifflin [23] for functionals and Qi and Sun [24] for vector-valued functions. Let U be an open set and Ψ be a locally Lipschitz continuous function on U. From the Rademacher's theorem, Ψ is almost everywhere Fréchet-differentiable in U. Let $\mathcal{D}_\Psi$ be the set of Fréchet-differentiable points of Ψ in U. Then, the Bouligand-subdifferential of Ψ at $x \in U$, denoted by $\partial_B\Psi(x)$, is characterized as $\partial_B\Psi(x) := \{\lim_{k\to\infty} \mathcal{J}\Psi(x^k) : x^k \in \mathcal{D}_\Psi, x^k \to x\}$, where $\mathcal{J}\Psi(x^k)$ denotes the Jacobian of Ψ at x^k.

Definition 2.2 (Semismoothness and strongly semismoothness): Let Ψ be a locally Lipschitz continuous function on a open set U. We say that Ψ is semismooth at $x \in U$ if Ψ is directionally differentiable at x and for any $\Delta x \in U$ and $V \in \partial\Psi(x+\Delta x)$ with $\Delta x \to 0$, $\Psi(x+\Delta x) - \Psi(x) - V(\Delta x) = o(\|\Delta x\|)$. Furthermore, Ψ is said to be strongly semismooth at $x \in U$ if Ψ is semismooth at x and for any $\Delta x \in U$, $V \in \partial\Psi(x+\Delta x)$ with $\Delta x \to 0$, $\Psi(x+\Delta x) - \Psi(x) - V(\Delta x) = O(\|\Delta x\|^2)$.

We now present two important lemmas about projection operator onto circular cone, which are needed in subsequent analysis. The interested readers can refer to [10, Lemma 3.1 and Theorem 3.3] for their proofs.

Lemma 2.3 (Bouligand-subdifferential of projection operator onto circular cone): *For any given $p = (p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{s-1}$ with the spectral decomposition (4). The Bouligand-subdifferential of projection operator onto circular cone is given as follows:*

(a) *If $p \notin \mathcal{L}_\theta \cup (-\mathcal{L}_\theta^*)$, then*

$$\partial_B\Pi_{\mathcal{L}_\theta}(p) = \frac{1}{\tan\theta + \cot\theta} \times \begin{pmatrix} \cot\theta & \bar{p}_2^T \\ \bar{p}_2 & \dfrac{p_1 + \|p_2\|\tan\theta}{\|p_2\|} \cdot I_{s-1} - \dfrac{p_1}{\|p_2\|} \cdot \bar{p}_2\bar{p}_2^T \end{pmatrix};$$

(b) *If $p \in \operatorname{int}\mathcal{L}_\theta$, then*

$$\partial_B\Pi_{\mathcal{L}_\theta}(p) = \{I_s\};$$

(c) *If $p \in \operatorname{bd}\mathcal{L}_\theta \setminus \{0_s\}$, then*

$$\partial_B\Pi_{\mathcal{L}_\theta}(p) = \left\{ I_s, I_s + \frac{1}{\tan\theta + \cot\theta} \begin{pmatrix} -\tan\theta & \bar{p}_2^T \\ \bar{p}_2 & -\cot\theta \cdot \bar{p}_2\bar{p}_2^T \end{pmatrix} \right\};$$

(d) *If* $p \in \operatorname{int}(-\mathcal{L}_\theta^*)$, *then*

$$\partial_B \Pi_{\mathcal{L}_\theta}(p) = \{0_{s\times s}\};$$

(e) *If* $p \in \operatorname{bd}(-\mathcal{L}_\theta^*) \setminus \{0_s\}$, *then*

$$\partial_B \Pi_{\mathcal{L}_\theta}(p) = \left\{0_{s\times s}, \frac{1}{\tan\theta + \cot\theta}\begin{pmatrix}\cot\theta & \bar{p}_2^T \\ \bar{p}_2 & \tan\theta \cdot \bar{p}_2\bar{p}_2^T\end{pmatrix}\right\};$$

(f) *If* $p = 0_s$, *then*

$$\partial_B \Pi_{\mathcal{L}_\theta}(x) = \left\{0_{s\times s}, I_s, \frac{1}{\tan\theta + \cot\theta}\begin{pmatrix}\cot\theta & w^T \\ w & a_\theta \cdot (I_{s-1} - ww^T) + \tan\theta \cdot ww^T\end{pmatrix}\right\}$$

with $a_\theta := (\tan\theta + \cot\theta)a$, $a \in [0, 1]$ *and* w *being any unity vector in* $\mathbb{R}^{s-1}$.

Lemma 2.4 (Strongly semismoothness of projection operator onto circular cone): *The projection operator* $\Pi_{\mathcal{L}_\theta}(\cdot)$ *defined as in* (6) *is strongly semismooth over* $\mathbb{R}^s$.

3. Properties of augmented Lagrangian function

In this section, we build up some properties of the augmented Lagrangian function $\mathfrak{L}_\tau$ defined as in (7). We first present its gradient formula and then derive its associated Bouligand-subdifferential.

Since f, h, G^j ($j \in \mathcal{I}_C$) are twice continuously differentiable functions, we know from the structure of $\mathfrak{L}_\tau(x, \mu, \Gamma)$ that $\mathfrak{L}_\tau(\cdot)$ is continuously differentiable as well. For any given $(x, \mu, \Gamma) \in \mathbb{R}^n \times \mathbb{R}^l \times \prod_{j\in\mathcal{I}_C} \mathbb{R}^{s_j}$, we denote

$$F_\tau^j(x, \mu, \Gamma) := \Gamma^j - \tau G^j(x), \quad j \in \mathcal{I}_C. \tag{9}$$

It is not hard to see that the Jacobian mapping $\mathcal{J}F_\tau^j(x, \mu, \Gamma) : \mathbb{R}^n \times \mathbb{R}^l \times \prod_{j\in\mathcal{I}_C} \mathbb{R}^{s_j} \to \mathbb{R}^{s_j}$ is onto for any given $(x, \mu, \Gamma) \in \mathbb{R}^n \times \mathbb{R}^l \times \prod_{j\in\mathcal{I}_C} \mathbb{R}^{s_j}$. Then, the gradient of $\mathfrak{L}_\tau(x, \mu, \Gamma)$ with respect to $x \in \mathbb{R}^n$, denoted by $\nabla_x \mathfrak{L}_\tau(x, \mu, \Gamma)$, has the following form

$$\nabla f(x) + \nabla h(x)(\mu + \tau h(x)) - \sum_{j\in\mathcal{I}_C} \nabla G^j(x)\Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x, \mu, \Gamma)), \quad \theta_j' := \frac{\pi}{2} - \theta_j,\ j \in \mathcal{I}_C, \tag{10}$$

where $F_\tau^j(x, \mu, \Gamma)$ is defined as in (9).

Similar to Lemma 2.1, we have two important lemmas for the KKT system (2), whose proofs are straightforward, we omit them here.

Lemma 3.1 (Property of the second part of the KKT system (2)): *Suppose that x_* is a stationary point of* $(\mathrm{P}_{\mathrm{NCCP}})$ *and* $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. *Then, for all $j \in \mathcal{I}_{\mathcal{C}}$, one of the following facts holds:*

(a) $\Gamma_*^j = 0_{s_j}$, $G^j(x_*) \in \mathcal{L}_{\theta_j}$;
(b) $\Gamma_*^j \in \operatorname{int} \mathcal{L}_{\theta_j}^*$, $G^j(x_*) = 0_{s_j}$;
(c) $\Gamma_*^j \in \operatorname{bd} \mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\}$, $G^j(x_*) = 0_{s_j}$;
(d) $\Gamma_*^j \in \operatorname{bd} \mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\}$, $G^j(x_*) \in \operatorname{bd} \mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}$ *and there exists $\sigma_j > 0$ such that* $\Gamma_*^j = \sigma_j(\mathcal{H}_j G^j(x_*))$, *where*

$$\mathcal{H}_j := \begin{pmatrix} \tan^2 \theta_j & 0_{s_j-1}^T \\ 0_{s_j-1} & -I_{s_j-1} \end{pmatrix}. \tag{11}$$

Lemma 3.2 (Property of the first relation in the KKT system (2)): *Suppose that x_* is a stationary point of* $(\mathrm{P}_{\mathrm{NCCP}})$. *For any $(\mu, \Gamma) \in \Lambda(x_*)$, we have* $\nabla_x \mathfrak{L}_\tau(x_*, \mu, \Gamma) = 0_n$.

In light of Lemma 2.4, the projection operator $\Pi_{\mathcal{L}_{\theta_j'}}(\cdot)$ is semismooth everywhere, we have $\partial_B(\Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x, \mu, \Gamma))) = \partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x, \mu, \Gamma)) \mathcal{J} F_\tau^j(x, \mu, \Gamma)$, where $F_\tau^j(x, \mu, \Gamma)$ is defined as in (9). Setting $\Psi_\tau^j(x, \mu, \Gamma) := \nabla G^j(x)$ $(\Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x, \mu, \Gamma)))$, the corresponding Bouligand-subdifferential $\partial_B \Psi_\tau^j(x, \mu, \Gamma)$ $(\Delta x, \Delta \mu, \Delta \Gamma)$ has an explicit expression as

$$\nabla^2 G^j(x)(\Delta x) \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x, \mu, \Gamma)) + \nabla G^j(x) \partial_B(\Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x, \mu, \Gamma)))(\Delta x, \Delta \mu, \Delta \Gamma). \tag{12}$$

Similarly, the following relation holds by direct calculations,

$$\begin{aligned} &\partial_B(\nabla_x \mathfrak{L}_\tau)(x, \mu, \Gamma) \\ &\quad = (\nabla^2 f(x), 0, 0, \ldots, 0) \\ &\qquad + \left(\sum_{i \in \mathcal{I}_{\mathcal{E}}} (\mu_i + \tau h_i(x)) \nabla^2 h_i(x) + \tau \nabla h(x) \mathcal{J} h(x), \nabla h(x), 0, \ldots, 0 \right) \\ &\qquad - \sum_{j \in \mathcal{I}_{\mathcal{C}}} \partial_B \Psi_\tau^j(x, \mu, \Gamma), \end{aligned}$$

where $\partial_B \Psi_\tau^j(x,\mu,\Gamma)$ is defined as in (12). Let $\Pi_x \partial_B(\nabla_x \mathfrak{L}_\tau)$ be the partial Bouligand-subdifferential of $\nabla_x \mathfrak{L}_\tau$ with respect to x, we compute that

$$\begin{aligned}
&(\Pi_x \partial_B(\nabla_x \mathfrak{L}_\tau))(x,\mu,\Gamma)(\Delta x) \\
&\quad = \nabla^2 f(x)(\Delta x) + \nabla^2 h(x)(\Delta x)(\mu + \tau h(x)) + \tau \nabla h(x) \mathcal{J} h(x)(\Delta x) \\
&\qquad - \sum_{j\in\mathcal{I}_C} (\Pi_x \partial_B \Psi_\tau^j)(x,\mu,\Gamma)(\Delta x) \\
&\quad = \nabla_{xx}^2 \mathfrak{L}(x, \mu + \tau h(x), \Pi_{\mathcal{L}_{\theta_1'}}(F_\tau^1(x,\mu,\Gamma)), \ldots, \Pi_{\mathcal{L}_{\theta_J'}}(F_\tau^J(x,\mu,\Gamma)))(\Delta x) \\
&\qquad + \tau \nabla h(x) \mathcal{J} h(x)(\Delta x) + \sum_{j\in\mathcal{I}_C} \tau \nabla G^j(x) \partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x,\mu,\Gamma)) \mathcal{J} G^j(x)(\Delta x).
\end{aligned}$$

To proceed, for any given $W := (W^1, \ldots, W^J) \in \prod_{j\in\mathcal{I}_C} \mathbb{R}^{s_j\times s_j}$, we define

$$\mathcal{A}_\tau(\mu,\Gamma,W) := \nabla_{xx}^2 \mathfrak{L}(x,\mu,\Gamma) + \tau \mathcal{J} h(x)^T \mathcal{J} h(x) + \sum_{j\in\mathcal{I}_C} \tau \mathcal{J} G^j(x)^T W^j \mathcal{J} G^j(x). \tag{13}$$

From all the above discussions, we achieve an explicit expression of the partial Bouligand-subdifferential of $\nabla_x \mathfrak{L}_\tau$ at a given KKT pair of ($\mathrm{P}_{\mathrm{NCCP}}$).

Lemma 3.3 (Partial Bouligand-subdifferential of $\nabla_x \mathfrak{L}_\tau$): *Suppose that x_* is a stationary point of* ($\mathrm{P}_{\mathrm{NCCP}}$) *and $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. Then, for any given $\Delta x \in \mathbb{R}^n$ and $W_* := (W_*^1, \ldots, W_*^J) \in \prod_{j\in\mathcal{I}_C} \mathbb{R}^{s_j\times s_j}$, we have*

$$(\Pi_x \partial_B(\nabla_x \mathfrak{L}_\tau))(x_*,\mu_*,\Gamma_*)(\Delta x) = \mathcal{A}_\tau(\mu_*,\Gamma_*,W_*)(\Delta x),$$

where $W_*^j \in \partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x_*,\mu_*,\Gamma_*))$, $\theta_j' := \frac{\pi}{2} - \theta_j$, $j \in \mathcal{I}_C$ and $\mathcal{A}_\tau(\mu,\Gamma,W)$ is defined as in (13).

In order to further characterize the term $\partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x_*,\mu_*,\Gamma_*))$ more precisely, we employ the following six index sets:

$$\begin{aligned}
\mathcal{I}_G(x_*) &:= \left\{ j \in \mathcal{I}_C : G^j(x_*) \in \operatorname{int} \mathcal{L}_{\theta_j} \right\}, \\
\mathcal{I}_{\Gamma_*} &:= \left\{ j \in \mathcal{I}_C : \Gamma_*^j \in \operatorname{int} \mathcal{L}_{\theta_j}^* \right\}, \\
\mathcal{B}_G(x_*) &:= \left\{ j \in \mathcal{I}_C : G^j(x_*) \in \operatorname{bd} \mathcal{L}_{\theta_j} \setminus \{0_{s_j}\} \right\}, \\
\mathcal{B}_{\Gamma_*} &:= \left\{ j \in \mathcal{I}_C : \Gamma_*^j \in \operatorname{bd} \mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\} \right\}, \\
\mathcal{Z}_G(x_*) &:= \left\{ j \in \mathcal{I}_C : G^j(x_*) = 0_{s_j} \right\}, \\
\mathcal{Z}_{\Gamma_*} &:= \left\{ j \in \mathcal{I}_C : \Gamma_*^j = 0_{s_j} \right\}.
\end{aligned} \tag{14}$$

After these preparations, we now present the Bouligand-subdifferential of $\Pi_{\mathcal{L}_{\theta'_j}}$ at $F^j_\tau(x_*, \mu_*, \Gamma_*)$.

Theorem 3.4 (Bouligand-subdifferential of $\Pi_{\mathcal{L}_{\theta'_j}}$): *Suppose that x_* is a stationary point of* (P$_{\text{NCCP}}$) *and $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. Then, the followings hold.*

(a) *If $j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$ and $-F^j_\tau(x_*, \mu_*, \Gamma_*) \notin (\mathcal{L}^\circ_{\theta_j} \cup \mathcal{L}_{\theta_j})$, then $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*))$ has the following form*

$$\frac{1}{\tan\theta_j + \cot\theta_j} \times \begin{pmatrix} \tan\theta_j & \overline{(\Gamma^j_*)_2}^T \\ \overline{(\Gamma^j_*)_2} & \dfrac{\sigma_j(\tan^2\theta_j + 1)}{(\tau + \sigma_j)\tan\theta_j} \cdot I_{s_j - 1} + \dfrac{\tau - \sigma_j \tan^2\theta_j}{(\tau + \sigma_j)\tan\theta_j} \cdot \overline{(\Gamma^j_*)_2}\,\overline{(\Gamma^j_*)_2}^T \end{pmatrix},$$

where $\overline{(\Gamma^j_*)_2} := (\Gamma^j_*)_2 / \|(\Gamma^j_*)_2\|$, if $(\Gamma^j_*)_2 \neq 0_{s_j-1}$.

(b) *If $j \in \mathcal{Z}_G(x_*) \cap \mathcal{I}_{\Gamma_*}$, then $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*))$ is equal to the s_j-dimensional identity matrix I_{s_j}.*

(c) *If $j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, then $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*))$ has the following two choices:*

$$I_{s_j}, \quad \begin{pmatrix} \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} & \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma^j_*)_2}^T \\ \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma^j_*)_2} & I_{s_j - 1} - \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma^j_*)_2}\,\overline{(\Gamma^j_*)_2}^T \end{pmatrix}.$$

(d) *If $j \in \mathcal{I}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$, then $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*))$ is equal to the s_j-dimensional zero matrix $0_{s_j \times s_j}$.*

(e) *If $j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$, then $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*))$ has the following two choices:*

$$0_{s_j \times s_j}, \quad \begin{pmatrix} \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} & -\dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{G^j_2(x_*)}^T \\ -\dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{G^j_2(x_*)} & \dfrac{\cot\theta_j}{\tan\theta_j + \cot\theta_j} \cdot \overline{G^j_2(x_*)}\,\overline{G^j_2(x_*)}^T \end{pmatrix},$$

where $\overline{G^j_2(x_*)} := G^j_2(x_*) / \|G^j_2(x_*)\|$, if $G^j_2(x_*) \neq 0_{s_j-1}$.

(f) *If* $j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$, *then* $\partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x_*, \mu_*, \Gamma_*))$ *has the following three choices:*

$$0_{s_j \times s_j},\ I_{s_j},\ \frac{1}{\tan\theta_j + \cot\theta_j}\begin{pmatrix} \tan\theta_j & w_j^T \\ w_j & a_{\theta_j}\cdot(I_{s_j-1} - w_j w_j^T) + \cot\theta_j \cdot w_j w_j^T \end{pmatrix}$$

with $a_{\theta_j} := (\tan\theta_j + \cot\theta_j)a,\ a \in [0,1]$, w_j *being any unity vector in* $\mathbb{R}^{s_j-1}$.

Proof: First, from Lemma 2.3, setting $\theta = \frac{\pi}{2} - \theta_j = \theta_j'$, $p = F_\tau^j(x_*, \mu_*, \Gamma_*) =: \Xi_*^j$ yields

$$(-\Xi_*^j) = \tau G^j(x_*) + (-\Gamma_*^j),\ G^j(x_*) \in \mathcal{L}_{\theta_j}, \quad -\Gamma_*^j \in \mathcal{L}_{\theta_j}^\circ,\ G^j(x_*)^T(-\Gamma_*^j) = 0. \tag{15}$$

(a) If $\Xi_*^j \notin \mathcal{L}_{\theta_j'} \cup -\mathcal{L}_{\theta_j'}^*$, from the relations in (3), we have $(-\Xi_*^j) \notin \mathcal{L}_{\theta_j}^\circ \cup \mathcal{L}_{\theta_j}$. In addition, using Lemma 2.3(a) implies that $\partial_B \Pi_{\mathcal{L}_{\theta_j'}}(\Xi_*^j)$ has the following form

$$\frac{1}{\tan\theta_j + \cot\theta_j} \times \begin{pmatrix} \tan\theta_j & \overline{(\Xi_*^j)_2}^T \\ \overline{(\Xi_*^j)_2} & \frac{(\Xi_*^j)_1 + \|(\Xi_*^j)_2\|\cot\theta_j}{\|(\Xi_*^j)_2\|}\cdot I_{s_j-1} - \frac{(\Xi_*^j)_1}{\|(\Xi_*^j)_2\|}\cdot \overline{(\Xi_*^j)_2}\,\overline{(\Xi_*^j)_2}^T \end{pmatrix}. \tag{16}$$

Applying the Moreau decomposition theorem and (15) indicate that $\Pi_{\mathcal{L}_{\theta_j}}(-\Xi_*^j) = \tau G^j(x_*) \in \operatorname{bd}\mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}$, $\Pi_{\mathcal{L}_{\theta_j}^\circ}(-\Xi_*^j) = -\Gamma_*^j \in \operatorname{bd}\mathcal{L}_{\theta_j}^\circ \setminus \{0_{s_j}\}$, which are equivalent to $G^j(x_*) \in \operatorname{bd}\mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}$, $\Gamma_*^j \in \operatorname{bd}\mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\}$, i.e. $j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$. Together with Lemma 2.1(d), there exists $\sigma_j > 0$ such that $\Gamma_*^j = \sigma_j(\mathcal{H}_j G^j(x_*))$, where $\mathcal{H}_j$ is defined as in (11). Then, it follows from above that

$$(-\Xi_*^j) = \tau G^j(x_*) - \sigma_j(\mathcal{H}_j G^j(x_*)) = \begin{pmatrix} (\tau - \sigma_j \tan^2\theta_j)\cdot G_1^j(x_*) \\ (\tau + \sigma_j)\cdot G_2^j(x_*) \end{pmatrix}. \tag{17}$$

Since $G^j(x_*) \in \operatorname{bd}\mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}$, we obtain $\|G_2^j(x_*)\| = G_1^j(x_*)\tan\theta_j$, $G_1^j(x_*) \neq 0$ and $G_2^j(x_*) \neq 0_{s_j-1}$. From the relation (17), we have $(\Xi_*^j)_1 = (\sigma_j \tan^2\theta_j - \tau)G_1^j(x_*)$, $(\Xi_*^j)_2 = -(\tau + \sigma_j)G_2^j(x_*)$. Together with the fact $\Gamma_*^j = \sigma_j(\mathcal{H}_j G^j(x_*))$

indicate that

$$(\Xi_*^j)_1 = (\sigma_j \tan^2\theta_j - \tau)G_1^j(x_*) = \frac{\sigma_j \tan^2\theta_j - \tau}{\sigma_j \tan^2\theta_j}(\Gamma_*^j)_1 \tag{18}$$

$$(\Xi_*^j)_2 = -(\tau + \sigma_j)G_2^j(x_*) = -(\tau + \sigma_j)\frac{(\Gamma_*^j)_2}{-\sigma_j} = \frac{\tau + \sigma_j}{\sigma_j}(\Gamma_*^j)_2. \tag{19}$$

In addition, from $\Gamma_*^j \in \mathrm{bd}\,\mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\}$ and $\mathcal{L}_{\theta_j}^* = \mathcal{L}_{\theta_j'}$, we have $(\Gamma_*^j)_1 = \|(\Gamma_*^j)_2\| \tan\theta_j$, $(\Gamma_*^j)_1 \neq 0$ and $(\Gamma_*^j)_2 \neq 0_{s_j-1}$. Therefore, it follows from the relations (18)-(19) that $(\Xi_*^j)_1 = \frac{\sigma_j \tan^2\theta_j - \tau}{\sigma_j \tan\theta_j}\|(\Gamma_*^j)_2\|$, $\|(\Xi_*^j)_2\| = \frac{\tau+\sigma_j}{\sigma_j}\|(\Gamma_*^j)_2\|$ and $\overline{(\Xi_*^j)_2} = \overline{(\Gamma_*^j)_2}$. Combining the above relations with (16), we achieve

$$\begin{aligned}&\frac{(\Xi_*^j)_1 + \|(\Xi_*^j)_2\|\cot\theta_j}{\|(\Xi_*^j)_2\|} \cdot I_{s_j-1} - \frac{(\Xi_*^j)_1}{\|(\Xi_*^j)_2\|} \cdot \overline{(\Xi_*^j)_2}\,\overline{(\Xi_*^j)_2}^T \\ &\quad = \frac{\sigma_j(\tan^2\theta_j + 1)}{(\tau + \sigma_j)\tan\theta_j} \cdot I_{s_j-1} + \frac{\tau - \sigma_j \tan^2\theta_j}{(\tau + \sigma_j)\tan\theta_j} \cdot \overline{(\Gamma_*^j)_2}\,\overline{(\Gamma_*^j)_2}^T,\end{aligned}$$

and $\partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x_*, \mu_*, \Gamma_*))$ has the following form

$$\frac{1}{\tan\theta_j + \cot\theta_j} \times \begin{pmatrix} \tan\theta_j & \overline{(\Gamma_*^j)_2}^T \\ \overline{(\Gamma_*^j)_2} & \frac{\sigma_j(\tan^2\theta_j + 1)}{(\tau + \sigma_j)\tan\theta_j} \cdot I_{s_j-1} + \frac{\tau - \sigma_j \tan^2\theta_j}{(\tau + \sigma_j)\tan\theta_j} \cdot \overline{(\Gamma_*^j)_2}\,\overline{(\Gamma_*^j)_2}^T \end{pmatrix}.$$

(b) If $\Xi_*^j \in \mathrm{int}\,\mathcal{L}_{\theta_j'}$, then $\Xi_*^j \in \mathrm{int}\,\mathcal{L}_{\theta_j}^*$. From the relation (3), we obtain $(-\Xi_*^j) \in \mathrm{int}\,\mathcal{L}_{\theta_j}^\circ$. Similar to part(a), it follows that $\Pi_{\mathcal{L}_{\theta_j}}(-\Xi_*^j) = \tau G^j(x_*) = 0_{s_j}$ and $\Pi_{\mathcal{L}_{\theta_j}^\circ}(-\Xi_*^j) = -\Xi_*^j = -\Gamma_*^j \in \mathrm{int}\,\mathcal{L}_{\theta_j}^\circ$, which are equivalent to $G^j(x_*) = 0_{s_j}$, $\Gamma_*^j \in \mathrm{int}\,\mathcal{L}_{\theta_j}^*$, i.e. $j \in \mathcal{Z}_G(x_*) \cap \mathcal{I}_{\Gamma_*}$. Hence, we conclude that $\partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x_*, \mu_*, \Gamma_*)) = \partial_B \Pi_{\mathcal{L}_{\theta_j'}}(\Xi_*^j) = \partial_B \Pi_{\mathcal{L}_{\theta_j}^*}(\Xi_*^j) = \partial_B(\Xi_*^j) = I_{s_j}$.

(c) If $\Xi_*^j \in \mathrm{bd}\,\mathcal{L}_{\theta_j'} \setminus \{0_{s_j}\}$, then $\Xi_*^j \in \mathrm{bd}\,\mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\}$. Similarly, we also obtain $(-\Xi_*^j) \in \mathrm{bd}\,\mathcal{L}_{\theta_j}^\circ \setminus \{0_{s_j}\}$, $\Pi_{\mathcal{L}_{\theta_j}}(-\Xi_*^j) = \tau G^j(x_*) = 0_{s_j}$, $\Pi_{\mathcal{L}_{\theta_j}^\circ}(-\Xi_*^j) = -\Xi_*^j = -\Gamma_*^j \in \mathrm{bd}\,\mathcal{L}_{\theta_j}^\circ \setminus \{0_{s_j}\}$, which are equivalent to $G^j(x_*) = 0_{s_j}$, $\Gamma_*^j \in \mathrm{bd}\,\mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\}$, i.e. $j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$. Then, using Lemma 2.3(c) and the fact $\overline{(\Xi_*^j)_2} = \overline{(\Gamma_*^j)_2}$ yields that $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*))$ can be set one of the following matrices

$$I_{s_j}, \quad \begin{pmatrix} \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} & \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma^j_*)_2}^T \\ \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma^j_*)_2} & I_{s_j-1} - \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma^j_*)_2}\,\overline{(\Gamma^j_*)_2}^T \end{pmatrix}.$$

(d) If $\Xi^j_* \in \operatorname{int}(-\mathcal{L}^*_{\theta'_j})$, then $(-\Xi^j_*) \in \operatorname{int}\mathcal{L}^*_{\frac{\pi}{2}-\theta_j}$. In addition, there have $(-\Xi^j_*) \in \operatorname{int}\mathcal{L}_{\theta_j}$, $\Pi_{\mathcal{L}_{\theta_j}}(-\Xi^j_*) = -\Xi^j_* = \tau G^j(x_*)$, $\Pi_{\mathcal{L}^\circ_{\theta_j}}(-\Xi^j_*) = -\Gamma^j_* = 0_{s_j}$, which are equivalent to $G^j(x_*) \in \operatorname{int}\mathcal{L}_{\theta_j}, \Gamma^j_* = 0_{s_j}$, i.e. $j \in \mathcal{I}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$. Hence, we obtain $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*)) = \partial_B \Pi_{\mathcal{L}_{\theta'_j}}(\Xi^j_*) = \partial_B \Pi_{-\mathcal{L}^\circ_{\theta_j}}(\Xi^j_*) = \partial_B(-\Pi_{\mathcal{L}^\circ_{\theta_j}}(-\Xi^j_*)) = 0_{s_j \times s_j}$.

(e) If $\Xi^j_* \in \operatorname{bd}(-\mathcal{L}^*_{\theta'_j}) \setminus \{0_{s_j}\}$, then $(-\Xi^j_*) \in \operatorname{bd}\mathcal{L}_{\theta_j}$. Similar to part(a), we also achieve $\Pi_{\mathcal{L}_{\theta_j}}(-\Xi^j_*) = -\Xi^j_* = \tau G^j(x_*)$, $\Pi_{\mathcal{L}^\circ_{\theta_j}}(-\Xi^j_*) = -\Gamma^j_* = 0_{s_j}$, which are equivalent to $G^j(x_*) \in \operatorname{bd}\mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}, \Gamma^j_* = 0_{s_j}$, i.e. $j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$. Moreover, Lemma 2.3(e) implies that $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*))$ can be set one of the following matrices

$$0_{s_j \times s_j}, \quad \begin{pmatrix} \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} & -\dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{G^j_2(x_*)}^T \\ -\dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{G^j_2(x_*)} & \dfrac{\cot\theta_j}{\tan\theta_j + \cot\theta_j} \cdot \overline{G^j_2(x_*)}\,\overline{G^j_2(x_*)}^T \end{pmatrix},$$

where the last equality is due to the below relations:

$$\begin{aligned} \Xi^j_* = -\tau G^j(x_*) \in \operatorname{bd}(-\mathcal{L}_{\theta_j}) \setminus \{0_{s_j}\} \;\Rightarrow\; &\overline{(\Xi^j_*)_2} \\ = \frac{(\Xi^j_*)_2}{\|(\Xi^j_*)_2\|} = -\frac{G^j_2(x_*)}{\|G^j_2(x_*)\|} = &-\overline{G^j_2(x_*)}. \end{aligned}$$

(f) If $\Xi^j_* = 0_{s_j}$, we have $G^j(x_*) = 0_{s_j}, \Gamma^j_* = 0_{s_j}$, i.e. $j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$. In this case, applying Lemma 2.3(f) yields that $\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_*, \mu_*, \Gamma_*))$ can be set one of the following matrices

$$0_{s_j \times s_j}, \quad I_{s_j}, \quad \frac{1}{\tan\theta_j + \cot\theta_j}\begin{pmatrix} \tan\theta_j & w_j^T \\ w_j & a_{\theta_j} \cdot (I_{s_j-1} - ww^T) + \cot\theta_j \cdot ww^T \end{pmatrix}.$$

In summary, the conclusion holds as verified in each case. ■

To make the statement of next theorem neat, we further introduce some notations:

$$a_j := \frac{1}{\sqrt{1+\tan^2\theta_j}}\begin{pmatrix}\tan\theta_j \\ (\Gamma_*^j)_2\end{pmatrix}, \quad j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}, \tag{20}$$

$$b_j := \frac{1}{\sqrt{1+\cot^2\theta_j}}\begin{pmatrix}\cot\theta_j \\ -(\Gamma_*^j)_2\end{pmatrix}, \quad j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}, \tag{21}$$

$$c_j := \frac{1}{\sqrt{1+\tan^2\theta_j}}\begin{pmatrix}\tan\theta_j \\ -G_2^j(x_*)\end{pmatrix}, \quad j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}, \tag{22}$$

$$D_j := \frac{\sigma_j}{\tau+\sigma_j}\begin{pmatrix}0 & 0_{s_j-1}^T \\ 0_{s_j-1} & I_{s_j-1} - \frac{(\Gamma_*^j)_2}{\|(\Gamma_*^j)_2\|}\frac{(\Gamma_*^j)_2}{\|(\Gamma_*^j)_2\|}^T\end{pmatrix}, \quad j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}, \tag{23}$$

$$\hat{W}_*^j := \begin{cases} 0_{s_j\times s_j} & \text{if } W_*^j = 0_{s_j\times s_j}, \\ I_{s_j} & \text{otherwise,}\end{cases} \quad j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}. \tag{24}$$

In light of Lemma 3.4, the term $\mathcal{A}_\tau$ in (13) possesses an explicit expression.

Theorem 3.5 (Explicit expression of $\mathcal{A}_\tau$ in (13)): *Suppose that x_* is a stationary point of (P) and $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. Then, we have*

$$\begin{aligned}
&\mathcal{A}_\tau(\mu_*, \Gamma_*, W_*) \\
&\quad = \nabla_{xx}^2\mathfrak{L}(x_*, \mu_*, \Gamma_*) + \tau \mathcal{J}h(x_*)^T\mathcal{J}h(x_*) \\
&\qquad + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau\mathcal{J}G^j(x_*)^T(a_ja_j^T + D_j)\mathcal{J}G^j(x_*) \\
&\qquad + \sum_{j\in\mathcal{Z}_G(x_*)\cap\mathcal{I}_{\Gamma_*}} \tau\mathcal{J}G^j(x_*)^T\mathcal{J}G^j(x_*) \\
&\qquad + \sum_{j\in\mathcal{Z}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau\mathcal{J}G^j(x_*)^T W_*^j\mathcal{J}G^j(x_*) \\
&\qquad + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{Z}_{\Gamma_*}} \tau\mathcal{J}G^j(x_*)^T c_j\hat{W}_*^j c_j^T\mathcal{J}G^j(x_*) \\
&\qquad + \sum_{j\in\mathcal{Z}_G(x_*)\cap\mathcal{Z}_{\Gamma_*}} \tau\mathcal{J}G^j(x_*)^T W_*^j\mathcal{J}G^j(x_*),
\end{aligned} \tag{25}$$

where $W_* := (W_*^1, \ldots, W_*^J) \in \prod_{j\in\mathcal{I}_C}\mathbb{R}^{s_j\times s_j}$, $W_*^j \in \partial_B\Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x_*, \mu_*, \Gamma_*))$, $\theta_j' := \frac{\pi}{2} - \theta_j$, $j \in \mathcal{I}_C$ *and* $a_j, c_j, D_j, \hat{W}_*^j$ *are defined as in* (20), (22), (23) *and* (24), *respectively.*

Proof: From the definition of $\mathcal{A}_\tau(x_*, \mu_*, \Gamma_*)$ in (13) and Lemma 3.1, we have

$$
\begin{aligned}
&\mathcal{A}_\tau(\mu_*, \Gamma_*, W_*) \\
&\quad = \nabla^2_{xx}\mathfrak{L}(x_*, \mu_*, \Gamma_*) + \tau \mathcal{J}h(x_*)^T \mathcal{J}h(x_*) \\
&\qquad + \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*) \\
&\qquad + \sum_{j \in \mathcal{Z}_G(x_*) \cap \mathcal{I}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*) \\
&\qquad + \sum_{j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*) \\
&\qquad + \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*) \\
&\qquad + \sum_{j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*), \\
&\qquad + \sum_{j \in \mathcal{I}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*),
\end{aligned} \tag{26}
$$

where $W_*^j \in \partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x_*, \mu_*, \Gamma_*))$, $\theta_j' := \frac{\pi}{2} - \theta_j$, $j \in \mathcal{I}_C$. In addition, by using Theorem 3.4, we discuss the following cases.

(a) If $j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, then

$$
\begin{aligned}
W_*^j &= \frac{1}{\tan^2\theta_j + 1}\begin{pmatrix} \tan^2\theta_j & \tan\theta_j \overline{(\Gamma_*^j)_2}^T \\ \tan\theta_j \overline{(\Gamma_*^j)_2} & \overline{(\Gamma_*^j)_2}\,\overline{(\Gamma_*^j)_2}^T \end{pmatrix} \\
&\quad + \frac{\sigma_j}{\tau + \sigma_j}\begin{pmatrix} 0 & 0_{s_j-1}^T \\ 0_{s_j-1} & I_{s_j-1} - \overline{(\Gamma_*^j)_2}\,\overline{(\Gamma_*^j)_2}^T \end{pmatrix} \\
&= a_j a_j^T + D_j,
\end{aligned}
$$

where a_j and D_j are defined as in (20) and (23). Hence, we conclude that

$$
\begin{aligned}
&\sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*) \\
&\quad = \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T (a_j a_j^T + D_j) \mathcal{J}G^j(x_*).
\end{aligned}
$$

(b) If $j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$, then

$$W_*^j = \begin{cases} 0_{s_j \times s_j} & \text{if } W_*^j = 0_{s_j \times s_j}, \\ \dfrac{1}{\tan^2\theta_j + 1}\begin{pmatrix} \tan^2\theta_j & -\tan\theta_j \cdot \overline{G_2^j(x_*)}^T \\ -\tan\theta_j \cdot \overline{G_2^j(x_*)} & \overline{G_2^j(x_*)}\,\overline{G_2^j(x_*)}^T \end{pmatrix} & \text{otherwise.} \end{cases}$$
$$= c_j \hat{W}_*^j c_j^T,$$

where c_j and $\hat{W}_*^j$ are defined as in (22) and (24). Hence, we obtain

$$\sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{Z}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*) = \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{Z}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T c_j \hat{W}_*^j c_j^T \mathcal{J}G^j(x_*).$$

(c) Similar to the above cases, we have

$$\sum_{j\in\mathcal{Z}_G(x_*)\cap\mathcal{I}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*) = \sum_{j\in\mathcal{Z}_G(x_*)\cap\mathcal{I}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T \mathcal{J}G^j(x_*),$$
$$\sum_{j\in\mathcal{I}_G(x_*)\cap\mathcal{Z}_{\Gamma_*}} \tau \mathcal{J}G^j(x_*)^T W_*^j \mathcal{J}G^j(x_*) = 0_{n\times n}.$$

From these discussions and Equation (26), the desired result (25) is true. ∎

In order to analyse the convergence of augmented Lagrangian method for solving nonlinear circular conic programs ($\mathrm{P}_{\mathrm{NCCP}}$), we need the following two assumptions.

(A1) *The constraint nondegeneracy condition (CNC) holds at $x_* \in \mathbb{R}^n$:*

$$\begin{pmatrix} \mathcal{J}h(x_*) \\ \mathcal{J}G^1(x_*) \\ \vdots \\ \mathcal{J}G^J(x_*) \end{pmatrix} \mathbb{R}^n + \begin{pmatrix} \{0_l\} \\ \mathrm{lin}\{\mathcal{T}_{\mathcal{L}_{\theta_1}}(G^1(x_*))\} \\ \vdots \\ \mathrm{lin}\{\mathcal{T}_{\mathcal{L}_{\theta_J}}(G^J(x_*))\} \end{pmatrix} = \begin{pmatrix} \mathbb{R}^l \\ \mathbb{R}^{s_1} \\ \vdots \\ \mathbb{R}^{s_J} \end{pmatrix},$$

where $\mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(x_*))$ is the tangent cone of $\mathcal{L}_{\theta_j}$ at $G^j(x_*)$ and $\mathrm{lin}\{\mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(x_*))\}$ denotes the linearity space of $\mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(x_*))$, which is the largest linear space contained in $\mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(x_*))$. Applying [6, Theorem 2.3], the tangent cone of $\mathcal{L}_\theta$ at $p \in \mathbb{R}^n$ is

$$\mathcal{T}_{\mathcal{L}_\theta}(p) := \begin{cases} \mathbb{R}^s & \text{if } p \in \mathrm{int}\,\mathcal{L}_\theta, \\ \mathcal{L}_\theta & \text{if } p = 0, \\ \{(h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{s-1} : h_2^T p_2 - h_1 p_1 \tan^2\theta \le 0\} & \text{if } p \in \mathrm{bd}\,\mathcal{L}_\theta \setminus \{0_s\}. \end{cases} \tag{27}$$

In addition, according to [6, Theorem 3.1], the constraint nondegeneracy condition is equivalent to the vectors

$$\begin{array}{ll} \mathcal{J}h^i(x_*)^T, & i \in \mathcal{I}_{\mathcal{E}}, \\ \mathcal{J}G^j(x_*)^T \mathcal{H}_j G^j(x_*), & j \in \mathcal{B}_G(x_*), \\ \mathcal{J}G^j(x_*)^T e_{s_j}^k,\ j \in \mathcal{Z}_G(x_*), & k = 1, 2, \ldots, s_j \end{array}$$

are linearly independent, where $e_{s_j}^k$ denotes the kth column vectors of I_{s_j}. At the same time, the Lagrange multiplier set $\Lambda(x_*)$ is a singleton.

(A2) The strong second-order sufficient condition (SSOSC) holds, i.e. let x_* be a stationary point of ($\mathrm{P_{NCCP}}$) and $(\mu_*, \Gamma_*) \in \Lambda(x_*)$, there exists a positive scalar η_0 such that

$$d^T \left(\nabla_{xx}^2 \mathfrak{L}(x_*, \mu_*, \Gamma_*) - \sum_{j \in \mathcal{I}_{\mathcal{C}}} A^j(x_*, \mu_*, \Gamma_*^j) \right) d \geq \eta_0 \|d\|^2, \quad \forall d \in \operatorname{aff}(\mathcal{C}(x_*)),$$

where $\operatorname{aff}(\mathcal{C}(x_*))$ denotes the affine hull of $\mathcal{C}(x_*)$, the critical cone at x_*, whose definition is given by

$$\mathcal{C}(x_*) := \left\{ d \in \mathbb{R}^n : \begin{array}{l} \mathcal{J}h(x_*)d = 0_l,\ \nabla f(x_*)^T d = 0, \\ \mathcal{J}G^j(x_*)d \in \mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(x_*)),\ j \in \mathcal{I}_{\mathcal{C}} \end{array} \right\}.$$

The matrix $A^j(x_*, \mu_*, \Gamma_*^j) \in \mathbb{R}^{n \times n}$ is the sigma term of ($\mathrm{P_{NCCP}}$), whose expression is (see Lemma 3.4 in [6] for details)

$$A^j(x_*, \mu_*, \Gamma_*^j) = \begin{cases} \dfrac{(\Gamma_*^j)_1}{G_1^j(x_*)} \cot^2 \theta_j \mathcal{J}G^j(x_*)^T \mathcal{H}_j \mathcal{J}G^j(x_*), & \text{if } j \in \mathcal{B}_G(x_*), \\ 0_{n \times n}, & \text{otherwise.} \end{cases} \tag{28}$$

With Lemma 3.1, the next lemma provides the explicit expressions of $\mathcal{C}(x_*)$ and $\operatorname{aff}(\mathcal{C}(x_*))$.

Lemma 3.6 (Explicit expressions of critical cone $\mathcal{C}(x_*)$ and its affine hull $\operatorname{aff}(\mathcal{C}(x_*))$): *Let x_* be a stationary point of* ($\mathrm{P_{NCCP}}$) *and $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. Then, we have*

$$\mathcal{C}(x_*) = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} (\mathcal{J}h(x_*)d)_i = 0, & i \in \mathcal{I}_{\mathcal{E}} \\ \mathcal{J}G^j(x_*)d \in \mathbb{R}^{s_j}, & j \in \mathcal{I}_G(x_*) \cap \mathcal{Z}_{\Gamma_*} \\ \mathcal{J}G^j(x_*)d \in \mathcal{L}_{\theta_j}, & j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*} \\ (-G_1^j(x_*) \tan^2 \theta_j, G_2^j(x_*)^T) & \\ \quad \mathcal{J}G^j(x_*)d \leq 0, & j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*} \\ \mathcal{J}G^j(x_*)d = 0_{s_j}, & j \in \mathcal{Z}_G(x_*) \cap \mathcal{I}_{\Gamma_*} \\ (\mathcal{J}G^j(x_*)d)^T a_j = 0, & j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*} \\ \mathcal{J}G^j(x_*)d \in \mathbb{R}_+(b_j), & j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*} \end{array} \right\},$$

where $a_j, b_j \in \mathbb{R}^n$ are respectively defined as in (20)–(21) and $\mathbb{R}_+(b_j) := \{\sigma b_j \mid \sigma \geq 0\}$. In addition, the associated affine hull aff$(\mathcal{C}(x_*))$ has the following form:

$$\operatorname{aff}(\mathcal{C}(x_*)) = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} (\mathcal{J}h(x_*)d)_i = 0, & i \in \mathcal{I}_{\mathcal{E}} \\ \mathcal{J}G^j(x_*)d = 0_{s_j}, & j \in \mathcal{Z}_G(x_*) \cap \mathcal{I}_{\Gamma_*} \\ (\mathcal{J}G^j(x_*)d)^T a_j = 0, & j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*} \\ B_j \mathcal{J}G^j(x_*)d = 0_{s_j}, & j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*} \end{array} \right\} \tag{29}$$

with $B_j := I_{s_j} - b_j b_j^T$, $j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$.

Proof: From [6, Theorem 3.3] and (14), the critical cone $\mathcal{C}(x_*)$ can be recast as follows:

$$\mathcal{C}(x_*) = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} (\mathcal{J}h(x_*)d)_i = 0, & i \in \mathcal{I}_{\mathcal{E}}, \\ \mathcal{J}G^j(x_*)d \in \mathcal{T}_{\mathcal{L}_{\theta_j}}(G^j(x_*)), & j \in \mathcal{Z}_{\Gamma_*}, \\ \mathcal{J}G^j(x_*)d = 0_{s_j}, & j \in \mathcal{Z}_G(x_*) \cap \mathcal{I}_{\Gamma_*} \\ (\mathcal{J}G^j(x_*)d)^T \Gamma_*^j = 0, & j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*} \\ \mathcal{J}G^j(x_*)d \in \mathbb{R}_+(\mathcal{H}_j^{-1}\Gamma_*^j), & j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}. \end{array} \right\}. \tag{30}$$

Using the relation (27), there have three possibilities:

(a) If $j \in \mathcal{I}_G(x_*)$, then $\mathcal{J}G^j(x_*)d \in \mathbb{R}^{s_j}$;
(b) If $j \in \mathcal{Z}_G(x_*)$, then $JG^j(x_*)d \in \mathcal{L}_{\theta_j}$;
(c) If $j \in \mathcal{B}_G(x_*)$, then $(-G_1^j(x_*)\tan^2\theta_j, G_2^j(x_*)^T)\mathcal{J}G^j(x_*)d \leq 0$.

Hence, the second line in the right-hand side of (30) is equivalent to

$$\begin{cases} \mathcal{J}G^j(x_*)d \in \mathbb{R}^{s_j}, & j \in \mathcal{I}_G(x_*) \cap \mathcal{Z}_{\Gamma_*} \\ \mathcal{J}G^j(x_*)d \in \mathcal{L}_{\theta_j}, & j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*} \\ (-G_1^j(x_*)\tan^2\theta_j, G_2^j(x_*)^T)\mathcal{J}G^j(x_*)d \leq 0, & j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*} \end{cases}$$

Furthermore, if $j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, we know

$$\begin{aligned} & \Gamma_*^j \in \operatorname{bd}\mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\} \\ \Longleftrightarrow\quad & \Gamma_*^j = \begin{pmatrix} (\Gamma_*^j)_1 \\ (\Gamma_*^j)_2 \end{pmatrix} = \|(\Gamma_*^j)_2\| \cdot \begin{pmatrix} \tan\theta_j \\ \frac{(\Gamma_*^j)_2}{\|(\Gamma_*^j)_2\|} \end{pmatrix} = \|(\Gamma_*^j)_2\| \cdot \begin{pmatrix} \tan\theta_j \\ \overline{(\Gamma_*^j)_2} \end{pmatrix} \\ \Longleftrightarrow\quad & \Gamma_*^j = \sqrt{1+\tan^2\theta_j}\|(\Gamma_*^j)_2\| \cdot a_j, \end{aligned}$$

where a_j is defined as in (20). This together with the fact $(\mathcal{J}G^j(x_*)d)^T\Gamma_*^j = 0$ show that

$$(\mathcal{J}G^j(x_*)d)^T a_j = 0, \quad j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}.$$

On the other hand, if $j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, we obtain $(\Gamma_*^j)_2 \neq 0_{s_j-1}$ and

$$\mathcal{H}_j^{-1}\Gamma_*^j = \begin{pmatrix} \cot^2\theta_j & 0_{s_j-1}^T \\ 0_{s_j-1} & -I_{s_j-1} \end{pmatrix} \cdot \|(\Gamma_*^j)_2\| \cdot \begin{pmatrix} \frac{\tan\theta_j}{(\Gamma_*^j)_2} \end{pmatrix}$$
$$= \sqrt{1+\cot^2\theta_j}\|(\Gamma_*^j)_2\| \cdot b_j,$$

where b_j is defined as in (21), which further implies that

$$\mathcal{J}G^j(x_*)d \in \mathbb{R}_+(\mathcal{H}_j^{-1}\Gamma_*^j) \Longleftrightarrow \mathcal{J}G^j(x_*)d \in \mathbb{R}_+(b_j). \tag{31}$$

Focusing on the above relations, its affine version of (31) can be rewritten as

$$B_j\mathcal{J}G^j(x_*)d = 0_{s_j}, \quad B_j := I_{s_j} - b_jb_j^T, \quad j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}. \tag{32}$$

Then, it follows from the relations (30)–(32) that the conclusion is true. ■

The next two lemmas connect some relations on the matrix $A^j(x_*, \mu_*, \Gamma_*^j)$ in the expression of sigma term (28) and the Hessian of the augmented Lagrangian function.

Lemma 3.7: *Suppose that x_* is a stationary point of* ($\mathrm{P_{NCCP}}$) *and* $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. *Then, the following relation holds for any given $d \in \mathrm{aff}(\mathcal{C}(x_*))$, i.e.*

$$\sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau(y^j)^T D_j y^j + \sum_{j\in\mathcal{I}_C} d^T A^j(x_*, \mu_*, \Gamma_*^j)d = \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \frac{\sigma_j^2}{\tau+\sigma_j}(y^j)^T\mathcal{H}_j y^j \tag{33}$$

with $y^j := \mathcal{J}G^j(x_*)d$, $j \in \mathcal{I}_C$ and D_j is defined as in (23).

Proof: From the definition of $A^j(x_*, \mu_*, \Gamma_*^j)$ in (28), we have

$$\begin{aligned}
&\sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau(y^j)^T D_j y^j + \sum_{j\in\mathcal{I}_C} d^T A^j(x_*, \mu_*, \Gamma_*^j)d \\
&= \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau(y^j)^T D_j y^j + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} d^T A^j(x_*, \mu_*, \Gamma_*^j)d \\
&\quad + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{Z}_{\Gamma_*}} d^T A^j(x_*, \mu_*, \Gamma_*^j)d \\
&= \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \frac{\tau\sigma_j}{\tau+\sigma_j}\left(\|y_2^j\|^2 - \|\overline{(\Gamma_*^j)_2}^T y_2^j\|^2\right) \\
&\quad + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \frac{(\Gamma_*^j)_1}{G_1^j(x_*)}\cot^2\theta_j(y^j)^T\mathcal{H}_j y^j.
\end{aligned} \tag{34}$$

In addition, if $j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, then $\Gamma_*^j \in \operatorname{bd} \mathcal{L}_{\theta_j}^* \setminus \{0_{s_j}\}$, $G^j(x_*) \in \operatorname{bd} \mathcal{L}_{\theta_j} \setminus \{0_{s_j}\}$ and there exists $\sigma_j > 0$ such that $\Gamma_*^j = \sigma_j(\mathcal{H}_j G^j(x_*))$, due to Lemma 3.1(d). By a simple calculation, we obtain $(\Gamma_*^j)_1 / G_1^j(x_*) = \sigma_j \tan^2 \theta_j$. Moreover, it follows from the relation (29) that $(y^j)^T a_j$ equals to 0 for any given $d \in \operatorname{aff}(\mathcal{C}(x_*))$ and $j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, which implies that $\|\overline{(\Gamma_*^j)_2}^T y_2^j\|^2 = \tan^2 \theta_j (y_1^j)^2$. Combining the above two equations with (34), we have

$$
\begin{aligned}
&\sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \tau (y^j)^T D_j y^j + \sum_{j \in \mathcal{I}_C} d^T A^j(x_*, \mu_*, \Gamma_*^j) d \\
&\quad = \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \frac{\tau \sigma_j}{\tau + \sigma_j} \left(\|y_2^j\|^2 - \tan^2 \theta_j (y_1^j)^2 \right) + \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \sigma_j (y^j)^T \mathcal{H}_j y^j. \\
&\quad = \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \frac{\sigma_j^2}{\tau + \sigma_j} (y^j)^T \mathcal{H}_j y^j,
\end{aligned}
$$

where the last equality is due to the definition of $\mathcal{H}_j$ in (11). ■

Lemma 3.8: *Suppose that x_* is a stationary point of* ($\mathrm{P_{NCCP}}$) *and* $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. *If Assumptions* (A1) *and* (A2) *hold at* (x_*, μ_*, Γ_*), *then there exist positive scalars τ_0, η_0 such that for any $\tau \geq \tau_0$,*

$$
d^T \nabla_{xx}^2 \mathfrak{L}(x_*, \mu_*, \Gamma_*) d + \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \tau (y^j)^T D_j y^j \geq \frac{\eta_0}{2} \|d\|^2, \quad \forall d \in \mathit{aff}(\mathcal{C}(x_*))
$$

with $y^j := \mathcal{J} G^j(x_*) d$, $j \in \mathcal{I}_C$ and D_j is defined as in (23).

Proof: From Assumption (A2), there exists a positive scalar η_0 such that

$$
d^T \left(\nabla_{xx}^2 \mathfrak{L}(x_*, \mu_*, \Gamma_*) - \sum_{j \in \mathcal{I}_C} A^j(x_*, \mu_*, \Gamma_*^j) \right) d \geq \eta_0 \|d\|^2, \quad \forall d \in \operatorname{aff}(\mathcal{C}(x_*)). \tag{35}
$$

In addition, let $\tau_0 > 0$ be satisfied that

$$
\sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \left\{ \frac{\sigma_j^2}{\tau + \sigma_j} \max_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \|\mathcal{J} G^j(x_*)\|^2 \max_{j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} \{\tan^2 \theta_j, 1\} \right\} \leq \frac{\eta_0}{2}.
$$

From the definition of $\mathcal{H}_j$ in (11) and the Equation (33), for any given $d \in \mathrm{aff}(\mathcal{C}(x_*))$, we have

$$\begin{aligned}
&\sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau (y^j)^T D_j y^j + \sum_{j=1}^{J} d^T A^j(x_*, \mu_*, \Gamma_*^j) d \\
&\geq - \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \\
&\quad \times \left\{ \frac{\sigma_j^2}{\tau + \sigma_j} \max_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \|\mathcal{J}G^j(x_*)\|^2 \max_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \{\tan^2\theta_j, 1\} \right\} \cdot \|d\|^2 \\
&\geq -\frac{\eta_0}{2}\|d\|^2. \qquad (36)
\end{aligned}$$

Hence, the conclusion is satisfied at (x_*, μ_*, Γ_*) by using the inequalities (35)–(36). ■

In light of Lemma 3.8 and the famous Debreu theorem [25], we can further establish an important inequality which will be used in the sequel.

Lemma 3.9: *Suppose that x_* is a stationary point of* ($\mathrm{P_{NCCP}}$) *and* $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. *If Assumptions* (A1) *and* (A2) *hold at* (x_*, μ_*, Γ_*), *then there exist* $\tau_1 \geq \tau_0$ *and* $\eta_1 \in (0, \eta_0/2)$ *such that for any* $\tau \geq \tau_1$ *and* $d \in \mathbb{R}^n$,

$$d^T \nabla^2_{xx}\mathfrak{L}(x_*, \mu_*, \Gamma_*) d + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau (y^j)^T D_j y^j + \tau_1 \mathcal{T}(x_*, y) \geq \eta_1 \|d\|^2,$$

where D_j *is defined as in* (23), $y := (y^1, y^2, \ldots, y^J)$, $y^j := \mathcal{J}G^j(x_*)d$, $j \in \mathcal{I}_C$, $\mathcal{T}(x_*, y)$ *is given by*

$$\begin{aligned}
\mathcal{T}(x_*, y) &:= \|\mathcal{J}h(x_*)d\|^2 + \sum_{j\in\mathcal{Z}_G(x_*)\cap\mathcal{I}_{\Gamma_*}} \|y^j\|^2 + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \left((y^j)^T a_j\right)^2 \\
&\quad + \sum_{j\in\mathcal{Z}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \|B_j y^j\|^2 \qquad (37)
\end{aligned}$$

and a_j *is defined as in* (20), $B_j := I_{s_j} - b_j b_j^T$, $j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$ *with* b_j *defined as in* (21).

Remark 3.1: Let us denote

$$\tilde{A} = \begin{pmatrix} -\mathcal{J}h_i(x_*), & i \in \mathcal{I}_{\mathcal{E}} \\ a_j^T \mathcal{J}G^j(x_*), & j \in \mathcal{BB} \\ \mathcal{J}G^j(x_*), & j \in \mathcal{ZI} \\ \mathcal{J}G^j(x_*), & j \in \mathcal{ZB} \\ c_j^T \mathcal{J}G^j(x_*), & j \in \mathcal{BZ} \\ \mathcal{J}G^j(x_*), & j \in \mathcal{ZZ} \end{pmatrix},$$

$$\tilde{E} = \begin{pmatrix} I_l & & & & & \\ & I_{|\mathcal{BB}|} & & & & \\ & & I_{|\mathcal{ZI}|} & & & \\ & & & W_*^{|\mathcal{ZB}|} & & \\ & & & & \hat{W}_*^{|\mathcal{BZ}|} & \\ & & & & & W_*^{|\mathcal{ZZ}|} \end{pmatrix}, \tag{38}$$

where

$$\begin{aligned} &\mathcal{BB} := \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}, \quad \mathcal{ZI} := \mathcal{Z}_G(x_*) \cap \mathcal{I}_{\Gamma_*}, \\ &\mathcal{ZB} := \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}, \quad \mathcal{BZ} := \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}, \\ &\mathcal{ZZ} := \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}, \quad W_*^{|\mathcal{ZB}|} = \mathrm{diag}(W_*^j), j \in \mathcal{ZB}, \\ &\hat{W}_*^{|\mathcal{BZ}|} := \mathrm{diag}(\hat{W}_*^j), j \in \mathcal{BZ}, \quad W_*^{|\mathcal{ZZ}|} = \mathrm{diag}(W_*^j), j \in \mathcal{ZZ}, \\ &W_*^j \in \partial_B \Pi_{\mathcal{L}_{\theta_j'}}(F_\tau^j(x_*, \mu_*, \Gamma_*)), \theta_j' := \frac{\pi}{2} - \theta_j, j \in \mathcal{I}_{\mathcal{C}} \end{aligned}$$

and $a_j, c_j, D_j, \hat{W}_*^j$ are defined as in (20), (22), (23) and (24), respectively. Setting

$$\mathcal{B}_{\tau_1,\tau}(\mu_*, \Gamma_*, W_*) := \nabla_{xx}^2 \mathcal{L}(x_*, \mu_*, \Gamma_*) + \tau_1 \tilde{A}^T \tilde{E} \tilde{A} + \sum_{j \in \mathcal{BB}} \tau \mathcal{J}G^j(x_*)^T D_j \mathcal{J}G^j(x_*) \tag{39}$$

From the definitions of $\mathcal{A}_\tau(\mu_*, \Gamma_*, W_*)$ and $\mathcal{B}_{\tau_1,\tau}(\mu_*, \Gamma_*, W_*)$ in (25) and (39), we obtain

$$d^T \mathcal{A}_\tau(\mu_*, \Gamma_*, W_*) d = d^T \mathcal{B}_{\tau_1,\tau}(\mu_*, \Gamma_*, W_*) d + (\tau - \tau_1) d^T \tilde{A}^T \tilde{E} \tilde{A} d. \tag{40}$$

Suppose that Assumption (A1) holds at (x_*, μ_*, Γ_*), we can further obtain the following property of the operator $\tilde{A}$.

Lemma 3.10: *Suppose that Assumption* (A1) *holds at* (x_*, μ_*, Γ_*), *the operator* $\tilde{A}$ *defined as in* (38) *is of full row rank.*

Proof: If $j \in \mathcal{B}_G(x_*)$, then $\|G_2^j(x_*)\| = G_1^j(x_*) \tan\theta_j$. In this case, we have

$$\mathcal{H}_j G^j(x_*) = \begin{pmatrix} \tan^2\theta_j G_1^j(x_*) \\ -G_2^j(x_*) \end{pmatrix} = \begin{pmatrix} \tan\theta_j \|G_2^j(x_*)\| \\ -G_2^j(x_*) \end{pmatrix}$$
$$= \|G_2^j(x_*)\| \begin{pmatrix} \tan\theta_j \\ \hline -G_2^j(x_*) \end{pmatrix}.$$

If $j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$, then it follows from the definition of c_j in (22) that

$$c_j = \frac{1}{\sqrt{1+\tan^2\theta_j}} \begin{pmatrix} \tan\theta_j \\ \hline -G_2^j(x_*) \end{pmatrix} \quad \text{is parallel to} \quad \mathcal{H}_j G^j(x_*).$$

If $j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, then $\|(\Gamma_*^j)_2\| = (\Gamma_*^j)_1 \cot\theta_j$. It follows from the definition of a_j in (20) and Lemma 2.1 that

$$a_j = \frac{1}{\sqrt{1+\tan^2\theta_j}} \begin{pmatrix} \tan\theta_j \\ \hline (\Gamma_*^j)_2 \end{pmatrix} \quad \text{is parallel to} \quad \|(\Gamma_*^j)_2\| \begin{pmatrix} \tan\theta_j \\ \hline (\Gamma_*^j)_2 \end{pmatrix}$$
$$= \Gamma_*^j = \sigma_j(\mathcal{H}_j G^j(x_*)).$$

In summary, we conclude that

(a) If $j \in \mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, then $a_j^T \mathcal{J} G^j(x_*)$ is parallel to $G^j(x_*)\mathcal{H}_j \mathcal{J} G^j(x_*)$.
(b) If $j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$, then $c_j^T \mathcal{J} G^j(x_*)$ is also parallel to $G^j(x_*)\mathcal{H}_j \mathcal{J} G^j(x_*)$.

Then, together with the fact that Assumption (A1) holds at (x_*, μ_*, Γ_*), it follows from [6, Theorem 3.1] and the structure of the operator $\tilde{A}$ defined in (38) that $\tilde{A}$ is of full row rank. ∎

Remark 3.2: By virtue of Lemma 3.10, we further assume that the operator $\tilde{A}$ has the following singular value decomposition:

$$\tilde{A} = \tilde{U} \begin{pmatrix} \tilde{\Sigma} & 0_{q\times(n-q)} \end{pmatrix} \tilde{V}^T, \ \tilde{U} \in \mathbb{R}^{q\times q}, \ \tilde{V} \in \mathbb{R}^{n\times n}, \tag{41}$$

where $q := l + |\mathcal{Z}_G(x_*)| + |\mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}| + |\mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}|$ and $\tilde{\Sigma} :=$ diag $(\sigma_i(\tilde{A}))_{1\le i\le q}$ with $\sigma_1(\tilde{A}) \ge \sigma_2(\tilde{A}) \ge \cdots \ge \sigma_q(\tilde{A})$.

Combining Lemmas 3.8 and 3.9, we next establish the second-order growth condition of $\mathcal{A}_\tau(\mu_*, \Gamma_*, W_*)$ in (25) and then discuss some properties on the corresponding inverse of $\mathcal{A}_\tau(\mu_*, \Gamma_*, W_*)$.

Theorem 3.11 (second-order growth condition of $\mathcal{A}_\tau(\mu_*, \Gamma_*, W_*)$): *Suppose that x_* is a stationary point of $(\mathrm{P}_{\mathrm{NCCP}})$ and $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. If Assumptions (A1) and (A2) hold at (x_*, μ_*, Γ_*), then there exist $\tau_1 \geq \tau_0$ and $\eta_1 \in (0, \eta_0/2)$ such that for any $\tau \geq \tau_1$ and $d \in \mathbb{R}^n$,*

$$d^T \mathcal{A}_\tau(\mu_*, \Gamma_*, W_*) d \geq d^T \mathcal{B}_{\tau_1,\tau}(\mu_*, \Gamma_*, W_*) d \geq \eta_1 \|d\|^2, \tag{42}$$

where $\mathcal{A}_\tau(\mu_*, \Gamma_*, W_*)$ and $\mathcal{B}_{\tau_1,\tau}(\mu_*, \Gamma_*, W_*)$ are defined as in (25) and (39), respectively. Consequently, every element in $\Pi_x \partial_B (\nabla_x \mathfrak{L}_\tau)(x_*, \mu_*, \Gamma_*)$ is positive definite.

Proof: It follows from the definition of $\mathcal{T}(x_*, y)$ in (37) that

$$\begin{aligned}
& d^T \tilde{A}^T \tilde{E} \tilde{A} d \\
&\quad = \mathcal{T}(x_*, y) + \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} (y^j)^T c_j \hat{W}_*^j c_j^T y^j \\
&\qquad + \sum_{j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} (y^j)^T W_*^j y^j + \sum_{j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} (y^j)^T (W_*^j - B_j^T B_j) y^j \\
&\quad \geq \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} (y^j)^T c_j \hat{W}_*^j c_j^T y^j + \sum_{j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} (y^j)^T W_*^j y^j \\
&\qquad + \sum_{j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}} (y^j)^T (W_*^j - B_j^T B_j) y^j. \\
&\quad \geq 0,
\end{aligned} \tag{43}$$

where the last inequality follows from the following discussions.

(a) If $j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$, from the definition of $\hat{W}_*^j$ in (24), then we have

$$\sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} \tau (y^j)^T c_j \hat{W}_*^j c_j^T y^j = \sum_{j \in \mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} \tau (c_j^T y^j)^T \hat{W}_*^j c_j^T y^j \geq 0.$$

(b) If $j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}$, from Lemma 3.4(f), then the matrix W_*^j can be set as one of the following matrices

$$\begin{aligned}
& 0_{s_j \times s_j}, \quad I_{s_j}, \quad \frac{1}{\tan\theta_j + \cot\theta_j} \\
& \qquad \times \begin{pmatrix} \tan\theta_j & w_j^T \\ w_j & a_{\theta_j} \cdot (I_{s_j - 1} - w_j w_j^T) + \cot\theta_j \cdot w_j w_j^T \end{pmatrix}.
\end{aligned}$$

Notice that the last matrix is also positive semidefinite, due to the Schur's complementarity theorem, since for any given $v \in \mathbb{R}^{s_j-1}$,

$$\begin{aligned} v^T \left(a_{\theta_j} \cdot (I_{s_j-1} - w_j w_j^T) + \cot\theta_j \cdot w_j w_j^T - \cot\theta_j \cdot w_j w_j^T \right) v \\ = a_{\theta_j} \cdot v^T (I_{s_j-1} - w_j w_j^T) v \geq 0, \end{aligned}$$

where the last inequality is due to the fact that w is any given unitary vector in $\mathbb{R}^{s_j-1}$. Therefore, we obtain $\sum_{j \in \mathcal{Z}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}} \tau (y^j)^T W_*^j y^j \geq 0$.

(c) If $j \in \mathcal{Z}_G(x_*) \cap \mathcal{B}_{\Gamma_*}$, from Lemma 3.4(c), then the matrix W_*^j can be set as one of the following matrices:

$$I_{s_j}, \quad \begin{pmatrix} \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} & \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2}^T \\ \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2} & I_{s_j-1} - \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2}\,\overline{(\Gamma_*^j)_2}^T \end{pmatrix}.$$

From the definition of B_j and b_j in (21), we have

$$\begin{aligned} B_j^T B_j &= (I_{s_j} - b_j b_j^T)(I_{s_j} - b_j b_j^T) \\ &= I_{s_j} - b_j b_j^T - b_j b_j^T + b_j b_j^T b_j b_j^T = I_{s_j} - b_j b_j^T = B_j \end{aligned}$$

and the matrix $W_*^j - B_j^T B_j$ is equal to either $b_j b_j^T$ or $0_{s_j \times s_j}$, since

$$\begin{aligned} &\begin{pmatrix} \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} & \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2}^T \\ \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2} & I_{s_j-1} - \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2}\,\overline{(\Gamma_*^j)_2}^T \end{pmatrix} + b_j b_j^T - I_{s_j} \\ &= \begin{pmatrix} \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} & \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2}^T \\ \dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2} & I_{s_j-1} - \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2}\,\overline{(\Gamma_*^j)_2}^T \end{pmatrix} \\ &\quad + \begin{pmatrix} \dfrac{\cot\theta_j}{\tan\theta_j + \cot\theta_j} & -\dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2}^T \\ -\dfrac{1}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2} & \dfrac{\tan\theta_j}{\tan\theta_j + \cot\theta_j} \cdot \overline{(\Gamma_*^j)_2}\,\overline{(\Gamma_*^j)_2}^T \end{pmatrix} - I_{s_j} \\ &= 0_{s_j \times s_j}. \end{aligned}$$

Thus, $\sum_{j\in\mathcal{Z}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau(y^j)^T(W_*^j - B_j^T B_j)y^j \ge 0$. Consequently, for any $\tau \ge \tau_1$, from (40) and (43), we obtain

$$
\begin{aligned}
& d^T\mathcal{A}_\tau(\mu_*,\Gamma_*,W_*)d \\
&\quad \ge d^T\mathcal{B}_{\tau_1,\tau}(\mu_*,\Gamma_*,W_*)d \\
&\quad = d^T\nabla^2_{xx}\mathcal{L}(x_*,\mu_*,\Gamma_*)d + \tau_1 d^T\tilde{A}^T\tilde{E}\tilde{A}d + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau(y^j)^T D_j y^j \\
&\quad \ge d^T\nabla^2_{xx}\mathcal{L}(x_*,\mu_*,\Gamma_*)d + \tau_1\mathcal{T}(x_*,y) + \sum_{j\in\mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}} \tau(y^j)^T D_j y^j \\
&\quad \ge \eta_1\|d\|^2,
\end{aligned}
$$

where the above last inequity is due to Lemma 3.9. Combining the above inequality and Lemma 3.3, we deduce that every element in $\Pi_x\partial_B(\nabla_x\mathfrak{L}_\tau)(x_*,\mu_*,\Gamma_*)$ is positive definite. ■

Remark 3.3: It follows from (23) that for $j \in \mathcal{B}_G(x_*)\cap\mathcal{B}_{\Gamma_*}$ we obtain

$$
D_j = \frac{\sigma_j}{\tau+\sigma_j}\begin{pmatrix} 0 & 0^T_{s_j-1} \\ 0_{s_j-1} & I_{s_j-1} - \frac{(\Gamma_*^j)_2}{\|(\Gamma_*^j)_2\|}\frac{(\Gamma_*^j)_2^T}{\|(\Gamma_*^j)_2\|} \end{pmatrix}, \quad \lim_{\tau\to\infty}\frac{\tau\sigma_j}{\sigma_j+\tau} = \sigma_j.
$$

Under the Assumptions (A1) and (A2), there exists a positive scalar η_2 such that

$$
d^T\mathcal{B}_{\tau_1,\tau}(\mu_*,\Gamma_*,W_*)d \le \eta_2\|d\|^2, \quad \forall\tau \ge \tau_1 \text{ and } d\in\mathbb{R}^n. \tag{44}
$$

After these preparations, under the singular decomposition (41), the inequality (44) with the Sherman-Morrison-Woodbury formula [26, Sect. 2.1], we further establish some properties on the inverse of the operator $\mathcal{A}_\tau(\mu_*,\Gamma_*,W_*)$ defined as in (25). The proof is similar to [20, Lemma 10], we omit it here.

Lemma 3.12 (Properties on the inverse of $\mathcal{A}_\tau(\mu_*,\Gamma_*,W_*)$): *Suppose that x_* is a stationary point of* ($\mathrm{P_{NCCP}}$) *and* $(\mu_*,\Gamma_*)\in\Lambda(x_*)$. *If Assumptions* (A1) *and* (A2) *hold at* (x_*,μ_*,Γ_*), *then for any* $\tau\ge\tau_1$ *we have*

$$
\begin{aligned}
(\mathcal{A}_\tau(\mu_*,\Gamma_*,W_*))^{-1} &\preceq \tilde{V} \\
&\times\begin{pmatrix} \tilde{\Sigma}^{-1}\tilde{U}^T\left(\hat{\sigma}_1\eta_1 I_q + (\tau-\tau_1)\tilde{E}\right)^{-1}\tilde{U}\tilde{\Sigma}^{-1} & 0_{q\times(n-q)} \\ 0_{(n-q)\times q} & \hat{\sigma}_1^{-1}\eta_1^{-1}I_{n-q} \end{pmatrix}\tilde{V}^T, \\
(\mathcal{A}_\tau(\mu_*,\Gamma_*,W_*))^{-1} &\succeq \tilde{V} \\
&\times\begin{pmatrix} \tilde{\Sigma}^{-1}\tilde{U}^T\left(\hat{\sigma}_2\eta_2 I_q + (\tau-\tau_1)\tilde{E}\right)^{-1}\tilde{U}\tilde{\Sigma}^{-1} & 0_{q\times(n-q)} \\ 0_{(n-q)\times q} & \hat{\sigma}_2^{-1}\eta_2^{-1}I_{n-q} \end{pmatrix}\tilde{V}^T
\end{aligned}
$$

and

$$\|\mathcal{A}_\tau(\mu_*, \Gamma_*, W_*)^{-1}\tilde{A}^T\tilde{E}w\| \le \sqrt{2}\left(\hat{\sigma}_2 + (\hat{\sigma}_1\eta_2)^{-2}(\hat{\sigma}_2\eta_2)^{-2}\right)(\tau - \tau_1)^{-1}\|w\|,$$

where $q := l + |\mathcal{Z}_G(x_*)| + |\mathcal{B}_G(x_*) \cap \mathcal{B}_{\Gamma_*}| + |\mathcal{B}_G(x_*) \cap \mathcal{Z}_{\Gamma_*}|$, $\hat{\sigma}_1 := \min\{1, \sigma_1^{-2}(\tilde{A})\}$, $\hat{\sigma}_2 := \max\{1, \sigma_q^{-2}(\tilde{A})\}$ *and* η_1, η_2 *are respectively defined as in* (42) *and* (44).

4. Local convergence analysis

This section is devoted to discussing the local convergence of augmented Lagrangian method for nonlinear circular cone programs ($\mathrm{P_{NCCP}}$).

From Theorem 3.11 and Clarke's implicit function theorem [27], there exist an open neighbourhood $\mathbb{B}_{\delta^0_\Lambda}(\mu_*, \Gamma_*)$ and a locally Lipschitz continuous function $x_\tau(\cdot)$ defined on $\mathbb{B}_{\delta^0_\Lambda}(\mu_*, \Gamma_*)$ such that for any given $(\mu, \Gamma) \in \mathbb{B}_{\delta^0_\Lambda}(\mu_*, \Gamma_*)$,

$$\nabla_x \mathfrak{L}_\tau(x_\tau(\mu, \Gamma), \mu, \Gamma) = 0_n. \tag{45}$$

In addition, since $\Pi_{\mathcal{L}_{\theta'_j}}(\cdot)$, ($\theta'_j = \frac{\pi}{2} - \theta_j$, $j \in \mathcal{I}_C$) is strongly semismooth everywhere (see Lemma 2.4), $x_\tau(\cdot)$ is semismooth (strongly semismooth if $\nabla^2 f$, $\nabla^2 h$, $\nabla^2 G$ are locally Lipschitz continuous and $\Pi_{\mathcal{L}_{\theta'_j}}(\cdot)$ is strongly semismooth everywhere) at any point in $B_{\delta^0_\Lambda}(\mu_*, \Gamma_*)$ and there exist two positive number $\epsilon^1, \delta^1_\Lambda \in (0, \delta^0_\Lambda]$ such that for any $x \in \mathbb{B}_{\epsilon^1}(x_*)$ and $(\mu, \Gamma) \in \mathbb{B}_{\delta^1_\Lambda}(\mu_*, \Gamma_*)$, it follows from Lemma 3.3 and Theorem 3.11 that every element in $\Pi_x\partial_B(\nabla_x\mathfrak{L}_\tau)(x, \mu, \Gamma)$ is positive definite and $x_\tau(\mu, \Gamma)$ is the unique minimizer of $\mathfrak{L}_\tau(x, \mu, \Gamma)$ over $\mathbb{B}_{\epsilon^1}(x_*)$, i.e.

$$x_\tau(\mu, \Gamma) = \operatorname{argmin}\left\{\mathfrak{L}_\tau(x, \mu, \Gamma) \mid x \in \mathbb{B}_{\epsilon^1}(x_*)\right\}. \tag{46}$$

Let $\vartheta_\tau : \mathbb{R}^l \times \prod_{j=1}^J \mathbb{R}^{s_j} \to \mathbb{R}$ be the optimal function of problem (46), i.e.

$$\vartheta_\tau(\mu, \Gamma) := \min_{x \in \mathbb{B}_{\epsilon^1}(x_*)} \mathfrak{L}_\tau(x, \mu, \Gamma), \quad (\mu, \Gamma) \in \mathbb{R}^l \times \prod_{j=1}^J \mathbb{R}^{s_j}.$$

Notice that $\mathfrak{L}_\tau(x, \cdot, \cdot)$ is a concave function for each fixed $x \in \mathbb{R}^n$, from the above definition of ϑ_τ, then $\vartheta_\tau(\mu, \Gamma)$ is also a concave function satisfying the following relation

$$\begin{aligned}\vartheta_\tau(\mu, \Gamma) &= \mathfrak{L}_\tau(x_\tau(\mu, \Gamma), \mu, \Gamma)\\ &= f(x_\tau(\mu, \Gamma)) + \mu^T h(x_\tau(\mu, \Gamma)) + \frac{\tau}{2}\|h(x_\tau(\mu, \Gamma))\|^2\\ &\quad + \frac{1}{2\tau}\sum_{j=1}^J \left(\|\Pi_{\mathcal{L}_{\theta'_j}}\left(\Gamma^j - \tau G^j(x_\tau(\mu, \Gamma))\right)\|^2 - \|\Gamma^j\|^2\right),\end{aligned} \tag{47}$$

where $\theta'_j = \frac{\pi}{2} - \theta_j, j \in \mathcal{I}_C$.

For any given $(\mu,\Gamma)\in\mathbb{B}_{\delta_\Lambda^1}(\mu_*,\Gamma_*)$, we denote

$$\begin{aligned}
&\mu_\tau(\mu,\Gamma):=\mu+\tau h(x_\tau(\mu,\Gamma)),\\
&\Gamma_\tau(\mu,\Gamma)):=\left(\Gamma_\tau^1(\mu,\Gamma),\quad \Gamma_\tau^2(\mu,\Gamma),\ldots,\Gamma_\tau^J(\mu,\Gamma)\right),\\
&\Gamma_\tau^j(\mu,\Gamma):=\Pi_{\mathcal{L}_{\theta_j'}}\left(\Gamma^j-\tau G^j(x_\tau(\mu,\Gamma))\right),\quad j\in\mathcal{C}.
\end{aligned}\tag{48}$$

From the relations (9)–(10) and (48), we obtain

$$\begin{aligned}
&F_\tau^j(x_\tau(\mu,\Gamma),\mu,\Gamma)=\Gamma^j-\tau G^j(x_\tau(\mu,\Gamma)),\\
&\Gamma_\tau^j(\mu,\Gamma)=\Pi_{\mathcal{L}_{\theta_j'}}\left(F_\tau^j(x_\tau(\mu,\Gamma),\mu,\Gamma)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\nabla_x\mathfrak{L}(x_\tau(\mu,\Gamma),\mu_\tau(\mu,\Gamma),\Gamma_\tau(\mu,\Gamma))\\
&\quad=\nabla f(x_\tau(\mu,\Gamma))+\nabla h(x_\tau(\mu,\Gamma))\mu_\tau(\mu,\Gamma)-\sum_{j\in\mathcal{I}_\mathcal{C}}\nabla G^j(x_\tau(\mu,\Gamma))\Gamma_\tau^j(\mu,\Gamma)\\
&\quad=\nabla_x\mathfrak{L}_\tau(x_\tau(\mu,\Gamma),\mu,\Gamma)\\
&\quad=0_n,
\end{aligned}\tag{49}$$

where the last equality follows from (45). In addition, using the definition of $\vartheta_\tau(\mu,\Gamma)$ in (47), we know

$$\begin{aligned}
\nabla_\mu\vartheta_\tau(\mu,\Gamma)&=\nabla_\mu x_\tau(\mu,\Gamma)\nabla_x\mathfrak{L}_\tau(x_\tau(\mu,\Gamma),\mu,\Gamma)\\
&\qquad+\nabla_\mu\mathfrak{L}_\tau(x_\tau(\mu,\Gamma),\mu,\Gamma)=h(x_\tau(\mu,\Gamma)),\\
\nabla_{\Gamma^j}\vartheta_\tau(\mu,\Gamma)&=\nabla_{\Gamma^j}x_\tau(\mu,\Gamma)\nabla_x\mathfrak{L}_\tau(x_\tau(\mu,\Gamma),\mu,\Gamma)+\nabla_{\Gamma^j}\mathfrak{L}_\tau(x_\tau(\mu,\Gamma),\mu,\Gamma)\\
&=-\tau^{-1}\Gamma^j+\tau^{-1}\Gamma_\tau^j(\mu,\Gamma).
\end{aligned}$$

Due to the above discussion, the following lemmas characterize the gradient of $\vartheta_\tau(\mu,\Gamma)$ and its Hessian on $\mathcal{D}_{\nabla\vartheta_\tau}$ (the set of all Fréchet differentiable point of $\nabla\vartheta_\tau(\cdot)$ in $\mathbb{B}_{\delta_\Lambda^1}(\mu_*,\Gamma_*)$), respectively.

Lemma 4.1: *Suppose that x_* is a stationary point of* ($\mathrm{P_{NCCP}}$) *and* $(\mu_*,\Gamma_*)\in\Lambda(x_*)$. *If Assumptions* (A1) *and* (A2) *hold at* (x_*,μ_*,Γ_*), *then for any $\tau\geq\tau_1$, the concave function $\vartheta_\tau(\cdot)$ is continuously differentiable on $\mathbb{B}_{\delta_\Lambda^1}(\mu_*,\Gamma_*)$ with*

$$\nabla\vartheta_\tau(\mu,\Gamma)=\begin{pmatrix}h(x_\tau(\mu,\Gamma))\\-\tau^{-1}\Gamma+\tau^{-1}\Gamma_\tau(\mu,\Gamma)\end{pmatrix},\quad(\mu,\Gamma)\in\mathbb{B}_{\delta_\Lambda^1}(\mu_*,\Gamma_*),$$

where $\Gamma_\tau(\mu,\Gamma))$ is defined as in (48). *Moreover, $\nabla\vartheta_\tau(\mu,\Gamma)$ is semismooth at any point in $\mathbb{B}_{\delta_\Lambda^1}(\mu_*,\Gamma_*)$ and it becomes a strongly semismooth function at any point* *in* $\mathbb{B}_{\delta^1_\Lambda}(\mu_*, \Gamma_*)$ *if* $\nabla^2 f$, $\nabla^2 h$, $\nabla^2 G$ *are locally Lipschitz continuous and* $\Pi_{\mathcal{L}_{\theta'_j}}(\cdot)$ *is strongly semismooth everywhere, where* $\theta'_j = \frac{\pi}{2} - \theta_j$, $j \in \mathcal{I}_C$.

Lemma 4.2: *Suppose that* x_* *is a stationary point of* ($\mathrm{P}_{\mathrm{NCCP}}$) *and* $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. *If Assumptions* (A1) *and* (A2) *hold at* (x_*, μ_*, Γ_*), *then for any* $\tau \geq \tau_1$, $(\mu, \Gamma) \in \mathcal{D}_{\nabla\vartheta_\tau}$ *and* $(\Delta\mu, \Delta\Gamma) \in \mathbb{R}^l \times \prod_{j=1}^J \mathbb{R}^{s_j}$,

$$\begin{aligned}
&\nabla^2\vartheta_\tau(\mu, \Gamma)(\Delta\mu, \Delta\Gamma) \\
&\in \left\{ \begin{pmatrix} \mathcal{J}h(x_\tau(\mu,\Gamma)) \\ -\hat{W}^1 \mathcal{J}G^1(x_\tau(\mu,\Gamma)) \\ -\hat{W}^2 \mathcal{J}G^2(x_\tau(\mu,\Gamma)) \\ \vdots \\ -\hat{W}^J \mathcal{J}G^J(x_\tau(\mu,\Gamma)) \end{pmatrix} \mathcal{A}_\tau(\mu, \Gamma, \hat{W})^{-1} \mathcal{T}((\mu,\Gamma); (\Delta\mu, \Delta\Gamma)) \right. \\
&\qquad \left. + \begin{pmatrix} 0_l \\ -\tau^{-1}\Delta\Gamma^1 + \tau^{-1}\hat{W}^1(\Delta\Gamma^1) \\ -\tau^{-1}\Delta\Gamma^2 + \tau^{-1}\hat{W}^2(\Delta\Gamma^2) \\ \vdots \\ -\tau^{-1}\Delta\Gamma^J + \tau^{-1}\hat{W}^J(\Delta\Gamma^J) \end{pmatrix} : \; \hat{W}^j \in \partial_B \Gamma^j_\tau(\mu, \Gamma) \right\} \\
&:= \mathcal{V}_\tau(\mu, \Gamma)(\Delta\mu, \Delta\Gamma),
\end{aligned} \tag{50}$$

where

$$\mathcal{T}((\mu,\Gamma); (\Delta\mu, \Delta\Gamma)) := -\nabla h(x_\tau(\mu,\Gamma))(\Delta\mu) + \sum_{j\in\mathcal{I}_C} \nabla G^j(x_\tau(\mu,\Gamma)) \hat{W}^j(\Delta\Gamma^j). \tag{51}$$

Proof: Let $(\Delta\mu, \Delta\Gamma)$ be any given point in $\mathbb{R}^l \times \prod_{j=1}^J \mathbb{R}^{s_j}$ and $(\mu, \Gamma) \in \mathcal{D}_{\nabla\vartheta_\tau}$, then for any $\tau \geq \tau_1$, the hessian $\nabla^2\vartheta_\tau(\mu, \Gamma)(\Delta\mu, \Delta\Gamma)$ has the following expression as

$$\begin{pmatrix} \mathcal{J}h(x_\tau(\mu,\Gamma)) x'_\tau((\mu,\Gamma); (\Delta\mu, \Delta\Gamma)) \\ -\tau^{-1}\Delta\Gamma^1 + \tau^{-1}\Pi'_{\mathcal{L}_{\theta'_1}} \\ \left(\Gamma^1 - \tau\mathcal{J}G^1(x_\tau(\mu,\Gamma)); \Delta\Gamma^1 - \tau\mathcal{J}G^1(x_\tau(\mu,\Gamma)) x'_\tau((\mu,\Gamma); (\Delta\mu, \Delta\Gamma))\right) \\ -\tau^{-1}\Delta\Gamma^2 + \tau^{-1}\Pi'_{\mathcal{L}_{\theta'_2}} \\ \left(\Gamma^2 - \tau\mathcal{J}G^2(x_\tau(\mu,\Gamma)); \Delta\Gamma^2 - \tau\mathcal{J}G^2(x_\tau(\mu,\Gamma)) x'_\tau((\mu,\Gamma); (\Delta\mu, \Delta\Gamma))\right) \\ \vdots \\ -\tau^{-1}\Delta\Gamma^J + \tau^{-1}\Pi'_{\mathcal{L}_{\theta'_J}} \\ \left(\Gamma^J - \tau\mathcal{J}G^J(x_\tau(\mu,\Gamma)); \Delta\Gamma^J - \tau\mathcal{J}G^J(x_\tau(\mu,\Gamma)) x'_\tau((\mu,\Gamma); (\Delta\mu, \Delta\Gamma))\right) \end{pmatrix},$$

where $\theta'_j = \frac{\pi}{2} - \theta_j$, $j \in \mathcal{I}_C$. In light of Lemma 2.4, the mapping $\Pi_{\mathcal{L}_{\theta'_j}}(\cdot)$ is semismooth everywhere. Hence, there exists an element $\hat{W}^j \in \partial_B \Gamma^j_\tau(\mu, \Gamma) =$

$\partial_B \Pi_{\mathcal{L}_{\theta'_j}}(F^j_\tau(x_\tau(\mu,\Gamma),\mu,\Gamma)) = \partial_B \Pi_{\mathcal{L}_{\theta'_j}}(\Gamma^j - \tau G^j(x_\tau(\mu,\Gamma)))$ such that

$$\begin{aligned}
&\Pi'_{\mathcal{L}_{\theta'_j}}\left(\Gamma^j - \tau \mathcal{J}G^j(x_\tau(\mu,\Gamma)); \Delta\Gamma^j - \tau \mathcal{J}G^j(x_\tau(\mu,\Gamma))x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))\right)\\
&\quad = \hat{W}^j\left(\Delta\Gamma^j - \tau \mathcal{J}G^j(x_\tau(\mu,\Gamma))x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))\right). \qquad (52)
\end{aligned}$$

Let $(\mu,\Gamma) \in \mathbb{B}_{\delta^1_\Lambda}(\mu_*,\Gamma_*)$, from the relations (49) and (52), we have

$$\begin{aligned}
0_{n\times n} = {} & \nabla^2_{xx}\mathfrak{L}(x_\tau(\mu,\Gamma),\mu_\tau(\mu,\Gamma),\Gamma_\tau(\mu,\Gamma))\,x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))\\
& + \nabla h(x_\tau(\mu,\Gamma))(\Delta\mu)\\
& + \tau\nabla h(x_\tau(\mu,\Gamma))\mathcal{J}h(x_\tau(\mu,\Gamma))x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))\\
& - \sum_{j\in\mathcal{I}_C}\nabla G^j(x_\tau(\mu,\Gamma))\hat{W}^j\\
& \times\left(\Delta\Gamma^j - \tau\mathcal{J}G^j(x_\tau(\mu,\Gamma))x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))\right)\\
= {} & \mathcal{A}_\tau(\mu,\Gamma,\hat{W})x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma)) + \nabla h(x_\tau(\mu,\Gamma))(\Delta\mu)\\
& - \sum_{j\in\mathcal{I}_C}\nabla G^j(x_\tau(\mu,\Gamma))\hat{W}^j(\Delta\Gamma^j),
\end{aligned}$$

where $\mathcal{A}_\tau(\mu,\Gamma,\hat{W})$ is defined as in (13). The above relation yields that

$$x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma)) = \mathcal{A}_\tau(\mu,\Gamma,\hat{W})^{-1}\mathcal{T}((\mu,\Gamma);(\Delta\mu,\Delta\Gamma)), \qquad (53)$$

where $\mathcal{T}((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))$ is defined as in (51). Hence, we have

$$\begin{aligned}
&\mathcal{J}h(x_\tau(\mu,\Gamma))x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))\\
&\quad = \mathcal{J}h(x_\tau(\mu,\Gamma))\mathcal{A}_\tau(\mu,\Gamma,\hat{W})^{-1}\mathcal{T}((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))
\end{aligned}$$

and

$$\begin{aligned}
&\tau^{-1}\Pi'_{\mathcal{L}_{\theta'_j}}\left(\Gamma^j - \tau\mathcal{J}G^j(x_\tau(\mu,\Gamma)); \Delta\Gamma^j - \tau\mathcal{J}G^j(x_\tau(\mu,\Gamma))x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))\right)\\
&\quad = \tau^{-1}\hat{W}^j\left(\Delta\Gamma^j - \tau\mathcal{J}G^j(x_\tau(\mu,\Gamma))x'_\tau((\mu,\Gamma);(\Delta\mu,\Delta\Gamma))\right)\\
&\quad = -\hat{W}^j\mathcal{J}G^j(x_\tau(\mu,\Gamma))\mathcal{A}_\tau(\mu,\Gamma,\hat{W})^{-1}\mathcal{T}((\mu,\Gamma);(\Delta\mu,\Delta\Gamma)) + \tau^{-1}\hat{W}^j(\Delta\Gamma^j),
\end{aligned}$$

where the last equality follows from the relation (53). Then, by applying these relations and the definition of $\nabla^2\vartheta_\tau(\mu,\Gamma)(\Delta\mu,\Delta\Gamma)$, the inclusion (50) holds. ■

Together with the continuity of $x_\tau(\cdot)$ and the semismooth property of $\nabla\vartheta_\tau(\cdot)$ in Lemma 4.1, we establish the following lemma.

Lemma 4.3: *Suppose that x_* is a stationary point of* ($\mathrm{P}_{\mathrm{NCCP}}$) *and* $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. *If Assumptions* (A1) *and* (A2) *hold at* (x_*, μ_*, Γ_*), *then for any* $\tau \geq \tau_1$ *and* $(\Delta\mu, \Delta\Gamma) \in \mathbb{R}^l \times \prod_{j=1}^{J} \mathbb{R}^{s_j}$, *there has*

$$\partial_B(\nabla\vartheta_\tau)(\mu_*, \Gamma_*)(\Delta\mu, \Delta\Gamma) \subseteq \mathcal{V}_\tau(\mu_*, \Gamma_*)(\Delta\mu, \Delta\Gamma),$$

where $\mathcal{V}_\tau(\mu, \Gamma)(\Delta\mu, \Delta\Gamma)$ *is defined as in* (50).

Finally, we first build up two important inequalities and then the linear convergence rate of augmented Lagrangian method for nonlinear circular conic programs can be achieved. Since the details of proof are similar to the procedure in [19, Proposition 3.1] and [20, Theorem 1], we omit them here.

Theorem 4.4: *Suppose that x_* is a stationary point of* ($\mathrm{P}_{\mathrm{NCCP}}$) *and* $(\mu_*, \Gamma_*) \in \Lambda(x_*)$. *If Assumptions* (A1) *and* (A2) *hold at* (x_*, μ_*, Γ_*), *then for any* $\tau \geq \tau_1$ *and* $(\Delta\mu, \Delta\Gamma) \in \mathbb{R}^l \times \prod_{j=1}^{J} \mathbb{R}^{s_j}$, *there exist positive scalars ρ and κ such that*

$$\|x'_\tau\ ((\mu_*, \Gamma_*); (\Delta\mu, \Delta\Gamma))\,\|^2 \leq \rho^2\tau^{-2}\|(\Delta\mu, \Delta\Gamma)\|^2 \tag{54}$$

and

$$\begin{aligned} &\left|-\left\langle V(\Delta\mu, \Delta\Gamma) + \tau^{-1}(\Delta\mu, \Delta\Gamma), (\Delta\mu, \Delta\Gamma)\right\rangle\right| \\ &\quad \leq \kappa\tau^{-2}\|(\Delta\mu, \Delta\Gamma)\|^2,\ \forall V(\Delta\mu, \Delta\Gamma) \in \mathcal{V}_\tau(\mu_*, \Gamma_*)(\Delta\mu, \Delta\Gamma). \end{aligned} \tag{55}$$

Furthermore, let η_1, η_2, ρ, κ be respectively defined as in (42), (44), (54), (55) *and τ_1 be given in Lemma* 3.9. *Define $\rho_1 := 2\rho$ and $\rho_2 = 4\kappa$. Then, for any $\tau \geq \tau_1$, there exist two positive scalars ϵ and δ (both depending on τ) such that for any $(\mu, \Gamma) \in \mathbb{B}_\delta(\mu_*, \Gamma_*)$, the problem*

$$\begin{aligned} \min \quad & \mathfrak{L}_\tau(x, \mu, \Gamma) \\ \text{s.t.} \quad & x \in \mathbb{B}_\epsilon(x_*) \end{aligned}$$

has a unique solution $x_\tau(\mu, \Gamma)$. The function $x_\tau(\cdot, \cdot)$ is semismooth at any point in $\mathbb{B}_\delta(\mu_*, \Gamma_*)$. Moreover, for any $(\mu, \Gamma) \in \mathbb{B}_\delta(\mu_*, \Gamma_*)$, we have

$$\begin{aligned} &\|x_\tau(\mu, \Gamma) - x^*\| \leq \rho_1\tau^{-1}\|(\mu, \Gamma) - (\mu^*, \Gamma^*)\|, \\ &\|(\mu_\tau(\mu, \Gamma), \Gamma_\tau(\mu, \Gamma)) - (\mu_*, \Gamma_*)\| \leq \rho_2\tau^{-1}\|(\mu, \Gamma) - (\mu^*, \Gamma^*)\|, \end{aligned}$$

where $\mu_\tau(\mu, \Gamma) := \mu + \tau h(x_\tau(\mu, \Gamma))$ and $\Gamma_\tau(\mu, \Gamma)$ is defined as

$$\begin{aligned} &\Gamma_\tau(\mu, \Gamma) \\ &\quad := \left(\Pi_{\mathcal{L}_{\theta'_1}}(\Gamma^1 - \tau G^1(x_\tau(\mu, \Gamma))), \ldots, \Pi_{\mathcal{L}_{\theta'_J}}(\Gamma^J - \tau G^J(x_\tau(\mu, \Gamma)))\right) \in \prod_{j \in \mathcal{I}_\mathcal{C}} \mathbb{R}^{s_j} \end{aligned}$$

with $\theta'_j := \frac{\pi}{2} - \theta_j$, $j \in I_\mathcal{C}$.

5. Concluding remarks

In this paper, we consider an augmented Lagrangian method for minimizing a class of nonlinear circular conic programs. In particular, a linear local convergence rate of generated iterations is established without requiring strict complementarity condition, whereas it is usually needed in the analysis of interior point method. On the other hand, due to the non-self duality property of circular cone under the standard inner product, our results show that ALM can deal with non-self dual conic programs based on Bouligand-subdifferential of the projection operator onto the given cone, the explicit expressions of the associated critical cone and its affine hull under the appropriate constraint qualifications, which enriches the algorithm design of non-interior-point method framework for handling non-self dual conic programs. In light of these results, it is hopeful to further discuss some stability issues such as the robust isolated calmness and the Jacobian uniqueness condition of the Karush-Kuhn-Tucker solution mapping of the given circular conic programs and ALM for solving another types of non-self dual conic programs (such as p-order conic programs, power conic programs and exponential conic programs). We leave it for our sequential researches.

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