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Research Article

Lipschitz Continuity of the Solution Mapping of Symmetric Cone Complementarity Problems

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This paper investigates the Lipschitz continuity of the solution mapping of symmetric cone (linear or nonlinear) complementarity problems (SCLCP or SCCP, resp.) over Euclidean Jordan algebras. We show that if the transformation has uniform Cartesian P-property, then the solution mapping of the SCCP is Lipschitz continuous. Moreover, we establish that the monotonicity of mapping and the Lipschitz continuity of solutions of the SCLCP imply ultra P-property, which is a concept recently developed for linear transformations on Euclidean Jordan algebra. For a Lyapunov transformation, we prove that the strong monotonicity property, the ultra P-property, the Cartesian P-property, and the Lipschitz continuity of the solutions are all equivalent to each other.

1. Introduction

Let $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ (we use $\mathbb{V}$ for short in subsequent content) be a Euclidean Jordan algebra and $\mathcal{K}$ be the symmetric cone in $\mathbb{V}$. Given a continuous transformation $F : \mathbb{V} \to \mathbb{V}$ and $q \in \mathbb{V}$, the symmetric cone complementarity problem denoted by $\mathrm{SCCP}(F, \mathcal{K}, q)$ is to find a vector $x \in \mathbb{V}$ such that

$$x \in \mathcal{K}, \quad F(x) + q \in \mathcal{K}, \quad \langle x, F(x) + q \rangle = 0. \tag{1.1}$$

When F reduces to a linear transformation L, the above problem is called the symmetric cone linear complementarity problem and is denoted by $\mathrm{SCLCP}(L, \mathcal{K}, q)$, that is, the symmetric cone linear complementarity problem is to find a vector $x \in \mathbb{V}$ such that

$$x \in \mathcal{K}, \quad L(x) + q \in \mathcal{K}, \quad \langle x, L(x) + q \rangle = 0. \tag{1.2}$$

These classes of symmetric cone complementarity problems provide a unified framework for the linear or nonlinear complementarity problems (LCP or NCP, resp.) over the nonnegative orthant cone in $\mathbb{R}^n$, that is, $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$ (see [1–4]), the second-order cone (linear or nonlinear) complementarity problems (SOCLCP or SOCCP, resp.), that is, $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathcal{K}^n$ (see [5–8]), and the semidefinite (linear or nonlinear) complementarity problems (SDLCP or SDCP, resp.), that is, $\mathbb{V} = S^n$ and $\mathcal{K} = S^n_+$ (see [9–12]). It is also known that the complementarity problem is special case of variational inequality problem which has a wide range of applications, see [3, 9].

One of the important issues in complementarity problems is to characterize the Lipschitz continuity of its solutions (or called the Lipschitz continuity of solution mapping) with respect to q. For $q \in \mathbb{V}$, let $\phi_F(q)$ be the set of all solutions to SCCP$(F, \mathcal{K}, q)$. Then, we intend to know under what conditions the multivalued solution mapping $\phi_F : q \mapsto \phi_F(q)$ of SCCP$(F, \mathcal{K}, q)$ is Lipschitz continuous. In other words, under what conditions, there will exist $\kappa > 0$ such that

$$\phi_F(q_1) \subseteq \phi_F(q_2) + \kappa \|q_1 - q_2\| B \tag{1.3}$$

for all $q_1, q_2 \in \mathbb{V}$ satisfying $\phi_F(q_1) \neq \emptyset$ and $\phi_F(q_2) \neq \emptyset$, where B is the closed unit ball in $\mathbb{V}$. That is, if $x_1 \in \phi_F(q_1)$ there exists $x_2 \in \phi_F(q_2)$ such that

$$\|x_1 - x_2\| \leq \kappa \|q_1 - q_2\|. \tag{1.4}$$

Note that the Lipschitz constant κ depends only on the continuous transformation F. Below is a brief history regarding this issue. For LCP(M, q), it is well known that the Lipschitz continuity of the solution mapping with respect to $q \in \mathbb{V}$ can be described in any one of the following ways:

(i) the matrix M is P-matrix (see [13, 14]);

(ii) LCP(M, q) has a unique solution for all $q \in \mathbb{R}^n$ (i.e., GUS-property of M);

(iii) for any $q \in \mathbb{R}^n$, the solution set $\phi_M(q) \neq \emptyset$ and the set-valued mapping $q \to \phi_M(q)$ are Lipschitzian.

In particular, Mangasarian and Shiau [14] showed that if M is a P-matrix, then solutions of linear inequalities, programs, and LCP are Lipschitz continuous. Murthy et al. [15] showed that M is a P-matrix if and only if the LCP(M, q) has a solution for all $q \in \mathbb{R}^n$ and the solution mapping is Lipschitzian. Gowda and Sznajder [16] generalized the above result to affine variational inequality problems, while Yen [17] studied Lipschitz continuity of the solution mapping of variational inequalities with a parametric polyhedral constraint. As for NCP, Levy [18] obtained that the solution mapping is locally single-valued and Lipschitz continuous under suitable conditions. How about when $\mathcal{K}$ is nonpolyhedral? Balaji et al. [19] proved that L being monotone and the Lipschitz continuity of the solution mapping of SDLCP imply the GUS-property, while Chen and Qi in [9] employed Cartesian P-property to guarantee the GUS-property and the locally Lipschitzian property of the solution mapping of SDLCP. These make a complete extension of (i)–(iii) to their counterparts in SDLCP. A natural question arises here: can the above results be extended to a general symmetric cone case which is a unified framework?

In fact, there has been some papers dealing with the SCLCP over Euclidean Jordan algebras. For example, Balaji [20] established the result that if L has the Lipschitzian Q-property, then L has the positive principal minor property. Gowda et al. [21] showed that if L has P-property, then SCLCP$(L, \mathcal{K}, q)$ has a nonempty compact set for all $q \in \mathbb{V}$. In addition, Tao and Gowda [22] used degree-theoretic arguments to show that under a certain R_0-type condition, every P_0 symmetric cone nonlinear complementarity problem SCCP$(F, \mathcal{K}, q)$ has a solution. However, it still remains open under what conditions the solution map $\phi_F : q \mapsto \phi_F(q)$ of SCCP$(F, \mathcal{K}, q)$ is Lipschitz continuous. In this paper, we explore new results regarding Lipschitz continuity of the solution mapping of the SCLCP$(L, \mathcal{K}, q)$ or SCCP$(F, \mathcal{K}, q)$ over Euclidean Jordan algebras. In Theorem 3.1, we show that if the transformation F has the uniform Cartesian P-property with modulus $\rho > 0$, then the solution mapping ϕ_F is Lipschitz continuous with respect to $q \in \mathbb{V}$. Meanwhile, we give examples to show that the solution mapping of nonstrong monotone SCLCP$(L, \mathcal{K}, q)$ is not Lipschitz continuous with respect to q, and GUS-property does not imply Lipschitz continuity of the solution mapping.

On the other hand, various P-properties and GUS-property have been investigated in the literature [4, 9, 10, 13, 16, 19, 21–24]. Relations among them are well studied as well. In [19, Theorem 2.2], it is proved that if the linear transformation L in SDLCP has the monotonicity property and ϕ_L is Lipschitzian, then L has the P_2-property and the GUS-property. The concept of P_2-property in S^n was extended to a general Euclidean Jordan algebra, called ultra P-property [23]. Hence, it is desirable to know whether [19, Theorem 2.2] can be true or not in SCLCP$(L, \mathcal{K}, q)$ if P_2-property is replaced by ultra P-property. In this paper, we answer this question positively, see Theorem 3.8. Further, for the Lyapunov transformation L_a, we present several equivalent conditions for the ultra P-property of L_a.

Next are a few words about notations and some basic concepts employed. For a vector $x \in \mathbb{V}$, the norm is denoted by $\|x\| := \sqrt{\langle x, x\rangle}$, where $\langle \cdot, \cdot\rangle$ denotes the Euclidean inner product. For the Euclidean Jordan algebra $\mathbb{V}$, let $\mathcal{L}(\mathbb{V})$ denote the set of all continuous linear transformation $L : \mathbb{V} \to \mathbb{V}$, and Aut$(\mathcal{K})$ denote the set of all (invertible) linear transformations $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(\mathcal{K}) = \mathcal{K}$. For the convex set $\mathcal{K}$, let int$(\mathcal{K})$ denote the interior of the $\mathcal{K}$. L^T means the adjoint operator of L. The identical transformation on $\mathbb{V}$ will be denoted by I. For the SCCP$(F, \mathcal{K}, q)$, the solution set of SCCP$(F, \mathcal{K}, q)$ is denoted by $\phi_F(q)$. For the SCLCP$(L, \mathcal{K}, q)$, the solution set of SCLCP$(L, \mathcal{K}, q)$ is denoted by SOL$(L, \mathcal{K}, q)$ or $\phi_L(q)$.

2. Preliminaries

In this section, we briefly recall some basic concepts and background materials in Euclidean Jordan algebras, which will be used in the subsequent analysis. More details can be found in [21–23, 25].

An *Euclidean Jordan algebra* is a triple $(\mathbb{V}, \circ, \langle \cdot, \cdot\rangle)$ ($\mathbb{V}$ for short), where $\mathbb{V}$ is a finite-dimensional inner product over $\mathbb{R}$ and $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a bilinear mapping satisfying the following three conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$, where $x^2 = x \circ x$;

(iii) $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ for all $x, y, z \in \mathbb{V}$.

We call $x \circ y$ the *Jordan product* of x and y. In addition, if there is an element $e \in \mathbb{V}$ such that $x \circ e = x$ for all $x \in \mathbb{V}$, the element e is called the *identity element* in $\mathbb{V}$. In a given Euclidean

Jordan algebra $\mathbb{V}$, the set of squares $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ is a *symmetric cone* [25, Theorem III.2.1]. In other words, $\mathcal{K}$ is a self-dual closed convex cone, and, for any two elements $x, y \in \operatorname{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(x) = y$ and $\Gamma(\mathcal{K}) = \mathcal{K}$. For any $x \in \mathbb{V}$, we write

$$x \in \mathcal{K} \quad (x \in \operatorname{int}(\mathcal{K})) \Longleftrightarrow x \succeq 0 \quad (x \succ 0). \tag{2.1}$$

An element $c \in \mathbb{V}$ such that $c^2 = c$ is called an *idempotent* in $\mathbb{V}$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \ldots, e_r\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$e_i \circ e_j = 0 \quad \text{for } i \neq j, \qquad \sum_{i=1}^{r} e_i = e, \tag{2.2}$$

where r is called the rank of $\mathbb{V}$. Now, we recall the spectral and Peirce decompositions of an element x in $\mathbb{V}$.

Theorem 2.1 ((spectral decomposition) [25, Theorem III.1.2]). *Let $\mathbb{V}$ be an Euclidean Jordan algebra. Then, there is a number r such that for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ with*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{2.3}$$

Here, the numbers λ_i for $i = 1, \ldots, r$ are the eigenvalues of x and the expression $\lambda_1 e_1 + \cdots + \lambda_r e_r$ is the spectral decomposition (or the spectral expansion) of x.

In an Euclidean Jordan algebra $\mathbb{V}$, corresponding to the convex cone $\mathcal{K}$, let $\Pi_{\mathcal{K}}$ denote the *metric projection* onto $\mathcal{K}$, namely, for an $x \in \mathbb{V}$, $x^* = \Pi_{\mathcal{K}}(x)$ if and only if $x^* \in \mathcal{K}$ and $\|x - x^*\| \leq \|x - y\|$ for all $y \in \mathcal{K}$. It is well known that x^* is unique. For any $x \in \mathbb{V}$, combining the spectral decomposition of x with the metric projection of x onto $\mathcal{K}$, we have the expression of metric projection $\Pi_{\mathcal{K}}(x)$ as follows (see [21]):

$$\Pi_{\mathcal{K}}(x) = \max\{0, \lambda_1\} e_1 + \cdots + \max\{0, \lambda_r\} e_r. \tag{2.4}$$

The Peirce Decomposition

Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in an Euclidean Jordan algebra $\mathbb{V}$. For $i, j \in \{1, 2, \ldots, r\}$, we define the following eigenspaces:

$$\begin{aligned} \mathbb{V}_{ii} &:= \{x \in \mathbb{V} \mid x \circ e_i = x\} = \mathbb{R} e_i, \\ \mathbb{V}_{ij} &:= \left\{x \in \mathbb{V} \mid x \circ e_i = \frac{1}{2} x = x \circ e_j\right\} \quad \text{for } i \neq j. \end{aligned} \tag{2.5}$$

Theorem 2.2 (see [25, Theorem IV.2.1]). *The space $\mathbb{V}$ is the orthogonal direct sum of spaces $\mathbb{V}_{ij}$ $(i \le j)$. Furthermore,*

$$\mathbb{V}_{ij} \circ \mathbb{V}_{ij} \subset \mathbb{V}_{ii} + \mathbb{V}_{jj},$$
$$\mathbb{V}_{ij} \circ \mathbb{V}_{jk} \subset \mathbb{V}_{ik}, \quad \text{if } i \neq k, \tag{2.6}$$
$$\mathbb{V}_{ij} \circ \mathbb{V}_{kl} = \{0\}, \quad \text{if } \{i,j\} \cap \{k,l\} = \emptyset.$$

Hence, given any Jordan frame $\{e_1, e_2, \dots, e_r\}$, we can write any element $x \in \mathbb{V}$ as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}, \tag{2.7}$$

where $x_i \in \mathbb{R}$ and $x_{ij} \in \mathbb{V}_{ij}$. The expression $\sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$ is called the Peirce decomposition of x.

Next, we recall concept of Lyapunov transformation and its relevant conclusions which will be used in our analysis later. In an Euclidean Jordan algebra $\mathbb{V}$, for any $x \in \mathbb{V}$, we define the corresponding Lyapunov transformation $L_x : \mathbb{V} \to \mathbb{V}$ by $L_x(z) = x \circ z$ for any $z \in \mathbb{V}$. As remarked in [21, page 209], traditionally, the notation $L(x)$ has been used the Lyapunov transformation [25]. As employed in [21], we also reserve the notation L_x for the Lyapunov transformation and write $L(x)$ to denote the image of an element $x \in \mathbb{V}$ under a linear transformation $L : \mathbb{V} \to \mathbb{V}$. We say that elements x and y *operator commute* if $L_x L_y = L_y L_x$. It is well known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame [25, Lemma X.2.2].

Property 1 (see [21, Proposition 6]). For $x, y \in \mathbb{V}$, the following conditions are equivalent:

(a) $x \succeq 0$, $y \succeq 0$, and $\langle x, y \rangle = 0$;

(b) $x \succeq 0$, $y \succeq 0$, and $x \circ y = 0$.

Moreover, in this case, elements x and y operator commute. That is, x and y have their spectral decompositions with respect to a common Jordan frame.

In fact, from Property 1 and definition of (1.1), it can be seen that SCCP$(F, \mathcal{K}, q)$ is equivalent to find a $x \in \mathbb{V}$ such that

$$x \in \mathcal{K}, \quad F(x) + q \in \mathcal{K}, \quad x \circ (F(x) + q) = 0. \tag{2.8}$$

In addition, if x is a solution of SCCP$(F, \mathcal{K}, q)$, then x and $F(x) + q$ operator commute. Now, we review various monotonicity and P-property for a continuous transformation $F : \mathbb{V} \to \mathbb{V}$.

Definition 2.3. Let $\mathbb{V}$ be an Euclidean Jordan algebra. A continuous transformation $F : \mathbb{V} \to \mathbb{V}$ is said to be

(a) monotone if $\langle F(x) - F(y), x - y \rangle \ge 0$, for all $x, y \in \mathbb{V}$;

(b) strictly monotone if $\langle F(x) - F(y), x - y \rangle > 0$, for all $x \neq y \in \mathbb{V}$;

(c) strongly monotone if there is $\alpha > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in \mathbb{V}. \tag{2.9}$$

It is said to have

(d) GUS-property if SCCP$(F, \mathcal{K}, q)$ has a unique solution for any $q \in \mathbb{V}$;

(e) P-property if

$$\left.\begin{array}{c} x - y \text{ and } F(x) - F(y) \text{ operator commute} \\ (x - y) \circ (F(x) - F(y)) \preceq 0 \end{array}\right\} \Longrightarrow x = y; \tag{2.10}$$

(f) Q-property if $\phi_F(q) \neq \emptyset$ for any $q \in \mathbb{V}$.

Remark 2.4. (i) When F is linear, strict monotonicity and strong monotonicity coincide. When F is nonlinear, strong monotonicity implies strict monotonicity.

(ii) Whether F is linear or nonlinear, we have the following implications [22–24]:

$$\begin{gathered} \text{strong monotonicity} \Longrightarrow \text{strict monotonicity} \Longrightarrow P\text{-property} \Longrightarrow Q\text{-property}, \\ \text{strong monotonicity} \Longrightarrow \text{GUS-property} \Longrightarrow P\text{-property}. \end{gathered} \tag{2.11}$$

(iii) When $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, GUS-property and P-property coincide. But, once $\mathbb{V}$ and $\mathcal{K}$ are the other cases, for example, $\mathbb{V} = \mathbb{R}^n$ and $\mathcal{K} = \mathcal{K}^n$, where $\mathcal{K}^n$ denotes the second-order cone, or $\mathbb{V} = \mathbb{S}^n$ and $\mathcal{K} = S^n_+$, and so forth. GUS-property is not equivalent to P-property.

Given an Euclidean Jordan algebra $\mathbb{V}$ with $\dim(\mathbb{V}) = n > 1$, from [25, Proposition III 4.4-4.5 and Theorem V.3.7], we know that any Euclidean Jordan algebra $\mathbb{V}$ and its corresponding symmetric cone $\mathcal{K}$ are, in a unique way, a direct sum of simple Euclidean Jordan algebras and the constituent symmetric cone therein, respectively, that is,

$$\mathbb{V} = \mathbb{V}_1 \times \cdots \times \mathbb{V}_m, \qquad \mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_m, \tag{2.12}$$

where every $\mathbb{V}_i$ is a simple Euclidean Jordan algebra (which cannot be direct sum of two Euclidean Jordan algebras) with the corresponding symmetric cone $\mathcal{K}_i$ for $i = 1, \ldots, m$, and $n = \sum_{i=1}^m n_i$ (n_i is the dimension of $\mathbb{V}_i$). Therefore, for any $x = (x_1, \ldots, x_m)^T$, $y = (y_1, \ldots, y_m)^T \in \mathbb{V}$ with $x_i, y_i \in \mathbb{V}_i$, there exist

$$x \circ y = (x_1 \circ y_1, \ldots, x_m \circ y_m)^T \in \mathbb{V}, \qquad \langle x, y \rangle = \langle x_1, y_1 \rangle + \cdots + \langle x_m, y_m \rangle. \tag{2.13}$$

Through the above description and Cartesian P-properties proposed by Chen and Qi [9] in the setting of semidefinite matrices, Kong et al. [26] introduced the concept of uniform Cartesian P-property for the general transformation F in the setting of Euclidean Jordan algebra. This concept is used to study the Lipschitz continuity of the solution mapping in SCCP.

Definition 2.5. Consider a linear or nonlinear transformation $F : \mathbb{V} \to \mathbb{V}$. We say that F has *the uniform Cartesian P-property* if for any $x, y \in \mathbb{V}$ and $x \neq y$, there exist an index $\nu \in \{1, 2, \ldots, m\}$ and a scalar $\rho > 0$ such that

$$\left\langle (x-y)_\nu, (F(x)-F(y))_\nu \right\rangle \geq \rho \|x-y\|^2. \tag{2.14}$$

Remark 2.6. It is easy to observe that when $m = 1$, the uniform Cartesian P-property becomes the strong monotonicity of transformation F. If $m = n$ and $\mathbb{V} = \mathbb{R}^n$, it becomes the P-property in the context of NCP.

When the continuous transformation $F : \mathbb{V} \to \mathbb{V}$ is linear (i.e., $F = L$), we will introduce another concept, the ultra P-property of L, which is a new concept recently developed for linear transformations on Euclidean Jordan algebra. In fact, the ultra P-property is an equivalently straightforward extension of P_2-property in the setting of the semidefinite matrices [23]. Since P_2-property involves the ordinary (associative) product of three square matrices and there may not have an associative (triple) product in an Euclidean Jordan algebra, for this reason, P_2-property cannot be extended in a natural way to an Euclidean Jordan algebra [23]. However, the P_2-property is introduced in Euclidean Jordan algebra using the concepts of principal subtransformation and cone automorphisms of $\mathbb{V}$ [23].

Given a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in Euclidean Jordan algebra $\mathbb{V}$, we define

$$\mathbb{V}^{(l)} = \mathbb{V}(e_1 + \cdots + e_l, 1) := \{x \in \mathbb{V} \mid x \circ (e_1 + \cdots + e_l) = x\} \quad \text{for } 1 \leq l \leq r. \tag{2.15}$$

It is known that $\mathbb{V}^{(l)}$ is a subalgebra of $\mathbb{V}$ with rank l, see [25, Proposition IV.1.1]. By means of Peirce decomposition, we have the following representation [21]:

$$\mathbb{V}^{(l)} = \mathbb{R}e_1 + \cdots + \mathbb{R}e_l + \sum_{i<j\leq l} \mathbb{V}_{ij}. \tag{2.16}$$

Let $P^{(l)}$ denote the orthogonal projection from $\mathbb{V}$ onto $\mathbb{V}^{(l)}$. For a linear transformation $L : \mathbb{V} \to \mathbb{V}$, let

$$L_l = L_{\{e_1,\ldots,e_l\}} := P^{(l)} L : \mathbb{V}^{(l)} \longrightarrow \mathbb{V}^{(l)}. \tag{2.17}$$

We call L_l a *principal subtransformation* of L. The determinant of L_l is called a principal minor of L.

Definition 2.7 (see [23]). Consider a linear transformation $L : \mathbb{V} \to \mathbb{V}$. We say that L has the ultra P-property if for any $\Gamma \in \mathrm{Aut}(\mathcal{K})$, every principal subtransformation of $\widehat{L} = \Gamma^T L \Gamma$ has the P-property.

3. Main Results

In this section, we first give several sufficient conditions for the Lipschitz continuity of the solution mapping ϕ_L in the SCLCP$(L, \mathcal{K}, q)$. For the classical LCP and SDLCP, the Lipschitz continuity results have been studied in [9, 13, 14, 19]. Along this direction, we generalize them to general SCCP$(F, \mathcal{K}, q)$ case where a weaker condition, uniform Cartesian P-property, is used. Furthermore, we also establish relationship between the Lipschitz continuity of the solution mapping and the ultra P-property.

Theorem 3.1. *Let $F : \mathbb{V} \to \mathbb{V}$ be a continuous linear or nonlinear transformation. If F has the uniform Cartesian P-property, then ϕ_F is Lipschitz continuous.*

Proof. Suppose that F has uniform Cartesian P-property. From [26, Theorem 6.2], we know that for any $q \in \mathbb{V}$, the problem (1.1) has a unique solution, that is, $\phi_F(q)$ is a single point set. Thus, we let $\{x\} = \phi_F(q_1)$ and $\{y\} = \phi_F(q_2)$ for any $q_1, q_2 \in \mathbb{V}$. If $x = y$, the inequality $\|x - y\| \leq \kappa\|q_1 - q_2\|$ is obvious, where $\kappa > 0$. If $x \neq y$, from definition of uniform Cartesian P-property, there exists an index $\nu \in \{1, \ldots, m\}$ such that

$$\begin{aligned}
\rho\|x-y\|^2 &\leq \left\langle (x-y)_\nu, (F(x)-F(y))_\nu \right\rangle \\
&= \left\langle x_\nu - y_\nu, F(x)_\nu - F(y)_\nu \right\rangle \\
&= \left\langle x_\nu - y_\nu, (F(x)+q_1)_\nu - (F(y)+q_2)_\nu \right\rangle - \left\langle x_\nu - y_\nu, (q_1)_\nu - (q_2)_\nu \right\rangle \\
&= \left\langle x_\nu - y_\nu, (q_2)_\nu - (q_1)_\nu \right\rangle - \left\langle x_\nu, (F(y)+q_2)_\nu \right\rangle - \left\langle y_\nu, (F(x)+q_1)_\nu \right\rangle \\
&\leq \left\langle x_\nu - y_\nu, (q_2)_\nu - (q_1)_\nu \right\rangle \\
&\leq \|x_\nu - y_\nu\| \left\| (q_1)_\nu - (q_2)_\nu \right\| \\
&\leq \|x-y\| \|q_1 - q_2\|,
\end{aligned} \tag{3.1}$$

where the third equality follows from $\langle x_\nu, (F(x)+q_1)_\nu \rangle = 0 = \langle y_\nu, (F(y)+q_2)_\nu \rangle$ because x and y are the solution of the problem (1.1) for $q_1, q_2 \in \mathbb{V}$, respectively. The second inequality is due to x_ν, y_ν, $(F(x)+q_1)_\nu$, and $(F(y)+q_2)_\nu \in \mathcal{K}_\nu$. This implies that $\rho\|x-y\| \leq \|q_1 - q_2\|$. Letting $\kappa = 1/\rho$ gives $\|x-y\| \leq \kappa\|q_1 - q_2\|$. Hence, ϕ_F is Lipschitzian. □

Remark 3.2. In Theorem 3.1, if the transformation F is linear, the condition of uniform Cartesian P-property reduces to the Cartesian P-property [26]. However, if we weaken the condition of uniform Cartesian P-property to the monotonicity for the linear transformation L, the conclusion of Theorem 3.1 is not true. The following example shows that the monotonicity property is not sufficient to conclude that the ϕ_L is Lipschitz continuous with respect to $q \in \mathbb{V}$.

Example 3.3. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$L := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{where } L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) := \begin{bmatrix} 0 \\ y \\ 2z \end{bmatrix}. \tag{3.2}$$

It is obvious that L has the monotonicity property. It can be seen that $\mathrm{SOL}(L, \mathcal{K}^3, e) = \{0\}$, where $\mathcal{K}^3 \subset \mathbb{R}^3$ is a second-order cone, and e is identity element in Euclidean Jordan algebra $\mathbb{R}^3$. Moreover, it is easy to verify that

$$\left\{(\alpha, 0, 0)^T : 0 < \alpha \in \mathbb{R}\right\} \subseteq \mathrm{SOL}\left(L, \mathcal{K}^3, 0\right). \tag{3.3}$$

It is an unbounded solution set. However, if the solution mapping ϕ_L of $\mathrm{SCLCP}(L, \mathcal{K}^3, 0)$ is Lipschitz continuous, then $\mathrm{SOL}(L, \mathcal{K}^3, 0)$ must be a bounded set, which is clearly a contradiction.

Kong et al. [26] proved that the strong monotonicity implies the uniform Cartesian P-property whether the transformation F is linear or nonlinear. Moreover, when $F = L$ is linear transformation, by [21, Theorem 21], if L is self-adjoint and has P-property, then L is strongly monotone. Hence, we have the following corollary.

Corollary 3.4. *Consider Euclidean Jordan algebra $\mathbb{V}$.*

(a) *Let $F : \mathbb{V} \to \mathbb{V}$ be a nonlinear transformation. If F is strongly monotone, then ϕ_F is Lipschitz continuous.*

(b) *Let $L : \mathbb{V} \to \mathbb{V}$ be a linear transformation. If L is either*

(i) *strictly monotone, or*

(ii) *self-adjoint and has P-property, or*

(iii) *P-property and $\mathcal{K}$ is polyhedral,*

then ϕ_L is Lipschitz continuous.

Remark 3.5. Even the transformation F is linear, the condition of uniform Cartesian P-property in Theorem 3.1 or strong monotonicity in Corollary 3.4 cannot be weakened to the GUS-property, otherwise the conclusion is not true. Example 4.2 will illustrate this point.

In the following theorem, we prove that if ϕ_L is Lipschitz continuous, then L has the ultra P-property provided the linear transformation L is monotone. To establish another main result of this paper, the following lemmas play important roles.

Lemma 3.6. (a) *Suppose that ϕ_L is Lipschitz continuous, and* $\mathrm{SOL}(L, \mathcal{K}, \overline{q}) = \{0\}$ for some $\overline{q} \succ 0$. Then, $\mathrm{SOL}(L, \mathcal{K}, q) = \{0\}$ for all $q \succeq 0$.

(b) If $\mathrm{SOL}(L, \mathcal{K}, e) = \{0\}$ and if L has R_0-property (i.e., $\mathrm{SOL}(L, \mathcal{K}, 0) = \{0\}$), then L has Q-property.

(c) If ϕ_L is Lipschitz continuous and L has Q-property, then for the every principal subtransformation L_l of L, ϕ_{L_l} is the Lipschitz continuous with respect to any Jordan frame of $\mathbb{V}$.

Proof. Please see [20, Lemma 5] for part (a), [20, Proposition 3] for part (b), and [20, Lemma 4] for part (c). □

Lemma 3.7. *If ϕ_L is Lipschitz continuous and L has Q-property, then*

(a) *the linear transformation L is invertible;*

(b) $\mathrm{SOL}(L, \mathcal{K}, q) = \{0\}$ for some $q \succ 0$.

Proof. Part (a) is from [20, Lemma 6], while part (b) is from [20, Lemma 1]. □

Theorem 3.8. *Let $L : \mathbb{V} \to \mathbb{V}$ be a linear transformation. Suppose L is monotone and the solution mapping ϕ_L of* SCLCP$(L, \mathcal{K}, q)$ is Lipschitz continuous. Then,

(a) L has the ultra P-property;

(b) L has the GUS-property.

Proof. (a) Consider any Jordan frame $\{e_1, \ldots, e_r\}$ of Euclidean Jordan algebra $\mathbb{V}$ and the principal subtransformation $\widehat{L}_l := \widehat{L}_{\{e_1,\ldots,e_l\}} : \mathbb{V}^{(l)} \to \mathbb{V}^{(l)}$, where $\widehat{L} = \Gamma^T L \Gamma$ for any $\Gamma \in \mathrm{Aut}(\mathcal{K})$. Note that

$$\begin{gathered} \left\langle z, \widehat{L}(z) \right\rangle = \left\langle z, \Gamma^T L \Gamma(z) \right\rangle = \langle \Gamma(z), L\Gamma(z) \rangle \quad \text{for any } z \in \mathbb{V}, \\ \left\langle z_1, \widehat{L}_l(z_1) \right\rangle = \left\langle z_1, \left(P^{(l)}\widehat{L}\right)(z_1) \right\rangle = \left\langle z_1, \widehat{L}(z_1) \right\rangle, \quad \text{where } z_1 \in \mathbb{V}^{(l)} \subseteq \mathbb{V}. \end{gathered} \tag{3.4}$$

Since L is monotone, it follows that the linear transformation $\widehat{L}$ and $\widehat{L}_l$ are both monotone. Thus, we have SOL$(\widehat{L}, \mathcal{K}, e) = \{0\}$ and SOL$(\widehat{L}_l, \mathcal{K}^{(l)}, e^{(l)}) = \{0\}$, where $\mathcal{K}^{(l)}$ and $e^{(l)}$ denote the symmetric cone and the identity element in $\mathbb{V}^{(l)}$, respectively. Furthermore, by direct calculation, it is not hard to prove that the solution mapping ϕ_L of SCLCP$(L, \mathcal{K}, q)$ is Lipschitz continuous if and only if the solution mapping $\phi_{\widehat{L}}$ of the corresponding SCLCP is Lipschitz continuous for the linear transformation $\widehat{L}$. Applying Lemma 3.6(a) and (b) yields that $\widehat{L}$ has Q-property. Then using Lemma 3.6(c), we obtain that the solution mapping $\phi_{\widehat{L}_l}$ of the corresponding SCLCP is Lipschitz continuous for the linear transformation $\widehat{L}_l$. It follows from SOL$(\widehat{L}_l, \mathcal{K}^{(l)}, e^{(l)}) = \{0\}$ and Lemma 3.6(a) again that $\widehat{L}_l$ has Q-property. This together with Lemma 3.7 says that the transformation $\widehat{L}_l$ is invertible.

Next, we want to prove that the transformation $\widehat{L}_l$ has P-property. Suppose that an element $0 \neq x \in \mathbb{V}^{(l)}$ operator commute with $\widehat{L}_l(x)$ and $x \circ \widehat{L}_l(x) \preceq 0$. Since $\widehat{L}_l$ is monotone by the above analysis, we have

$$0 \leq \left\langle x, \widehat{L}_l(x) \right\rangle = \left\langle x \circ \widehat{L}_l(x), e^{(l)} \right\rangle \leq 0, \tag{3.5}$$

which means that $\langle \widehat{L}_l(x) \circ x, e^{(l)} \rangle = 0$. Together with Property 1, it is easy to verify that $\widehat{L}_l(x) \circ x = 0$, and $\widehat{L}_l(x)$ and x have the same Jordan frame. Since $\widehat{L}_l(x) \circ x = 0$, we write

$$x = \sum_{i=1}^{k} \lambda_i f_i, \qquad \widehat{L}_l(x) = \sum_{i=k+1}^{l} \mu_i f_i, \tag{3.6}$$

where $\{f_1, f_2, \ldots, f_l\}$ is a Jordan frame in $\mathbb{V}^{(l)}$, $\lambda_i \neq 0$ $(i = 1, \ldots, k)$ and $1 \leq k \leq l$. Let $Q^{(k)}$ denote the projection operator from $\mathbb{V}^{(l)}$ onto the eigenspace $\mathbb{W}^{(k)}$ of $f_1 + \cdots + f_k$. Then,

$$0 = Q^{(k)}\widehat{L}_l(x) = Q^{(k)}\widehat{L}_{\{e_1,\ldots,e_l\}}(x) = Q^{(k)}P^{(l)}\widehat{L}(x). \tag{3.7}$$

Let $T_k := Q^{(k)}\widehat{L}_l : \mathbb{W}^{(k)} \to \mathbb{W}^{(k)}$ be the principal subtransformation of $\widehat{L}_l$ corresponding to $\{1, \ldots, k\}$. From the definition of T_k, it follows that $T_k(x) = Q^{(k)}\widehat{L}_l(x) = 0$. By the same arguments as above, we know that T_k has Q-property, and the solution mapping ϕ_{T_k} of the corresponding SCLCP is Lipschitz continuous for the transformation T_k. Hence, from Lemma 3.7, we get that T_k is invertible. This together with $T_k(x) = 0$ yields $x = 0$, which gives a contradiction to $x \neq 0$. Therefore, we have proved that L has the ultra P-property.

(b) This is immediate by [23, Theorem 6.2]. □

It was shown in [19, Theorem 2.2] that if $L : S^n \to S^n$ is monotone and ϕ_L is Lipschitz continuous, then L has the P_2-property. Note that P_2-property in S^n is equivalent to the ultra P-property in S^n (see [23]). Therefore, the result of Theorem 3.8 is a natural extension of [19, Theorem 2.2] to the setting of Euclidean Jordan algebra.

4. A Special Linear Transformation

In this section, we specialize to a special linear transformation which is studied in the SCLCP setting, see [19, 23]. For $a \in \mathbb{V}$, we consider the corresponding Lyapunov transformation L_a. We will give several equivalent conditions regarding the ultra P-property of Lyapunov transformation L_a.

Theorem 4.1. *For the Lyapunov transformation L_a $(a \in \mathbb{V})$, the following statements are equivalent:*

(a) $a \succ 0$;

(b) L_a *is strongly monotone;*

(c) L_a *has (uniform) Cartesian P-property;*

(d) L_a *has GUS-property;*

(e) L_a *has P-property;*

(f) L_a *has the ultra P-property;*

(g) L_a *has Q-property and the solution mapping* ϕ_{L_a} *of the SCLCP*(L_a, K, q) *is Lipschitz continuous with respect to* $q \in \mathbb{V}$.

Proof. (a)⇒(b) For any $0 \neq x \in V$, we have $\langle x, L_a(x)\rangle = \langle x, a \circ x\rangle = \langle a, x^2\rangle$. Since $a \succ 0$ and $x^2 \in \mathcal{K}$, $\langle a, x^2\rangle > 0$ (see [25, Proposition I.1.4]). Thus, L_a has the strong monotonicity property.

(b)⇒(c) It is straightforward by the definitions.

The implication (c)⇒(d) follows from [26, Theorem 6.2].

(d)⇒(e) This follows from [21, Theorem 14].

(e)⇒(a) Suppose that the Lyapunov transformation L_a has P-property. Let $a = \sum_{i=1}^{r} \lambda_i(a) e_i$ and $I = \{i : \lambda_i(a) \leq 0\}$, where $\{e_1, \ldots, e_r\}$ is a Jordan frame of $\mathbb{V}$. Note that $a \succ 0$ if and only if $I = \emptyset$. Suppose that $I \neq \emptyset$. Let $x = \sum_{i \in I} e_i \neq 0$. Then, x and $L_a(x)$ operator commute, and $x \circ L_a(x) = \sum_{i \in I} \lambda_i(a) e_i \preceq 0$. Therefore, by the P-property of L_a, we have $x = 0$ which leads to $a \succ 0$.

(b)⇒(f) It follows from [23, Theorem 6.1].

(f)⇒(e) It is obvious.

(b)⇒(h) For any linear transformation, the strong monotonicity is equivalent to the strict monotonicity. Then, it follows from Corollary 3.4 that the solution mapping ϕ_{L_a} of the SCLCP$(L_a, \mathcal{K}, q)$ is Lipschitz continuous with respect to $q \in \mathbb{V}$. Moreover, it is true that the strong monotonicity implies Q-property for any linear transformation, see [21]. Hence, the conclusion of (h) is obtained.

(h)⇒(b) Suppose that the solution mapping ϕ_{L_a} of the SCLCP$(L_a, \mathcal{K}, q)$ is Lipschitz continuous with respect to $q \in \mathbb{V}$, and L_a has Q-property. Let $\{e_1, \ldots, e_r\}$ be a Jordan frame of $\mathbb{V}$ and $x = \sum_{i=1}^{r} \lambda_i(x) e_i$. Note that

$$\langle x, L_a(x) \rangle = \sum_{i,j=1}^{r} \lambda_i(x)\lambda_j(x)\langle e_i, L_a(e_j) \rangle = \sum_{i=1}^{r} \lambda_i^2(x)\langle e_i, L_a(e_i) \rangle. \tag{4.1}$$

Since L_a has the Q-property and the solution map ϕ_{L_a} is Lipschitz continuous, $\langle L_a(e_i), e_i \rangle > 0$ (see [27, Theorem 3.1]). It follows from (4.1) that $\langle L_a(x), x \rangle > 0$ for all $0 \neq x \in \mathbb{V}$. Therefore, the linear transformation L_a has the strong monotonicity. The proof is complete. □

In general, the above result may fail to hold. The following example shows that ϕ_L is not Lipschitz continuous, but L has the GUS-property. Meanwhile, this example also shows that for Theorem 3.1 and Corollary 3.4, if weaken the condition of strong monotonicity to GUS-property, the conclusions of Theorem 3.1 and Corollary 3.4 are not true.

Example 4.2. Let $\mathbb{V} = S^2$ and $\mathcal{K} = S_+^2$. For

$$A := \begin{bmatrix} 0 & -3 \\ 3 & 3 \end{bmatrix}, \tag{4.2}$$

consider the corresponding Lyapunov transformation defined by

$$L_A(\mathrm{X}) := AX + XA^T. \tag{4.3}$$

It is easy to prove that A is positive stable and positive semidefinite, and L_A is a linear transformation. From [10, Theorem 9], we have that L_A has GUS-property. On the other hand, since A is not a positive definite matrix, it follows from [19, Theorem 3.3] that ϕ_{L_A} is not Lipschitz continuous.

5. Concluding Remarks

In this paper, we have studied the Lipschitz continuity of the solution mapping for symmetric cone linear or nonlinear complementarity problems over Euclidean Jordan algebras and provided several sufficient conditions for the Lipschitz continuity of the solution mapping. We have established the relationship between the Lipschitz continuity of the solution mapping and ultra P-property. Furthermore, for Lyapunov transformation, we have shown that the strong monotonicity property, the ultra P-property, GUS-property, the Lipschitz continuity of the solution mapping, and so forth are all equivalent to each other.

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