Construction of merit functions for ellipsoidal cone complementarity problem

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Abstract: Nonsymmetric cone program and its corresponding complementarity problem have long been mysterious to optimization researchers because of no unified analysis technique to handle these cones. Nonetheless, merit function approach is a popular method to deal with general conic complementarity problem, for which the key lies on constructing appropriate merit functions. In this paper, we focus on a special class of nonsymmetric cone complementarity problem, that is, the ellipsoidal cone complementarity problem (ECCP). We not only show the readers how to construct merit functions for solving the ellipsoidal cone complementarity problem, but also we study the conditions under which the level sets of the corresponding merit functions are bounded. In addition, we assert that these merit functions provide an error bound for the ellipsoidal cone complementarity problem. All these results build up a theoretical basis for the merit function method for solving ellipsoidal cone complementarity problem.

Keywords: Ellipsoidal cone complementarity problem, merit function, error bound
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1 Introduction

Nonsymmetric cone program and its corresponding complementarity problem have been mysterious to optimization researchers because of no unified analysis techniques to handle these conic optimizations. Nonetheless, merit function approach is a popular way to deal with general conic complementarity problem, for which the key lies on constructing appropriate merit functions. Recently, we have investigated how to construct merit functions for two special nonsymmetric cone complementarity problems in [20, 21], including circular cone complementarity problem and p-order cone complementarity problem. Following this line, we focus on the ellipsoidal cone complementarity problem (ECCP) in this paper.

The class of ellipsoidal cones [1, 15, 16, 28], as an important prototype in closed convex cones, covers several practical instances such as second-order cone, circular cone, and elliptic cone. Mathematically, it is described as

$$\mathcal{K}_E := \{x \in \mathbb{R}^n \,|\, x^T Q x \leq 0,\ u_n^T x \geq 0\}, \tag{1.1}$$

where $Q \in \mathbb{R}^{n\times n}$ is a real-valued nonsingular symmetric matrix with a single negative eigenvalue $\lambda_n \in \mathbb{R}$, and $u_n \in \mathbb{R}^n$ is a unit eigenvector of λ_n. By natural feature, it belongs to the category of nonsymmetric cones because it is non-self-dual under standard inner product. Nonetheless, it can be converted to a second-order cone, which is symmetric, by a transformation and vice versa, please refer to [15] for more details. As shown in [15], the ellipsoidal cone $\mathcal{K}_E$ is a closed convex cone and its dual cone is given by

$$\mathcal{K}_E^* = \{y \in \mathbb{R}^n \,|\, y^T Q^{-1} y \leq 0,\ u_n^T y \geq 0\}.$$

Clearly, from the expression of $\mathcal{K}_E^*$, we know that the dual cone $\mathcal{K}_E^*$ is also a closed convex cone.

The ellipsoidal cone complementarity problem (ECCP) indeed fits in the general conic complementarity problem, which is to find an element $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \quad F(x) \in \mathcal{K}^* \quad \text{and} \quad \langle x, F(x)\rangle = 0, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable mapping, $\mathcal{K}$ represents a closed convex cone, and $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$ given by $\mathcal{K}^* := \{v \in \mathbb{R}^n \,|\, \langle v, x\rangle \geq 0,\ \forall x \in \mathcal{K}\}$.

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In other words, when $\mathcal{K}$ represents the ellipsoidal cone, the problem (1.2) reduces to the ellipsoidal cone complementarity problem (ECCP). In particular, as remarked in [15, 16], when $Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$ with I_{n-1} being the unit matrix and $u_n = (0, 0, \cdots, 1)^T$, the ellipsoidal cone is exactly the well known second-order cone $\mathcal{K}^n$ [2, 5, 7, 8], described by

$$\mathcal{K}^n := \{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|\bar{x}\| \le x_1\},$$

or another form

$$\mathcal{K}^n := \{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}\| \le x_n\}.$$

Hence, the ellipsoidal cone can be viewed as a generalization of the second-order cone $\mathcal{K}^n$ in $\mathbb{R}^n$. When $Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2\theta \end{bmatrix}$ and $u_n = (0, 0, \cdots, 1)^T$, the ellipsoidal cone reduces to the circular cone [19, 31], which is also a special nonsymmetric cone:

$$\mathcal{L}_\theta := \{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}\| \le \tan\theta \, x_n\}.$$

Moreover, when $Q = \begin{bmatrix} M^T M & 0 \\ 0 & -1 \end{bmatrix}$ with M being any nonsingular matrix of order $n-1$ and $u_n = (0, 0, \cdots, 1)^T$, the ellipsoidal cone reduces to the elliptic cone [1]:

$$\mathcal{K}^n_M := \{x = (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, x_n \ge \|M\bar{x}\|\}.$$

Here $\|\cdot\|$ denotes the standard Euclidean norm. In summary, the ellipsoidal cone complementarity problem (ECCP) covers a range of nonsymmetric cone complementarity problems.

When $\mathcal{K}$ is a symmetric cone, the complementarity problem (1.2) is called the symmetric cone complementarity problem, which has been extensively studied from different views [12, 13, 14, 25, 27], including the second-order cone complementarity problem [2, 3, 4, 5, 6, 7, 8, 9, 11]. Recently, more and more nonsymmetric cones appears in plenty of real applications. In contrast to symmetric cone programming and symmetric cone complementarity problem, we are not quite familiar with their nonsymmetric counterparts. Referring the reader to [20, 21, 22, 24, 26, 30] and the bibliographies therein, we observe that there is no any unified way to handle nonsymmetric cone constraints, and the study of such problems usually uses certain specific features of the nonsymmetric cones under consideration. This is one motivation that we continuously pay attention to the series of investigation on nonsymmetric cone complementarity problem in this paper and [20, 21].

There exist many popular ways for dealing with various complementarity problems, which include the interior-point methods, the smoothing Newton methods, the semismooth Newton methods, and the merit function methods, see [2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 17, 24, 29] and references therein. As seen in the literature, almost all the attention was paid to symmetric cone complementarity problems, including nonlinear complementarity problem (NCP), positive semi-definite complementarity problem (SDCP), and second-order cone complementarity problem (SOCCP). However, the study about nonsymmetric cone complementarity problem is very limited. The main difficulty is that there is no unified framework to deal with general nonsymmetric cone complementarity problems. Nonetheless, we believe that the merit function approach, in which the complementarity problem is recast as an unconstrained minimization via merit function or complementarity function, may be appropriately viewed as a unified way to deal with nonsymmetric cone complementarity problem.

More specifically, a merit function is used to recast the problem (1.2) as an unconstrained smooth minimization problem or a system of nonsmooth equations. In other words, through a *merit function* $h : \mathbb{R}^n \to \mathbb{R}_+$ for the ECCP, there usually holds

$$h(x) = 0 \quad \Longleftrightarrow \quad x \;\text{ solves the ECCP (1.2).}$$

Hence, solving the problem (1.2) is also equivalent to handling the unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} h(x)$$

with the optimal value zero. Until now, for solving symmetric cone complementarity problem, a large number of merit functions have been proposed. Among them, one of the most popular merit functions is the natural residual (NR) merit function $\Psi_{\rm NR} : \mathbb{R}^n \to \mathbb{R}_+$, which is defined as

$$\Psi_{\rm NR}(x) := \frac{1}{2}\|\phi_{\rm NR}(x, F(x))\|^2 = \frac{1}{2}\left\|x - (x - F(x))_+^{\mathcal{K}}\right\|^2,$$

where $(\cdot)_+^{\mathcal{K}}$ denotes the projection onto the symmetric cone $\mathcal{K}$. Then, we know that $\Psi_{\rm NR}(x) = 0$ if and only if x is a solution to the symmetric cone complementarity problem. As remarked in [21], this function $\Psi_{\rm NR}$ (or $\phi_{\rm NR}$) can also serve as merit function (or complementarity function) for general conic complementarity problem. Hence, it is also applicable to ellipsoidal cone complementarity problem. Under this setting, for any $x \in \mathbb{R}^n$, we denote x_+ be the projection of x onto the ellipsoidal cone $\mathcal{K}_E$, and x_- be the projection of $-x$ onto the dual cone $\mathcal{K}_E^*$ of $\mathcal{K}_E$. By properties of projection onto the closed convex cone, it can be verified that $x = x_+ - x_-$. Besides the NR merit function $\Psi_{\rm NR}$, are there any other types of merit functions for the ECCP? The answer is yes. In this paper, we will present other types of merit functions for the ECCP. Moreover, we investigate the properties of these proposed merit functions, and study conditions under which these merit functions provide bounded level sets. Note that such properties will guarantee that the sequence generated by descent methods has at least one accumulation point, and build up a theoretical basis for designing the merit function method for solving ellipsoidal cone complementarity problem.

2 Preliminaries

In this section, for subsequent needs, we briefly review some basic concepts and background materials about the ellipsoidal cone, and define one type of product associated with the ellipsoidal cone, which will be extensively used in subsequent analysis.

As mentioned in Section 1, we know that the ellipsoidal cone $\mathcal{K}_E$ given as in (1.1), i.e.,

$$\mathcal{K}_E := \{x \in \mathbb{R}^n \mid x^T Q x \leq 0,\ u_n^T x \geq 0\},$$

is a pointed closed convex cone. Besides, the dual cone $\mathcal{K}_E^*$ is given as

$$\mathcal{K}_E^* = \{y \in \mathbb{R}^n \mid y^T Q^{-1} y \leq 0,\ u_n^T y \geq 0\},$$

and it is also a closed convex cone. It is obvious that the ellipsoidal cone $\mathcal{K}_E$ is not a symmetric cone. Since the matrix Q is real-valued nonsingular symmetric matrix, there exist an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that

$$U^T Q U = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} := \Lambda,$$

where $U := [u_1\ u_2\ \cdots\ u_n]$ with eigen-pairs (λ_i, u_i) $(i = 1, \cdots, n)$ satisfying the following conditions:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0 > \lambda_n \quad \text{and} \quad u_i^T u_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then, we have $Q = U \Lambda U^T$. Let $\alpha := [\alpha_1, \alpha_2, \cdots, \alpha_n]^T = U^T x$, i.e., $\alpha_i = u_i^T x$ for $i = 1, 2, \cdots, n$, it follows that

$$x^T Q x = \alpha^T \Lambda \alpha = \sum_{i=1}^n \lambda_i \alpha_i^2 \quad \text{and} \quad \alpha_n = u_n^T x,$$

which indicates the ellipsoidal cone $\mathcal{K}_E$ can be expressed as follows:

$$\mathcal{K}_E = \left\{ U\alpha \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \lambda_i \alpha_i^2 \leq 0,\ \alpha_n \geq 0 \right\}.$$

On the other hand, by letting $\beta := [\beta_1, \beta_2, \cdots, \beta_n]^T = U^T y$, i.e., $\beta_i = u_i^T y$ for $i = 1, 2, \cdots, n$, the dual cone $\mathcal{K}_E^*$ can be expressed as

$$\mathcal{K}_E^* = \left\{ U\beta \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \lambda_i^{-1} \beta_i^2 \leq 0,\ \beta_n \geq 0 \right\}.$$

The following lemma explains the relationship between ellipsoidal cone $\mathcal{K}_E$ and second-order cone $\mathcal{K}^n$.

Lemma 2.1. *For the ellipsoidal cone $\mathcal{K}_E$ and second-order cone $\mathcal{K}^n$, the following relations hold.*

(a) $\mathcal{K}_E = UD\,\mathcal{K}^n$ *and* $\mathcal{K}^n = D^{-1}U^T \mathcal{K}_E$ *with* $D := \operatorname{diag}\left[\lambda_1^{-\frac{1}{2}}, \lambda_2^{-\frac{1}{2}}, \cdots, (-\lambda_n)^{-\frac{1}{2}}\right]$.

(b) $\mathcal{K}_E^* = UD^{-1}\,\mathcal{K}^n$ *and* $\mathcal{K}^n = DU^T \mathcal{K}_E^*$.

(c) $(\mathcal{K}_E^*)^* = \mathcal{K}_E$.

Proof. Please see [15, Theorem2.1, Theorem 3.1]. $\square$

It is well known that Jordan product plays a critical role in the study of symmetric cone programming or symmetric cone complementarity problems. More specifically, for any $x = (x_0, \bar{x}), y = (y_0, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ (or $x = (\bar{x}, x_n), y = (\bar{y}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$), in the setting of the SOC, the so-called **Jordan product** of x and y is defined as

$$x \circ y := \begin{bmatrix} \langle x, y \rangle \\ y_0\bar{x} + x_0\bar{y} \end{bmatrix} \quad \left(\text{ or } \ x \circ y := \begin{bmatrix} y_n\bar{x} + x_n\bar{y} \\ \langle x, y \rangle \end{bmatrix} \right).$$

The Jordan product "$\circ$", unlike scalar or matrix multiplication, is not associative. The identity element under Jordan product is $e = (1, 0, \cdots, 0)^T \in \mathbb{R}^n$ (or $e = (0, 0, \cdots, 1)^T$). With the Jordan product associated with SOC, we have the following conclusion.

Lemma 2.2. *For any $x, y \in \mathbb{R}^n$, the following holds:*

$$x \in \mathcal{K}^n, \ y \in \mathcal{K}^n \text{ and } \ \langle x, y \rangle = 0 \iff x \in \mathcal{K}^n, \ y \in \mathcal{K}^n \text{ and } \ x \circ y = 0.$$

Proof. Please see [11, Proposition 2.1]. $\square$

Applying Lemma 2.1 and Lemma 2.2 yields the following theorem, which indicates the relationship between the SOCCP and the ECCP.

Theorem 2.3. *For any $x, y \in \mathbb{R}^n$, the following are equivalent:*
(a) $x \in \mathcal{K}_E, \ y \in \mathcal{K}_E^*$ *and* $\langle x, y \rangle = 0$.
(b) $D^{-1}U^T x \in \mathcal{K}^n, \ DU^T y \in \mathcal{K}^n$ *and* $\langle D^{-1}U^T x, DU^T y \rangle = \langle x, y \rangle = 0$.
(c) $D^{-1}U^T x \in \mathcal{K}^n, \ DU^T y \in \mathcal{K}^n$ *and* $(D^{-1}U^T x) \circ (DU^T y) = 0$.

In the literature, there are two popular and well-known merit functions for the SOCCP, which are Fischer-Burmeister (FB) merit function and natural residual (NR) merit function:

$$\begin{aligned} \Psi_{\rm FB}(x) = \frac{1}{2}\|\phi_{\rm FB}(x, F(x))\|^2 &= \frac{1}{2}\left\|(x^2 + (F(x))^2)^{1/2} - (x + F(x))\right\|^2, \\ \Psi_{\rm NR}(x) = \frac{1}{2}\|\phi_{\rm NR}(x, F(x))\|^2 &= \frac{1}{2}\left\|x - (x - F(x))_+^{\mathcal{K}^n}\right\|^2. \end{aligned}$$

In view of this and Theorem 2.3, we can modify the above two kinds of merit functions for the SOCCP to formulate merit functions for the ECCP. Consequently, we obtain the following merit functions for the ECCP.

$$\begin{aligned} \widetilde{\Psi_{\rm FB}}(x) &:= \frac{1}{2}\left\|\left[(D^{-1}U^T x)^2 + [DU^T F(x)]^2\right]^{\frac{1}{2}} - (D^{-1}U^T x + DU^T F(x))\right\|^2, \\ \widetilde{\Psi_{\rm NR}}(x) &:= \frac{1}{2}\left\|D^{-1}U^T x - [D^{-1}U^T x - DU^T F(x)]_+^{\mathcal{K}^n}\right\|^2, \end{aligned}$$

where z^2 and $z^{\frac{1}{2}}$ means $z^2 := z \circ z$ and $z = z^{\frac{1}{2}} \circ z^{\frac{1}{2}}$, respectively, which are computed by the Jordan product associated with SOC. By this, we can solve the ECCP by solving the SOCCP. In fact, the NR function $\phi_{\rm NR} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $\phi_{\rm NR}(x, y) := x - (x - y)_+^{\mathcal{K}}$ always serves as a complementarity function for general conic complementarity problem, see [9, Proposition 1.5.8], i.e., the function $\Psi_{\rm NR}$ is a merit function for general conic complementarity problem. In other words, the function

$$\Psi_{\rm NR}(x) = \frac{1}{2}\|\phi_{\rm NR}(x, F(x))\|^2 = \frac{1}{2}\left\|x - (x - F(x))_+^{\mathcal{K}_E}\right\|^2$$

can also be a merit function for the ECCP. Besides, the function $\Psi_{\rm NR}$ possesses the following property.

Lemma 2.4. ([20, Lemma 3.1]) *Let $x, y \in \mathbb{R}^n$ and $\phi_{\rm NR}(x, y) = x - (x - y)_+^{\mathcal{K}}$. For any closed convex cone $\mathcal{K}$, we have*

$$\|\phi_{\rm NR}(x, y)\| \geq \max\left\{\|x_-^{\mathcal{K}^*}\|, \|(-y)_+^{\mathcal{K}}\|\right\},$$

where $z_+^{\mathcal{K}}$ denotes the projection of z onto the closed convex cone $\mathcal{K}$, and $z_-^{\mathcal{K}^*}$ means the projection of $-z$ onto its dual cone $\mathcal{K}^*$.

To the contrast, it is unfortunately that $\Psi_{\rm FB}(x)$ cannot serve as a merit function for the ECCP because x^2 is not well-defined associated with the ellipsoidal cone $\mathcal{K}_E$ for any $x \in \mathbb{R}^n$. Indeed, there is no product corresponding to the standard inner product for the setting of the ellipsoidal cone so far. This inspires us to define an appropriate product associated with ellipsoidal cone $\mathcal{K}_E$, and find more merit functions for the ECCP.

Based on the experience in [20, 21] and looking the structure of the ellipsoidal cone, we present the new product directly as below. For any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, let

$$\begin{aligned} \alpha &:= [\alpha_1, \cdots, \alpha_n]^T = U^T x \\ \beta &:= [\beta_1, \cdots, \beta_n]^T = U^T y, \end{aligned}$$

i.e., $\alpha_i = u_i^T x$ and $\beta_i = u_i^T y$ for $i = 1, 2, \cdots, n$. The product of x and y associated with the ellipsoidal cone $\mathcal{K}_E$ is defined by

$$x \bullet y = \begin{bmatrix} w \\ \langle x, y \rangle \end{bmatrix} \quad \text{where} \quad w := (w_1, \cdots, w_{n-1})^T \quad \text{with} \quad w_i = \beta_n \lambda_i^{\frac{1}{2}} \alpha_i - \lambda_n \alpha_n \lambda_i^{-\frac{1}{2}} \beta_i. \tag{2.1}$$

According to the definition of the product "$\bullet$" defined as in (2.1), it is easy to see that the product "$\bullet$" is not commutative. Nonetheless, there have the following results.

Theorem 2.5. *For any $x, y \in \mathbb{R}^n$, the following statements are equivalent:*

(a) *$x \in \mathcal{K}_E$, $y \in \mathcal{K}_E^*$ and $\langle x, y \rangle = 0$.*

(b) *$x \in \mathcal{K}_E$, $y \in \mathcal{K}_E^*$ and $x \bullet y = 0$.*

In each case, x and y satisfy the condition that there is $c \leq 0$ such that $\beta_i = c \cdot \lambda_i \alpha_i$ or $\alpha_i = c \cdot \lambda_i^{-1} \beta_i$ for any $i = 1, \cdots, n$.

Proof. "(b) $\Rightarrow$ (a)" The implication is obvious from the definition of product $x \bullet y$.

"(a) $\Rightarrow$ (b)" Because $x \in \mathcal{K}_E$, it follows that

$$\begin{aligned} x \in \mathcal{K}_E \quad &\Longleftrightarrow \quad x^T Q x \leq 0 \ \text{ and } \ u_n^T x \geq 0 \\ &\Longleftrightarrow \quad x^T U \Lambda U^T x \leq 0 \ \text{ and } \ u_n^T x \geq 0 \\ &\Longleftrightarrow \quad \sum_{i=1}^{n} \lambda_i (u_i^T x)^2 \leq 0 \ \text{ and } \ u_n^T x \geq 0 \\ &\Longleftrightarrow \quad \sum_{i=1}^{n-1} \lambda_i \alpha_i^2 \leq (-\lambda_n) \alpha_n^2 \text{ and } \alpha_n \geq 0 \\ &\Longleftrightarrow \quad \left(\sum_{i=1}^{n-1} (\lambda_i^{\frac{1}{2}} \alpha_i)^2 \right)^{\frac{1}{2}} \leq (-\lambda_n)^{\frac{1}{2}} \alpha_n. \end{aligned} \tag{2.2}$$

Similarly, we have

$$y \in \mathcal{K}_E^* \iff \left(\sum_{i=1}^{n-1} (\lambda_i^{-\frac{1}{2}} \beta_i)^2 \right)^{\frac{1}{2}} \leq (-\lambda_n)^{-\frac{1}{2}} \beta_n. \tag{2.3}$$

Hence, by using $\langle x, y \rangle = 0$, we further obtain

$$\begin{aligned} 0 \quad &= \quad \langle x, y \rangle \ = \ y^T U U^T x \ = \ \sum_{i=1}^{n} y^T u_i u_i^T x \\ &= \quad \sum_{i=1}^{n-1} y^T u_i u_i^T x + y^T u_n u_n^T x \\ &= \quad \sum_{i=1}^{n-1} y^T u_i \lambda_i^{-\frac{1}{2}} \lambda_i^{\frac{1}{2}} u_i^T x + y^T u_n (-\lambda_n)^{-\frac{1}{2}} (-\lambda_n)^{\frac{1}{2}} u_n^T x \\ &= \quad \sum_{i=1}^{n-1} \lambda_i^{-\frac{1}{2}} \beta_i \lambda_i^{\frac{1}{2}} \alpha_i + (-\lambda_n)^{-\frac{1}{2}} \beta_n (-\lambda_n)^{\frac{1}{2}} \alpha_n \\ &\geq \quad - \left(\sum_{i=1}^{n-1} (\lambda_i^{\frac{1}{2}} \alpha_i)^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n-1} (\lambda_i^{-\frac{1}{2}} \beta_i)^2 \right)^{\frac{1}{2}} + (-\lambda_n)^{-\frac{1}{2}} \beta_n (-\lambda_n)^{\frac{1}{2}} \alpha_n \\ &\geq \quad 0, \end{aligned}$$

where the first inequality holds by Cauchy-inequality, and the second inequality holds due to (2.2) and (2.3). This implies

$$(-\lambda_n)^{\frac{1}{2}}\alpha_n = \left(\sum_{i=1}^{n-1}(\lambda_i^{\frac{1}{2}}\alpha_i)^2\right)^{\frac{1}{2}}, \quad (-\lambda_n)^{-\frac{1}{2}}\beta_n = \left(\sum_{i=1}^{n-1}(\lambda_i^{-\frac{1}{2}}\beta_i)^2\right)^{\frac{1}{2}}$$

and $\lambda_i^{-\frac{1}{2}}\beta_i = c\cdot\lambda_i^{\frac{1}{2}}\alpha_i$ or $\lambda_i^{\frac{1}{2}}\alpha_i = c\cdot\lambda_i^{-\frac{1}{2}}\beta_i$ with some $c\le 0$ for any $i=1,\cdots,n-1$, i.e., $\beta_i = c\cdot\lambda_i\alpha_i$ or $\alpha_i = c\cdot\lambda_i^{-1}\beta_i$ with some $c\le 0$ for any $i=1,\cdots,n-1$. In summary, $\beta_n = c\cdot\lambda_n\alpha_n$ or $\alpha_n = c\cdot\lambda_n^{-1}\beta_n$.

Next, we only consider the case $\beta_i = c\cdot\lambda_i\alpha_i$ for any $i=1,\cdots,n-1$, and the same arguments apply for the case $\alpha_i = c\cdot\lambda_i^{-1}\beta_i$. For any $k=1,\cdots,n-1$, by $\beta_k = c\cdot\lambda_k\alpha_k$, it follows that

$$\begin{aligned}
w_k &= \beta_n\lambda_k^{\frac{1}{2}}\alpha_k - \lambda_n\alpha_n\lambda_k^{-\frac{1}{2}}\beta_k \\
&= (-\lambda_n)^{\frac{1}{2}}\left[(-\lambda_n)^{-\frac{1}{2}}\beta_n\lambda_k^{\frac{1}{2}}\alpha_k + (-\lambda_n)^{\frac{1}{2}}\alpha_n\lambda_k^{-\frac{1}{2}}\beta_k\right] \\
&= (-\lambda_n)^{\frac{1}{2}}\left[\left(\sum_{i=1}^{n-1}(\lambda_i^{-\frac{1}{2}}\beta_i)^2\right)^{\frac{1}{2}}\lambda_k^{\frac{1}{2}}\alpha_k + \left(\sum_{i=1}^{n-1}(\lambda_i^{\frac{1}{2}}\alpha_i)^2\right)^{\frac{1}{2}}\lambda_k^{-\frac{1}{2}}\beta_k\right] \\
&= (-\lambda_n)^{\frac{1}{2}}\left[-c\cdot\left(\sum_{i=1}^{n-1}(\lambda_i^{\frac{1}{2}}\alpha_i)^2\right)^{\frac{1}{2}}\lambda_k^{\frac{1}{2}}\alpha_k + c\cdot\left(\sum_{i=1}^{n-1}(\lambda_i^{\frac{1}{2}}\alpha_i)^2\right)^{\frac{1}{2}}\lambda_k^{\frac{1}{2}}\alpha_k\right] \\
&= 0.
\end{aligned}$$

Hence, we have $w=0$. Clearly, it leads to $x\bullet y = 0$ and the proof is complete. □

In the next part, we introduce some concepts on the monotonicity of F, which will be needed in subsequent analysis. A function $F:\mathbb{R}^n\to\mathbb{R}^n$ is said to be *monotone* if, for any $x,y\in\mathbb{R}^n$, there holds

$$\langle x-y, F(x)-F(y)\rangle \ge 0;$$

and *strictly monotone* if, for any $x\ne y$, the above inequality holds strictly; and *strongly monotone* with modulus $\rho>0$ if, for any $x,y\in\mathbb{R}^n$, the following inequality holds

$$\langle x-y, F(x)-F(y)\rangle \ge \rho\|x-y\|^2.$$

Using the monotonicity of the function F, we may achieve the following technical result for the general conic complementarity problem, which is crucial for establishing the property of bounded level sets.

Theorem 2.6. *Suppose that the general conic complementarity problem has a strictly feasible point z, i.e., $z\in\mathrm{int}(\mathcal{K})$ and $F(z)\in\mathrm{int}(\mathcal{K}^*)$ and that F is a monotone function. Then, for any sequence $\{x^k\}$ satisfying*

$$\left\|x^k\right\|\to\infty,\quad \limsup_{k\to\infty}\left\|(x^k)_-^{\mathcal{K}^*}\right\|<\infty \quad \textit{and} \quad \limsup_{k\to\infty}\left\|(-F(x^k))_+^{\mathcal{K}}\right\|<\infty,$$

we have $\left\langle x^k, F(x^k)\right\rangle\to\infty$.

Proof. Since F is monotone, for any $x^k\in\mathbb{R}^n$, we have

$$\left\langle x^k - z, F(x^k)-F(z)\right\rangle \ge 0,$$

which leads to

$$\left\langle x^k, F(x^k)\right\rangle + \langle z, F(z)\rangle \ge \left\langle x^k, F(z)\right\rangle + \left\langle z, F(x^k)\right\rangle. \tag{2.4}$$

Using properties of projection, we write $x^k = (x^k)_+^{\mathcal{K}} - (x^k)_-^{\mathcal{K}^*}$ and $F(x^k) = (-F(x^k))_-^{\mathcal{K}^*} - (-F(x^k))_+^{\mathcal{K}}$. In light of this and (2.4), we obtain

$$\begin{aligned}
&\left\langle x^k, F(x^k)\right\rangle + \langle z, F(z)\rangle \\
\ge\ &\left\langle (x^k)_+^{\mathcal{K}}, F(z)\right\rangle - \left\langle (x^k)_-^{\mathcal{K}^*}, F(z)\right\rangle + \left\langle z, (-F(x^k))_-^{\mathcal{K}^*}\right\rangle - \left\langle z, (-F(x^k))_+^{\mathcal{K}}\right\rangle.
\end{aligned} \tag{2.5}$$

By the properties of projection onto the closed convex cone, we know $x^k = (x^k)_+^{\mathcal{K}} - (x^k)_-^{\mathcal{K}^*}$. This leads to $\|(x^k)_+^{\mathcal{K}}\| \ge \|x^k\| - \|(x^k)_-^{\mathcal{K}^*}\|$. From the assumptions $\|x^k\|\to\infty$ and $\limsup_{k\to\infty}\|(x^k)_-^{\mathcal{K}^*}\|<\infty$, it follows that $\|(x^k)_+^{\mathcal{K}}\| \to \infty$. Since $(x^k)_+^{\mathcal{K}} \in \mathcal{K}$ and $F(z) \in \text{int}(\mathcal{K}^*)$, by the properties of the closed convex cone and its dual cone again, we have $\langle (x^k)_+^{\mathcal{K}}, F(z)\rangle > 0$. On the other hand, from the condition $F(z) \in \text{int}(\mathcal{K}^*)$, there is $r > 0$ such that for any $h \in \mathbb{R}^n$, $H = F(z) + rh \in \text{int}(\mathcal{K}^*)$. This yields that

$$0 < \left\langle (x^k)_+^{\mathcal{K}}, H \right\rangle = \left\langle (x^k)_+^{\mathcal{K}}, F(z) \right\rangle + r \left\langle (x^k)_+^{\mathcal{K}}, h \right\rangle.$$

Choosing $h = -\frac{(x^k)_+^{\mathcal{K}}}{\|(x^k)_+^{\mathcal{K}}\|}$ gives

$$\left\langle (x^k)_+^{\mathcal{K}}, F(z) \right\rangle \geq r \left\langle (x^k)_+^{\mathcal{K}}, \frac{(x^k)_+^{\mathcal{K}}}{\|(x^k)_+^{\mathcal{K}}\|} \right\rangle = r \left\| (x^k)_+^{\mathcal{K}} \right\| \to \infty.$$

Thus, we conclude that

$$\langle (x^k)_+^{\mathcal{K}}, F(z)\rangle \to \infty \quad \text{as} \quad k \to \infty. \tag{2.6}$$

With similar arguments, we also obtain $\langle z, (-F(x^k))_-^{\mathcal{K}^*}\rangle \geq 0$. Besides, we see that

$$\begin{aligned}
\limsup_{k\to\infty} \langle (x^k)_-^{\mathcal{K}^*}, F(z)\rangle &\leq \limsup_{k\to\infty} \|(x^k)_-^{\mathcal{K}^*}\| \|F(z)\| < \infty, \\
\limsup_{k\to\infty} \langle z, (-F(x^k))_+^{\mathcal{K}}\rangle &\leq \limsup_{k\to\infty} \|z\| \|(-F(x^k))_+^{\mathcal{K}}\| < \infty.
\end{aligned}$$

All of these results together with (2.5) and (2.6) yield

$$\left\langle x^k, F(x^k) \right\rangle \to \infty.$$

Then, the proof is complete. $\square$

Remark 2.7. We point out that Theorem 2.6 is a stronger version of [20, Proposition 2.2] and [21, Lemma 4.1], which also covers the ECCP setting.

3 Merit functions for ECCP

In this section, based on the product (2.1) of x and y associated with ellipsoidal cone $\mathcal{K}_E$, we propose several classes of merit functions for the ellipsoidal cone complementarity problem (ECCP) and investigate their favorable properties, respectively.

3.1 The first class of merit functions

In this subsection, we focus on a more general NR merit function, whose format is as bellow:

$$\Psi_\alpha(x) = \frac{1}{2} \|x - (x - \alpha F(x))_+\|^2, \qquad (\alpha > 0).$$

where $(\cdot)_+$ denotes the projection function on the ellipsoidal cone $\mathcal{K}_E$. As mentioned in Section 2, the NR merit function

$$\Psi_{\text{NR}}(x) = \frac{1}{2}\|\phi_{\text{NR}}(x, F(x))\|^2 = \frac{1}{2}\|x - (x - F(x))_+\|^2$$

serves a merit function for the ECCP. In fact, the function Ψ_α is also a merit function for the ECCP. For the general NR merit function Ψ_α, Lu and Huang have showed the property of error bound under the strong monotonicity and the global Lipschitz continuity of F in [17].

Theorem 3.1. *[17, Theorem 3.3] Suppose that F is strongly monotone with modulus $\rho > 0$ and is Lipschitz continuous with constant $L > 0$, Then for any fixed $\alpha > 0$, the following inequality holds*

$$\frac{1}{2 + \alpha L} \sqrt{\Psi_\alpha(x)} \leq \|x - x^*\| \leq \frac{1 + \alpha L}{\alpha \rho} \sqrt{\Psi_\alpha(x)},$$

where x^* is the unique solution of the general conic complementarity problems.

As mentioned in [20], for the NR merit function $\Psi_{\rm NR}$, we cannot guarantee the boundedness of the level set for the function $\phi_{\rm NR}$ under the same conditions used in Theorem 2.6. In order to establish the boundedness of the level set for the function $\phi_{\rm NR}$ or the merit function Ψ_α, we need the definition of strongly coercive property.

Definition 3.2. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} \frac{\langle F(x), x-y\rangle}{\|x-y\|} = \infty$$

holds for all $y \in \mathbb{R}^n$.

Based on the strongly coercive property of F, with the same skills as in [21, Theorem 4.2], we have the boundedness property of the level set.

Theorem 3.3. *Suppose that F is strongly coercive. Then, the level set*

$$\mathscr{L}_{\rm NR}(\gamma) = \{x \in \mathbb{R}^n \mid \|\phi_{\rm NR}(x, F(x))\| \le \gamma\}$$

or

$$\mathscr{L}_{\Psi_\alpha}(\gamma) = \{x \in \mathbb{R}^n \mid \Psi_\alpha(x) \le \gamma\}$$

is bounded for all $\gamma \ge 0$.

Proof. The proof is similar to [21, Theorem 4.2]. Hence, we omit it here. □

3.2 The second class of merit functions

In this subsection, for any $x \in \mathbb{R}^n$, we consider the LT (standing for Luo-Tseng) merit function f_{LT} associated with the ellipsoidal cone complementarity problem, whose mathematical formula is given as follows:

$$f_{LT}(x) := \varphi(\langle x, F(x)\rangle) + \frac{1}{2}\left[\|x_-\|^2 + \|(-F(x))_+\|^2\right], \tag{3.1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}_+$ is an arbitrary smooth function satisfying

$$\varphi(t) = 0,\ \forall t \le 0 \quad \text{and} \quad \varphi'(t) > 0,\ \forall t > 0.$$

The merit function was proposed by Luo and Tseng [18]. It is easy to see that $\varphi(t) \ge 0$ for all $t \in \mathbb{R}$ from the above condition. This class of functions has been considered by Tseng for the positive semidefinite complementarity problem in [29], for the second-order cone complementarity problem by Chen in [3], for the general SCCP case by Pan and Chen in [25], and for the p-order cone complementarity problem by Miao, Chang and Chen in [20], respectively. For the setting of general closed convex cone complementarity problems, the LT merit function has also been studied by Lu and Huang in [17], with some favorable properties shown as below.

Property 3.1. ([17, Lemma 3.1 and Theorem 3.4]) *Let $f_{LT} : \mathbb{R}^n \to \mathbb{R}$ be given as in (3.1). Then, the following results hold.*

(a) *For all $x \in \mathbb{R}^n$, we have $f_{LT}(x) \ge 0$; and $f_{LT}(x) = 0$ if and only if x solves the ellipsoidal cone complementarity problem.*

(b) *If $F(\cdot)$ is differentiable, then so is $f_{LT}(\cdot)$. Moreover,*

$$\nabla f_{LT}(x) = \nabla\varphi(\langle x, F(x)\rangle)[F(x) + x\nabla F(x)] - x_- - \nabla F(x)(-F(x))_+$$

for all $x \in \mathbb{R}^n$.

Theorem 3.4. ([17, Theorem 3.6]) *Let f_{LT} be given as in (3.1). Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a strongly monotone mapping and that the ellipsoidal cone complementarity problem has a solution x^*. Then, there exists a constant $\tau > 0$ such that*

$$\tau\|x - x^*\|^2 \le \max\{0, \langle x, F(x)\rangle\} + \|x_-\| + \|(-F(x))_+\|, \quad \forall x \in \mathbb{R}^n.$$

Moreover,

$$\tau\|x - x^*\|^2 \le \varphi^{-1}(f_{LT}(x)) + 2[f_{LT}(x)]^{\frac{1}{2}}, \quad \forall x \in \mathbb{R}^n.$$

The following theorem present the condition which ensures the boundedness of level sets for the LT merit function f_{LT} to solve the ellipsoidal cone complementarity problem.

Theorem 3.5. *Suppose that the ellipsoidal cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level set*

$$\mathscr{L}_{f_{LT}}(\gamma) := \{x \in \mathbb{R}^n \mid f_{LT}(x) \leq \gamma\}$$

is bounded for all $\gamma \geq 0$.

Proof. The proof is similar to [20, Theorem 3.3]. Hence, we omit it here. □

3.3 The third class of merit functions

Based on the merit function f_{LT} and the closed convex cone Ω, we make a slight modification on the LT merit function f_{LT} associated with the ellipsoidal cone complementarity problem, which leads to the third class of merit functions as follows:

$$\widehat{f_{LT}}(x) := \frac{1}{2}\|x \bullet F(x)\|^2 + \frac{1}{2}\left[\|x_-\|^2 + \|(-F(x))_+\|^2\right], \tag{3.2}$$

where $x \bullet y$ is defined as in (2.1), x_- denotes the projection of $-x$ onto $\mathcal{K}_E^*$, and $(-F(x))_+$ is the projection of $-F(x)$ onto $\mathcal{K}_E$. As shown in the following theorem, we see that the function $\widehat{f_{LT}}$ is also a type of merit functions for the ellipsoidal cone complementarity problem.

Theorem 3.6. *Let the function $\widehat{f_{LT}}$ be given as in (3.2). Then, for all $x \in \mathbb{R}^n$, we have*

$$\widehat{f_{LT}}(x) = 0 \iff x \in \mathcal{K}_E, \quad F(x) \in \mathcal{K}_E^* \quad \text{and} \quad \langle x, F(x)\rangle = 0.$$

Proof. Combining with Theorem 2.5, the proof is similar to [20, Theorem 3.4]. Hence, we omit it here. □

The following conclusions show the error bound property and the boundedness property of level sets of the merit function $\widehat{f_{LT}}$ for the ellipsoidal cone complementarity problem.

Theorem 3.7. *Let the function $\widehat{f_{LT}}$ be given as in (3.2). Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone mapping and that x^* is a solution to the ECCP. Then, there exists a scalar $\tau > 0$ such that*

$$\tau\|x - x^*\|^2 \leq \left(2 + \sqrt{2}\right)\left[\widehat{f_{LT}}(x)\right]^{\frac{1}{2}}.$$

Proof. Combining with the expression of the product $x \bullet F(x)$, the proof is similar to [20, Theorem 3.5]. Hence, we omit it here. □

Theorem 3.8. *Let the merit function $\widehat{f_{LT}}$ be given as in (3.2). Suppose that the ellipsoidal cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level set*

$$\mathscr{L}_{\widehat{f_{LT}}}(\gamma) = \left\{x \in \mathbb{R}^n \,\middle|\, \widehat{f_{LT}}(x) \leq \gamma\right\}$$

is bounded for all $\gamma \geq 0$.

Proof. The proof is similar to [20, Theorem 3.6]. Hence, we omit it here. □

Remark 3.9. If the term $x^k \bullet F(x^k)$ in the expression of $\widehat{f_{LT}}$ is replaced by $(x^k \bullet F(x^k))_+^{\mathcal{L}_\theta}$, where $(x^k \bullet F(x^k))_+^{\mathcal{L}_\theta}$ denotes the projection of $x^k \bullet F(x^k)$ onto the circular cone $\mathcal{L}_\theta$. Similar to the proof of Theorem 3.4, Theorem 3.5 and Theorem 3.6 in [20], all Theorem 3.6, Theorem 3.7 and Theorem 3.8 still hold in this paper.

3.4 The fourth class of merit function

In this subsection, in light of the product $x \bullet y$ and the NR merit function $\Psi_{\rm NR}$, we consider another merit function as below:

$$f_r(x) := \frac{1}{2}\left\|\phi_{\rm NR}(x, F(x))\right\|^2 + \frac{1}{2}\left\|x \bullet F(x)\right\|^2. \tag{3.3}$$

The following result will show that the function $f_r(x)$ is also a merit function for the ellipsoidal cone complementarity problem.

Theorem 3.10. *Let the function f_r be given as in (3.3). Then, for all $x \in {\rm I\!R}^n$, we have*

$$f_r(x) = 0 \iff x \in \mathcal{K}_E, \quad F(x) \in \mathcal{K}_E^* \quad \text{and} \quad \langle x, F(x)\rangle = 0.$$

Proof. By the definition of f_r given as in (3.3), we have

$$\begin{aligned} f_r(x) = 0 \quad &\iff \quad \|x \bullet F(x)\|^2 = 0 \;\text{ and }\; \Psi_{\rm NR}(x) = \frac{1}{2}\left\|\phi_{\rm NR}(x, F(x))\right\|^2 = 0, \\ &\iff \quad x \in \mathcal{K}_E, \; F(x) \in \mathcal{K}_E^* \;\text{ and }\; \langle x, F(x)\rangle = 0, \end{aligned}$$

where the second equivalence holds because the NR function $\Psi_{\rm NR}$ is a merit function for the ECCP. Thus, the proof is complete. □

From Theorem 3.10, it is easy to verify that if the squared exponent in the expression of f_r is deleted, i.e.,

$$\widetilde{f_r}(x) := \left\|\phi_{\rm NR}(x, F(x))\right\| + \left\|x \bullet F(x)\right\|, \tag{3.4}$$

the function $\widetilde{f_r}$ is also a merit function for the ECCP. In fact, there has no big differences between the merit functions f_r and $\widetilde{f_r}$ except the nature of f_r is better than $\widetilde{f_r}$. Next, we will establish the error bound properties for f_r and $\widetilde{f_r}$.

Theorem 3.11. *Let f_r and $\widetilde{f_r}$ be given as in (3.3) and (3.4), respectively. Suppose that $F : {\rm I\!R}^n \to {\rm I\!R}^n$ is strongly monotone mapping and that x^* is a solution to the ellipsoidal cone complementarity problem. Then, there exists a scalar $\tau > 0$ such that*

$$\tau\|x - x^*\|^2 \le 3\sqrt{2}\big[f_r(x)\big]^{\frac{1}{2}} \quad \textit{and} \quad \tau\|x - x^*\|^2 \le 2\widetilde{f_r}(x).$$

Proof. Since the product "$\bullet$" defined in (2.1) plays the same role, the proof is totally the same as the one for [20, Theorem 3.8]. Hence, we omit it here. □

Likewise, we can easily achieve the boundedness of the level sets of the functions $\widetilde{f_r}$ and f_r.

Theorem 3.12. *Let f_r and $\widetilde{f_r}$ be given as in (3.3) and (3.4), respectively. Suppose that that the ellipsoidal cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level sets*

$$\mathscr{L}_{f_r}(\gamma) = \{x \in {\rm I\!R}^n \mid f_r(x) \le \gamma\}$$

and

$$\mathscr{L}_{\widetilde{f_r}}(\gamma) = \left\{x \in {\rm I\!R}^n \;\middle|\; \widetilde{f_r}(x) \le \gamma\right\}$$

are both bounded for all $\gamma \ge 0$.

Proof. The proof is similar to [20, Theorem 3.9]. Hence, we omit it here. □

3.5 The fifth class of merit functions

In this subsection, we introduce the implicit Lagrangian merit associated with the ECCP. For any $x \in \mathbb{R}^n$ and $\alpha > 0$, the implicit Lagrangian merit function is given by

$$M_\alpha(x) := \langle x, F(x) \rangle + \frac{1}{2\alpha} \left\{ \|(x - \alpha F(x))_+\|^2 - \|x\|^2 + \|(\alpha x - F(x))_-\|^2 - \|F(x)\|^2 \right\}. \tag{3.5}$$

This class of functions was first introduced by Mangasarian and Solodov [23] for solving nonlinear complementarity problems, and was extended by Kong, Tuncel and Xiu [14] to the setting of symmetric cone complementarity problems. Moreover, for the setting of general closed convex cone complementarity problems in Hilbert space, Lu and Huang [17] further investigated this merit function. Hence, the corresponding results in [17] can be applied to the the setting of the ECCP. For completeness, as below, the error bound property of the merit function M_α for the ECCP is also presented.

Property 3.2. ([17, Theorem 3.9]) *Let M_α be given as in (3.5). Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a strongly monotone mapping with modulus $\rho > 0$ and is Lipschitz continuous with $L > 0$. Assume that the ellipsoidal cone complementarity problem has a solution x^*. Then, for any fixed $\alpha > 0$, the following inequality holds*

$$\frac{1}{(\alpha - 1)(2 + L)^2} M_\alpha(x) \le \|x - x^*\| \le \frac{\alpha(1 + L)^2}{(\alpha - 1)\rho^2} M_\alpha(x).$$

The following theorem will present the boundedness property of the level sets on the merit function M_α for solving the ECCP.

Theorem 3.13. *Suppose that the ellipsoidal cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level set*

$$\mathscr{L}_{M_\alpha}(\gamma) := \{x \in \mathbb{R}^n \mid M_\alpha(x) \le \gamma\}$$

is bounded for all $\gamma \ge 0$.

Proof. Since the merit function $M_\alpha(x)$ given as in (3.5) does not involve the product "$\bullet$" defined in (2.1), the proof is totally the same as the one for [20, Theorem 3.10]. Hence, we omit it here. □

4 Concluding Remarks

We point out that the product "$\bullet$" defined as in (2.1) and its corresponding property in Theorem 2.5 play the key in the whole paper. With them, all analysis for merit functions investigated in Section 3 is routine work. Compared to [20, 21], one may ask whether there exist unified rules for defining a "product" for general nonsymmetric cone complementarity problem. For symmetric cone complementarity problem, the so-called Jordan product "$\circ$" (see [10]) contributes to the construction of merit functions associated with symmetric cones, see [2, 3, 4, 6, 7, 8] for more details. To the contrast, there seem no such unified role for nonsymmetric cone complementarity problems, to the best of our knowledge. To see this, we provide some "observations" as below to elaborate why we think the product "$\bullet$" is related to the structure of nonsymmetric cone. This indicates that there is no unified way to define a suitable product "$\bullet$" associated with general nonsymmetric cone.

For circular cone complementarity problem, we define two types of "$\bullet$" in [21]:

$$x \bullet y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \bullet \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \langle x, y \rangle \\ \max\{\tan^2\theta, 1\}\, x_1 y_2 + \max\{\cot^2\theta, 1\}\, y_1 x_2 \end{bmatrix},$$

or

$$x \bullet y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \bullet \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \langle x, y \rangle \\ \min\{\tan^2\theta, 1\}\, x_1 y_2 + \min\{\cot^2\theta, 1\}\, y_1 x_2 \end{bmatrix},$$

where $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. It is well known that the feature of circular cone depends on the angle θ. The above two products are clearly defined in light of the angle and possess special property [21, Lemma 3.1] up to the angle θ greater than $\frac{\pi}{4}$ or less than $\frac{\pi}{4}$. That is why the proposed product enables Theorem 2.5 (respectively [21, Theorem 3.1]) valid and the merit functions can be constructed successfully.

For p-order cone complementarity problem, we define "$\bullet$" in [20] as follows:

$$x \bullet y = \begin{bmatrix} \langle x, y \rangle \\ w \end{bmatrix} \quad \text{where} \quad w := (w_2, \cdots, w_n)^T \quad \text{with} \quad w_i = |x_1|^{\frac{p}{q}} |y_i| - |y_1| |x_i|^{\frac{p}{q}},$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$. Apparently, the feature of p-order cone depends on $\| \cdot \|_p$-norm. The product in this setting is defined by exploiting the structures of p-order cone and its dual cone (an q-order cone with $\frac{1}{p} + \frac{1}{q} = 1$), which enables Theorem 2.5 (respectively [20, Proposition2.1]) valid and the merit functions can be constructed successfully.

For ellipsoidal cone complementarity problem, we define "$\bullet$" as in (2.1)

$$x \bullet y = \begin{bmatrix} w \\ \langle x, y \rangle \end{bmatrix} \quad \text{where} \quad w := (w_1, \cdots, w_{n-1})^T \quad \text{with} \quad w_i = \beta_n \lambda_i^{\frac{1}{2}} \alpha_i + \lambda_n \alpha_n \lambda_i^{-\frac{1}{2}} \beta_i.$$

As mentioned, the class of ellipsoidal cones include second-order cone, circular cone, and elliptic cone as special cases. The feature of ellipsoidal cone depends on the representation of matrix Q and its eigenvalues. In light of these, the product in this setting is defined accordingly to enable Theorem 2.5 valid. Hence, the merit functions can be constructed successfully. Moreover, looking into the above product associated with the p-order cone and the ellipsoidal cone, respectively, we observe that they have similar format. However, there exists a big difference in nature between these two products. The w_i only involves the components x_i and y_i in the product associated with p-order cone, whereas the w_i combines the whole vectors x and y in the product associated with the ellipsoidal cone because $\alpha_n := u_n^T x$ and $\beta_n := u_n^T y$. This makes the proof of Theorem 2.5 a bit harder than the other cases.

Another observation is described as follows. When $Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2 \theta \end{bmatrix}$ and $u_n = (0, 0, \cdots, 1)^T$, the product $x \bullet y$ of x and y associated with ellipsoidal cone in (2.1) reduces to the product $x \bullet y$ in the setting of the circular cone. When $Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$ with I_{n-1} being the unit matrix and $u_n = (0, 0, \cdots, 1)^T$, the product $x \bullet y$ of x and y associated with ellipsoidal cone in (2.1) reduces to the product $x \circ y$ in the setting of second-order cone. Thus, this product $x \bullet y$ is a generalization of the product $x \circ y$ associated with the circular cone or second-order cone in $\mathbb{R}^n$. Nonetheless, in dealing with problems using the product $x \bullet y$, it is much more difficult than the setting of second-order cone. Again, this is because $\alpha_n := u_n^T x$ and $\beta_n := u_n^T y$ involve the whole vector x and y in the product $x \bullet y$, while x_n is only the n-th component of x in the setting of second-order cone or circular cone or p-order cone. More specifically, for $x := (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, if $x \in \mathcal{K}^n$ (or circular cone $\mathcal{L}_\theta$ or p-order cone $\mathcal{K}_p$), the n-th component x_n satisfies $\|\bar{x}\| \leq x_n$ (or $\|\bar{x}\| \leq \tan \theta \, x_n$ or $\|\bar{x}\|_p \leq x_n$). Such are not the cases for the ellipsoidal cone, no component information of x is employed. This may lead to harder analysis when designing solution methods for ellipsoidal cone complementarity problem.

In summary, for each nonsymmetric cone complementarity problem, one needs to exploit the structure of its nonsymmetric cone to define a suitable product "$\bullet$" which makes Theorem 2.5 valid under the proposed product. Indeed, we also believe that a suitable product "$\bullet$" can be defined whenever the general cone and its dual cone share the same shape (needs more specific definitions and descriptions). Therefore, some merit functions can be constructed accordingly to deal with its corresponding complementarity problem. We leave this as our future direction.

Another direction is about the comparison of these five classes of merit functions. In general, these five classes of merit functions are based on the NR merit function or the projection function on ellipsoidal cone or its dual cone. We can guarantee the boundedness of the level set for all five merit functions under different conditions. Especially, the merit function $f_r(\cdot)$ needs weaker condition to this end. Combining with the product $x \bullet y$ of x and y, the construction of the third class of merit function follows from the second class of merit function F_{LT} and obtain a simpler expression about the error bound property for the ECCP. How about comparison in numerical side? We tried very hard to find a way to do some numerical implementations as suggested by one reviewer. It seems that there are a few suitable algorithms based on on our constructed merit functions, which we can employ to do numerical simulations. This viewpoint is true, however, there is a big hurdle in practice so far. The main difficulty lies on the expressions of spectral decomposition associated with ellipsoidal cone. Although two types of spectral decomposition associated with ellipsoidal cone are studied and provided in [16], there exist implicit expressions in some parts. Therefore, it is not possible to clearly and explicitly write out subdifferential, gradients, Jacobian, projections, etc, which are the main ingredients in coding. It definitely needs further investigation for building up the aforementioned items for practical implementations.

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