Operations Research Letters 48 (2020) 115–121

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

On construction of new NCP functions

Jan Harold Alcantara, Chen-Han Lee, Chieu Thanh Nguyen, Yu-Lin Chang, Jein-Shan Chen *

Department of Mathematics, National Taiwan Normal University, Taiwan

ARTICLE INFO

Article history:
Received 5 November 2019
Accepted 17 January 2020
Available online 28 January 2020

Keywords:
Complementarity functions
Nonlinear complementarity problem
NCP-function

ABSTRACT

We report a new method to construct complementarity functions for the nonlinear complementarity problem (NCP). Basic properties related to growth behavior, convexity and semismoothness of the newly discovered NCP functions are proved. We also present some variants, generalizations and other transformations of these NCP functions. Finally, we propose some interesting research directions that can be explored in the NCP research.

1. Motivation

Nonlinear complementarity problems (NCPs) are an important class of variational inequalities often encountered when dealing with Karush–Kuhn–Tucker conditions of optimization problems [7]. Apart from these, NCP provides an important framework for the study of equilibrium problems which usually arises from different areas such as operations research, engineering and economics [7,9,10].

Given a function $F : \mathbb{R}^n \to \mathbb{R}^n$, the problem of finding a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \text{and} \quad \langle x, F(x) \rangle = 0, \tag{1}$$

is precisely the nonlinear complementarity problem. Various approaches to solving this problem have been proposed in the past years. One class of methods utilizes a so-called NCP function, that is, a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a \geq 0, \quad b \geq 0, \quad \text{and} \quad ab = 0.$$

An NCP function is useful in solving NCP (1) as it naturally exploits the structure of the problem. In particular, defining $\Phi_{\mathrm{F}} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Phi_{\mathrm{F}}(x) = \begin{pmatrix} \phi(x_1, F_1(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix}, \tag{2}$$

it is clear to see that NCP (1) is equivalent to solving the system of equations $\Phi_{\mathrm{F}}(x) = 0$. Moreover, the NCP-function also gives rise to a merit function, namely $\Psi_{\mathrm{F}}(x) := \frac{1}{2}\|\Phi_{\mathrm{F}}(x)\|^2$. That is, the global minimizers of Ψ_{F} and the solutions of (1) coincide. Consequently, designing solution methods for solving (1) usually involves these NCP functions.

Due to their usefulness, numerous NCP functions have been proposed and extensively studied in the literature [11]. Among them, the Fischer–Burmeister (FB) function given by

$$\phi_{\mathrm{FB}}(a, b) = \sqrt{a^2 + b^2} - (a + b) \tag{3}$$

had gained significant attention and had been widely used in several studies because of its desirable numerical properties. In addition, as noted in [11], it is remarkable that several NCP functions are akin to the FB function. For instance, the generalized FB function

$$\phi^p_{\mathrm{FB}}(a, b) = \|(a, b)\|_p - (a + b), \quad p > 1 \tag{4}$$

is an interesting generalization of ϕ_{FB} which can be efficiently used in solving NCPs. Here, $\| \cdot \|_p$ denotes the l_p-norm, and the tunable parameter p has been shown to possibly improve numerical performance of some algorithms [4,6].

A general way to construct NCP functions was first given by Mangasarian in [16], and another method was formulated by Luo and Tseng [15] and Kanzow, Yamashita, and Fukushima [13]. More recently, a rigorous discussion on how to construct NCP functions was presented by Galantai in [11]. On the other hand, the purpose of this paper is to present another general method to construct NCP functions which is new to the literature. The very useful generalized FB function ϕ^p_{FB} is one among the functions that our method can generate. We also discuss some analytic properties and geometric views of the proposed functions. We present some variants and generalizations of these NCP functions, and we also suggest some possibly important extensions. Finally, we report some possible research directions that are worth exploring in the future.

* Correspondence to: Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan.

E-mail address: jschen@math.ntnu.edu.tw (J. Chen).

https://doi.org/10.1016/j.orl.2020.01.002

2. New NCP functions

In this section, we present a new method to construct continuous NCP functions. Let $\theta : \mathbb{R} \to \mathbb{R}$ be continuous and define $\phi_\theta^p : \mathbb{R}^2 \to \mathbb{R}$ as

$$\phi_\theta^p(a, b) = \|(a, b)\|_p - (\theta(b)a + \theta(a)b), \quad p \geq 1. \tag{5}$$

Note that ϕ_θ^p is a continuous symmetric function; that is, $\phi_\theta^p(a, b) = \phi_\theta^p(b, a)$. For some suitable choice of θ, the above function yields an NCP function. We divide our discussion into two cases, depending on the value of p.

2.1. The case $p = 1$

We first consider the case of $p = 1$, that is,

$$\phi_\theta^1(a, b) = |a| + |b| - (\theta(b)a + \theta(a)b). \tag{6}$$

Proposition 2.1. *Let $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(0) = 1$, $\theta(t) > 1$ for all $t > 0$, and $-1 < \theta(t) < 1$ for all $t < 0$. Then, ϕ_θ^1 defined by (6) is an NCP function. Moreover, $\phi_\theta^1(a, b) \leq 0$ if and only if $(a, b) \in \mathbb{R}^2_+$.*

Proof. Observe that we may rewrite ϕ_θ^1 as

$$\phi_\theta^1(a, b) = a\big(\mathrm{sgn}(a) - \theta(b)\big) + b\big(\mathrm{sgn}(b) - \theta(a)\big),$$

where

$$\mathrm{sgn}(t) := \begin{cases} 1 & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t = 0, \\ -1 & \text{if} \quad t < 0. \end{cases}$$

Then, it is easy to verify that

$$\phi_\theta^1(a, b) = \begin{cases} 0 & \text{if} \quad a, b \geq 0 \;\&\; ab = 0, \\ a(1 - \theta(b)) + b(1 - \theta(a)) & \text{if} \quad a > 0 \;\&\; b > 0, \\ -a(1 + \theta(b)) + b(1 - \theta(a)) & \text{if} \quad a < 0 \;\&\; b \geq 0, \\ -a(1 + \theta(b)) - b(1 + \theta(a)) & \text{if} \quad a < 0 \;\&\; b < 0. \end{cases} \tag{7}$$

By our hypotheses on θ, we see that $\phi_\theta^1(a, b) < 0$ for the second case, and $\phi_\theta^1(a, b) > 0$ for the third and last cases. Finally, by symmetry of ϕ_θ^1, we have $\phi_\theta^1(a, b) > 0$ when $a > 0$ and $b < 0$ as in the third case. In other words, $\phi_\theta^1(a, b) = 0$ if and only if $a, b \geq 0$ and $ab = 0$. This says that ϕ_θ^1 is an NCP function. □

An important consequence of Proposition 2.1 is given by the following result, which describes the growth behavior of the NCP function ϕ_θ^1. This corollary plays an important role in establishing coerciveness of Φ_{F} given by (2) (see [8]), which in turn is helpful in convergence analysis of algorithms. We omit the proof of the following corollary since it easily follows from the formula of ϕ_θ^1 given in (7). We do note that the strict inequality assumptions on the limits of θ at $\pm\infty$ are important to avoid indeterminate products.

Corollary 2.1. *Let θ satisfy the hypothesis of Proposition 2.1 such that $\lim_{t\to\infty} \theta(t) > 1$ and $-1 < \lim_{t\to-\infty} \theta(t) < 1$. Then, $|\phi_\theta^1(a^k, b^k)| \to \infty$ as $k \to \infty$ for any sequence $\{(a^k, b^k)\} \subseteq \mathbb{R}^2$ with $|a^k| \to \infty$ and $|b^k| \to \infty$.*

In the remaining parts of the paper, we assume that θ satisfies the conditions in Proposition 2.1 whenever $p = 1$. Note that a simple choice of θ is any monotonically increasing function whose range is contained in $(-1, \infty)$, passes through $(0, 1)$, and is strictly monotonic in some neighborhood of 0.

Example 2.1. The functions $\theta_1(t) = e^t$, $\theta_2(t) = \frac{\sqrt{t^2+4}+t}{2}$, and $\theta_3(t) = \frac{2}{1+e^{-t}}$ clearly satisfy the conditions of Proposition 2.1 and Corollary 2.1. The graphs of $\phi_{\theta_i}^1(a, b)$ for $i = 1, 2, 3$ are shown in Figs. 1(a), 2(a), and 3(a). For each i, it is evident that the function $\phi_{\theta_i}^1$ is non-positive on $\mathbb{R}^2_+$ and has the growth behavior as described in Corollary 2.1. In addition, $\phi_{\theta_i}^1$ is a nonsmooth nonconvex function for all i. In particular, the function has sharp trace curves corresponding to $a = 0$ and $b = 0$, which are the points of non-differentiability of ϕ_θ^1.

2.2. The case $p > 1$

Now, we consider ϕ_θ^p with $p > 1$ and provide conditions on θ which will make ϕ_θ^p an NCP function. The conditions are almost similar to those given in Proposition 2.1. However, we do not require strict inequality at $t = 1$, but we need a higher lower bound for $\theta(t)$ on $(-\infty, 0)$.

Proposition 2.2. *Let $p > 1$. Suppose $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(0) = 1$, $\theta(t) \geq 1$ for all $t > 0$, and $-2^{\frac{1-p}{p}} \leq \theta(t) \leq 1$ for all $t < 0$. Then, ϕ_θ^p defined by (5) is an NCP function. Moreover, $\phi_\theta^p(a, b) \leq 0$ if and only if $(a, b) \in \mathbb{R}^2_+$.*

Proof. Since ϕ_θ^p is symmetric w.r.t. the line $a = b$, it suffices to check the values of ϕ_θ^p on the region $a \leq b$. We carefully consider four cases.

(i) If $a = 0$ and $b > 0$, then $\phi_\theta^p(a, b) = |b| - \theta(0)b = 0$ since $\theta(0) = 1$.

(ii) Suppose $a > 0$ and $b > 0$. Due to $p > 1$, we have $\|(a, b)\|_p = (a^p + b^p)^{\frac{1}{p}} < a + b$ which in turn yields

$$\phi_\theta^p(a, b) < a + b - (\theta(b)a + \theta(a)b) = a(1 - \theta(b)) + b(1 - \theta(a)).$$

Because $\theta(t) \geq 1$ for any $t > 0$ it follows that $\phi_\theta^p(a, b) < 0$.

(iii) Suppose $a < 0$ and $b \geq 0$. In this case, we have that $\|(a, b)\|_p > a + b$. Thus,

$$\phi_\theta^p(a, b) > a + b - (\theta(b)a + \theta(a)b) = a(1 - \theta(b)) + b(1 - \theta(a)).$$

Since $b \geq 0$, we have $1 - \theta(b) \leq 0$ and so the term $a(1 - \theta(b))$ is nonnegative. On the other hand, $1 - \theta(a) > 0$ since $a < 0$ which means that the term $b(1 - \theta(a))$ is likewise nonnegative. Hence, $\phi_\theta^p(a, b) > 0$.

(iv) Finally, suppose that $a < 0$ and $b < 0$. The function $t \mapsto t^p$ is strictly convex on $[0, \infty)$ since $p > 1$. Thus,

$$\|(a, b)\|_p^p = |a|^p + |b|^p > 2^{1-p}(|a| + |b|)^p,$$

which implies that $\|(a, b)\|_p > 2^{\frac{1-p}{p}}(|a| + |b|) = -2^{\frac{1-p}{p}}(a + b)$. Consequently,

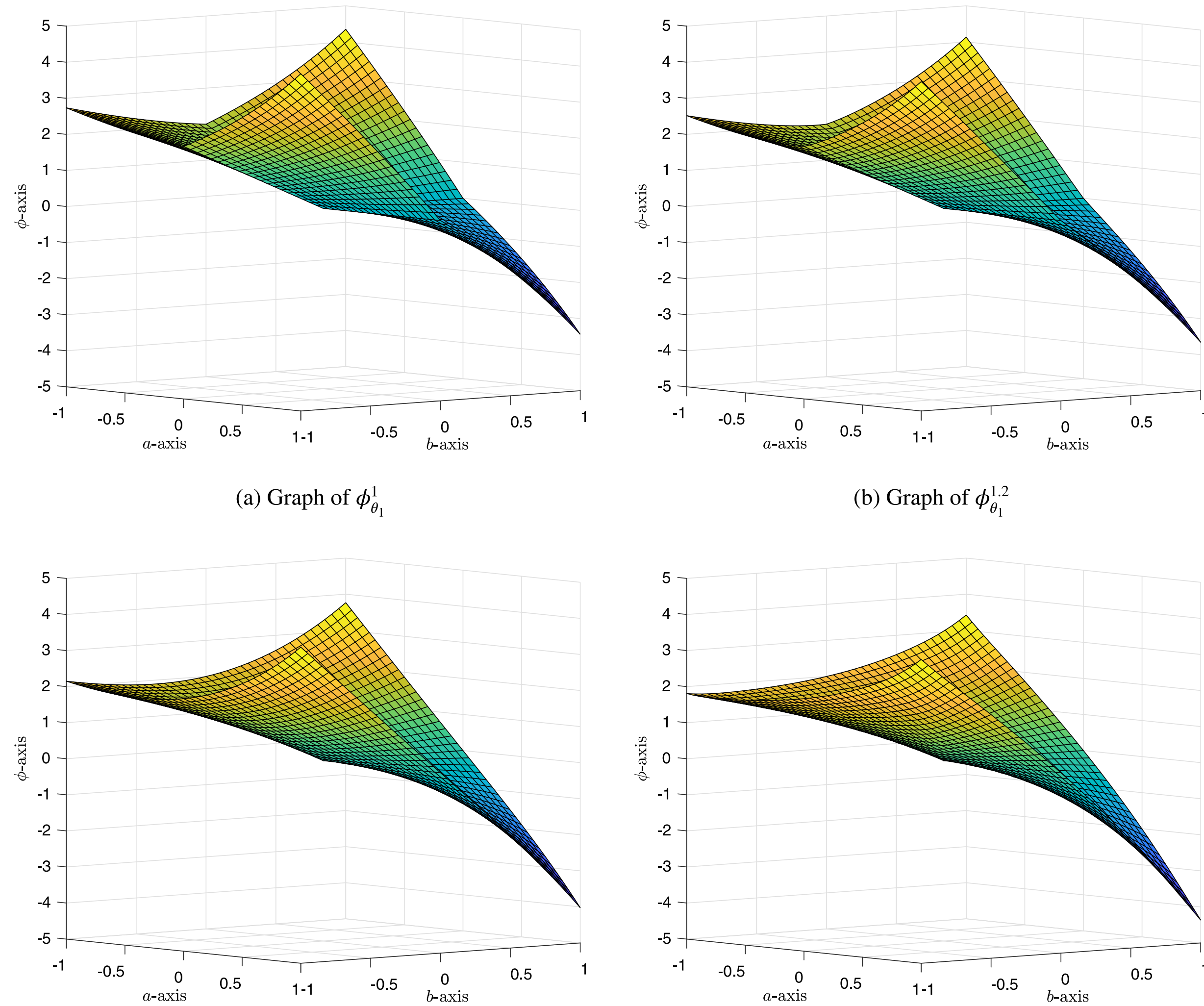
$$\begin{aligned} \phi_\theta^p(a, b) &> -2^{\frac{1-p}{p}}(a + b) - (\theta(b)a + \theta(a)b) \\ &= -a(2^{\frac{1-p}{p}} + \theta(b)) - b(2^{\frac{1-p}{p}} + \theta(a)) \\ &\geq 0 \end{aligned}$$

where the last inequality follows from the assumption that $\theta(t) \geq -2^{\frac{1-p}{p}}$ for all $t \leq 0$.

From the above four cases, it is clear that $\phi_\theta^p(a, b) \leq 0$ only on $\mathbb{R}^2_+$. This completes the proof. □

We now state a consequence of (5), similar to Corollary 2.1.

Corollary 2.2. *Let θ satisfy the hypothesis of Proposition 2.2 such that $\lim_{t\to\infty} \theta(t) > 1$ and $-2^{\frac{1-p}{p}} < \lim_{t\to-\infty} \theta(t) < 1$. Then, $|\phi_\theta^p(a^k, b^k)| \to \infty$ as $k \to \infty$ for any sequence $\{(a^k, b^k)\} \subseteq \mathbb{R}^2$ with $|a^k| \to \infty$ and $|b^k| \to \infty$.*



(a) Graph of $\phi^{1}_{\theta_1}$

(b) Graph of $\phi^{1.2}_{\theta_1}$

(c) Graph of $\phi^{2}_{\theta_1}$

(d) Graph of $\phi^{10}_{\theta_1}$

Fig. 1. Graphs of $\phi^p_{\theta_1}$ for different values of p where $\theta_1(t) = e^t$.

Proof. The result follows from the inequalities obtained from cases (ii), (iii) and (iv) in the proof of Proposition 2.2. $\square$

Whenever $p > 1$, we always assume that θ satisfies the conditions of Proposition 2.2 for the remaining parts of the paper. We now show some examples.

Example 2.2. Observe that by taking $\theta(t) \equiv 1$, we obtain the generalized FB function (4). Hence, the family of NCP functions given by (5) subsumes the class of generalized FB functions.

Example 2.3. As in Example 2.1, consider θ_i for $i = 1, 2, 3$. Then, for any $p > 1$, the function $\phi^p_{\theta_i}$ is an NCP function by Proposition 2.2. Notice from Figs. 1–3 (subfigures (b)–(d)) that the graphs of $\phi^p_{\theta_i}$ $(p > 1)$ look "smoother" than that of $\phi^1_{\theta_i}$. In particular, ϕ^p_θ is not differentiable only at the origin. Finally, $\phi^p_{\theta_i}$ is also nonconvex similar to $\phi^1_{\theta_i}$ in Example 2.1.

It is known that the differentiability and convexity of any complementarity function cannot be held simultaneously [12,18]. Nonetheless, it could be neither differentiable nor convex. The below two propositions indicate that this is the case for ϕ^p_θ. First, as we have observed from Examples 2.1 and 2.3, ϕ^p_θ is not convex. We claim that this is indeed the case in general.

Proposition 2.3. *Suppose that θ is strictly increasing on some interval $I = [0, t_0)$. Then, ϕ^p_θ is not convex.*

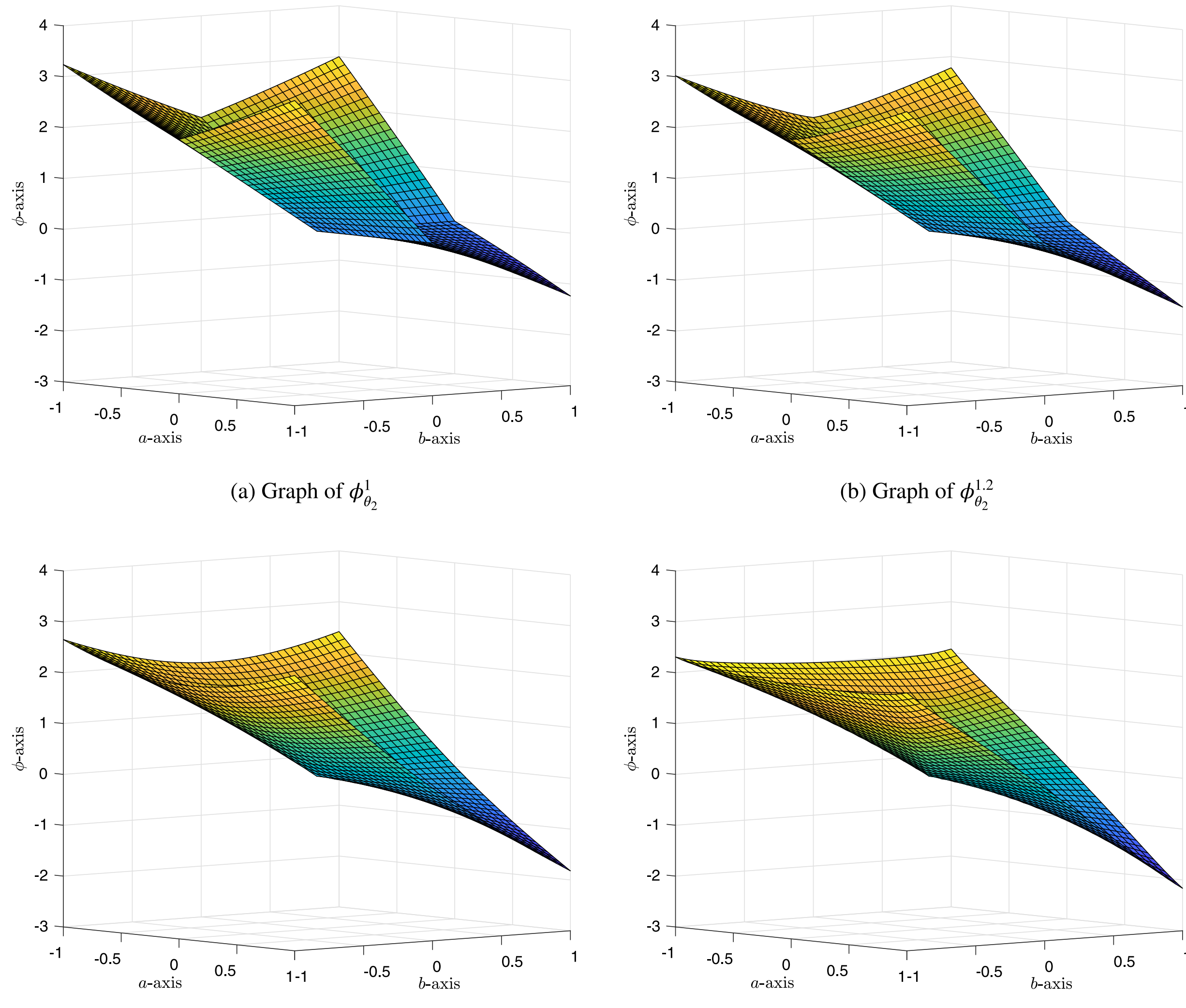
Proof. Suppose that ϕ^p_θ is convex, due to $\phi^p_\theta(0, 0) = 0$, it must be the case that $\phi^p_\theta(\lambda a, \lambda b) \le \lambda \phi^p_\theta(a, b)$ for any $\lambda \in [0, 1]$ and any $u, v \in \mathbb{R}$. Taking any $a, b \in I$ yields

$$\begin{aligned}
&\phi^p_\theta(\lambda a, \lambda b) - \lambda \phi^p_\theta(a, b) \\
&= \|(\lambda a, \lambda b)\|_p - (\lambda\theta(\lambda b)a + \lambda\theta(\lambda a)b) - \lambda(\|(a, b)\|_p \\
&\quad - (\theta(b)a + \theta(a)b)) \\
&= \lambda a(\theta(b) - \theta(\lambda b)) + \lambda b(\theta(a) - \theta(\lambda a)).
\end{aligned}$$

Since $\lambda \in [0, 1]$, we have that $\lambda a, \lambda b \in I$. By the strict monotonicity assumption on θ in I, there has $\phi^p_\theta(\lambda a, \lambda b) - \lambda \phi^p_\theta(a, b) > 0$. Hence, ϕ^p_θ is not convex. $\square$

We close this section by showing the semismoothness of ϕ^p_θ. The concept of semismoothness was introduced by Mifflin [17] for functionals, and was later extended by Qi and Sun [19] for vector-valued functions.

Proposition 2.4. *Suppose that θ is continuously differentiable and satisfies the conditions of Proposition 2.1 if $p = 1$ or Proposition 2.2*



(a) Graph of $\phi^1_{\theta_2}$

(b) Graph of $\phi^{1.2}_{\theta_2}$

(c) Graph of $\phi^2_{\theta_2}$

(d) Graph of $\phi^{10}_{\theta_2}$

Fig. 2. Graphs of $\phi^p_{\theta_2}$ for different values of p where $\theta_2(t) = \frac{\sqrt{t^2+4}+t}{2}$.

if $p > 1$. Then, ϕ^p_θ is semismooth. Moreover, the generalized gradient of ϕ^1_θ is described by

$$\partial\phi^1_\theta(a,b) = \begin{cases} \{\ [\mathrm{sgn}(a) - \theta'(a)b - \theta(b),\ \mathrm{sgn}(b) - \theta'(b)a - \theta(a)]^T\ \} & \text{if } a \neq 0 \ \&\ b \neq 0 \\ \{\ [0,\ 2\lambda - 1 - a\theta'(0) - \theta(a)]^T\ :\ \lambda \in [0,1]\ \} & \text{if } a > 0 \ \&\ b = 0 \\ \{\ [2\lambda - 1 - b\theta'(0) - \theta(b),\ 0]^T\ :\ \lambda \in [0,1]\ \} & \text{if } a = 0 \ \&\ b > 0 \\ \{\ [-2,\ 2\lambda - 1 - a\theta'(0) - \theta(a)]^T\ :\ \lambda \in [0,1]\ \} & \text{if } a < 0 \ \&\ b = 0 \\ \{\ [-2,\ 2\lambda - 1 - b\theta'(0) - \theta(b)]^T\ :\ \lambda \in [0,1]\ \} & \text{if } a = 0 \ \&\ b < 0 \\ \{\ [\xi, \zeta]^T\ :\ \xi, \zeta \in [-2, 0]\ \} & \text{if } a = b = 0 \end{cases}$$

and for $p > 1$, we have

$$\partial\phi^p_\theta(a,b) = \begin{cases} \left\{\ \left[\frac{\mathrm{sgn}(a)|a|^{p-1}}{\|(a,b)\|_p^{1-p}} - \theta(b) - b\theta'(a),\ \frac{\mathrm{sgn}(b)|b|^{p-1}}{\|(a,b)\|_p^{1-p}} - \theta(a) - a\theta'(b)\right]^T\ \right\} \\ \qquad \text{if } (a,b) \neq (0,0) \\ \left\{[\xi - 1, \zeta - 1]^T\ :\ |\xi|^{\frac{p}{p-1}} + |\zeta|^{\frac{p}{p-1}} \leq 1\right\} \\ \qquad \text{if } a = b = 0. \end{cases}$$

Proof. Note that the mapping $f : (a,b) \mapsto \|(a,b)\|_p$ is a convex map and is therefore semismooth. Because $g : (a,b) \mapsto -(\theta(b)a + \theta(a)b)$ is smooth (and hence semismooth), their sum $f + g = \phi^p_\theta$ is semismooth. Now, we compute the generalized gradient of ϕ^1_θ. It is clear that ϕ^1_θ is differentiable only on $D := \{(a,b)\ :\ a \neq 0 \text{ and } b \neq 0\}$. Then, its gradient

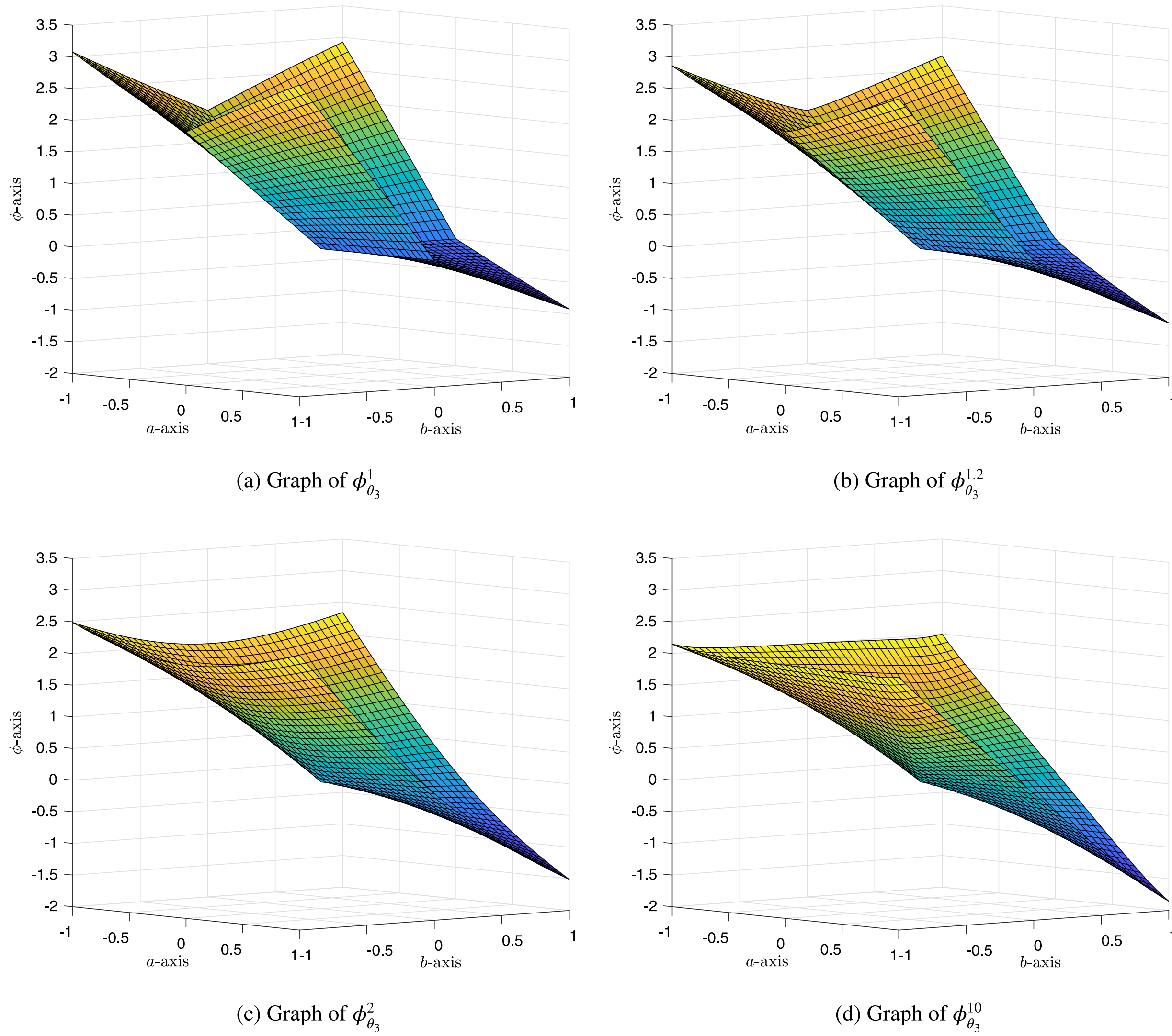
$$\nabla\phi^1_\theta(a,b) = \begin{bmatrix} \mathrm{sgn}(a) - \theta'(a)b - \theta(b) \\ \mathrm{sgn}(b) - \theta'(b)a - \theta(a) \end{bmatrix} \quad \forall (a,b) \in D,$$

coincides with the generalized gradient on D. Suppose then that $(a,b) \notin D$. First, we consider the case when $a > 0$ and $b = 0$. By definition of Clarke's generalized gradient $\partial\phi^1_\theta(a,b) = \mathrm{conv}\left(\partial_B\phi^1_\theta(a,b)\right)$, i.e., the convex hull of the B-subdifferential

$$\partial_B\phi^1_\theta(a,b) = \Big\{g \in \mathbb{R}^2 \mid \exists\{(a_k,b_k)\}_{k=1}^\infty \subseteq D \text{ s.t. } (a_k,b_k) \to (a,b) \text{ and } \nabla\phi^1_\theta(a_k,b_k) \to g\Big\}.$$

Let $\{(a_k,b_k)\}_{k=1}^\infty \subseteq D$ such that $(a_k,b_k) \to (a,0)$. For all sufficiently large k, we have $a_k > 0$. If $b_k > 0$ for all k sufficiently large, then

$$\begin{aligned} \lim_{k\to\infty} \nabla\phi^1_\theta(a_k,b_k) &= \lim_{k\to\infty} \begin{bmatrix} \mathrm{sgn}(a_k) - \theta'(a_k)b_k - \theta(b_k) \\ \mathrm{sgn}(b_k) - \theta'(b_k)a_k - \theta(a_k) \end{bmatrix} \\ &= \begin{bmatrix} 1 - \theta'(a)\cdot 0 - \theta(0) \\ 1 - \theta'(0)\cdot a - \theta(a) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 - a\theta'(0) - \theta(a) \end{bmatrix}, \end{aligned}$$



(a) Graph of $\phi^1_{\theta_3}$

(b) Graph of $\phi^{1.2}_{\theta_3}$

(c) Graph of $\phi^2_{\theta_3}$

(d) Graph of $\phi^{10}_{\theta_3}$

Fig. 3. Graphs of $\phi^p_{\theta_3}$ for different values of p where $\theta_3(t) = \frac{2}{1+e^{-t}}$.

where we used the fact that θ is continuously differentiable and that $\theta(0) = 1$. If $b_k < 0$ for all k sufficiently large, then

$$\lim_{k\to\infty} \nabla\phi^1_\theta(a_k, b_k) = \left[\begin{array}{c} 1-\theta'(a)\cdot 0-\theta(0) \\ -1-\theta'(0)\cdot a-\theta(a) \end{array}\right] = \left[\begin{array}{c} 0 \\ -1-a\theta'(0)-\theta(a) \end{array}\right].$$

In other cases, $\nabla\phi^1_\theta(a_k, b_k)$ has no limit. Hence,

$$\partial_B\phi^1_\theta(a, 0) = \left\{[0, 1-a\theta'(0)-\theta(a)]^T, [0, -1-a\theta'(0)-\theta(a)]^T\right\}$$

and the result for the case $a > 0$ and $b = 0$ follows by taking the convex hull. We omit the proof of the other cases as the arguments are similar. Finally, note that ϕ^p_θ is differentiable on $\mathbb{R}^2$ except at $(0, 0)$. The computation of the generalized gradient $\phi^p_\theta(0, 0)$ is similar to the computation of $\partial\phi^p_{\text{FB}}(0, 0)$ shown as in [3]. This completes the proof. □

3. Some extensions

In this section, we discuss some variants and generalizations of ϕ^p_θ. We also suggest some specific functions which can be used to derive new NCP functions from old ones. To proceed, we denote by t^+ the projection onto $[0, \infty)$, i.e.,

$$t^+ := \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

For convenience, we define $\hat{\phi}^{p,i}_\theta$ for $i = 1, 2, 3$ as follows:

$$\hat{\phi}^{p,1}_\theta(a, b) = \phi^p_\theta(a, b) - \alpha a^+ b^+$$
$$\hat{\phi}^{p,2}_\theta(a, b) = \phi^p_\theta(a, b) - \alpha(a^+ b^+)^2$$
$$\hat{\phi}^{p,3}_\theta(a, b) = \phi^p_\theta(a, b) - \alpha(a^+)^2(b^+)^2$$

where $\alpha > 0$. For any $p \geq 1$ and $(a, b) \in \mathbb{R}^2_{++}$, we know from Propositions 2.1 and 2.2 that $\hat{\phi}^{p,i}_\theta(a, b) < 0$. Moreover, $\hat{\phi}^{p,i}_\theta(a, b) = \phi^p_\theta(a, b) > 0$ for all $(a, b) \notin \mathbb{R}^2_+$. Consequently, these three variants are easily to be seen as NCP functions as well.

Proposition 3.1. *The functions $\hat{\phi}^{p,i}_\theta$ are all NCP functions for any $\alpha > 0$ and $i = 1, 2, 3$.*

Recently, "continuous" and "discrete" generalizations of NCP functions have gained some attention, see [2,3,5]. These generalizations involve a tunable parameter q, which have been shown to play important role in achieving better numerical performance of some NCP functions-based algorithms [1,4,6]. Moreover, the extension results to an NCP function with possibly different analytic properties [2,5]. For instance, the generalized FB function (4) is considered a continuous generalization of the FB function (3) in the sense that p takes on values from the interval $(1,\infty)$, and the FB function can be obtained by taking $p=2$. On the other hand, discrete generalizations have also been studied recently. For instance, the natural residual (NR) function

$$\phi_{\mathrm{NR}}(a,b)=\min\{a,b\}=a-(a-b)^+$$

is another popular NCP function apart from the FB function. A discrete generalization of this function proposed in [5] is given by

$$\phi^q_{\mathrm{NR}}(a,b)=a^q-[(a-b)^+]^q \tag{8}$$

where q is a positive odd integer. For $q=1$, the above function reduces to the original NR function. The generalization is "discrete" in the sense that q can only take on positive odd integral values. An interesting property of the generalized NR function (8) is that it possesses twice differentiability for $q>3$, which is not the case for the NR function. This makes ϕ^q_{NR} suitable for algorithms needing differentiability.

We wish to point out that the technique employed in the second type of generalization discussed above can always be adopted for NCP functions of the form

$$\phi(a,b)=\bar{\phi}_1(a,b)-\bar{\phi}_2(a,b). \tag{9}$$

In other words, the function

$$\phi^q(a,b):=[\bar{\phi}_1(a,b)]^q-[\bar{\phi}_2(a,b)]^q$$

is always a discrete generalization of ϕ given in (9), where q is a positive odd integer. As a matter of fact, we can further extend such technique by considering any family of injective functions $\{f_q\}$. More precisely, the function

$$\phi_{f_q}(a,b):=f_q(\bar{\phi}_1(a,b))-f_q(\bar{\phi}_2(a,b)) \tag{10}$$

is easily seen to be an NCP function whenever f_q is injective and ϕ is an NCP function as in (9). The transformation (10) has also been noted in [11]. For instance, the discrete generalized NR function (8) can be realized by transforming the NR function as in (10) using the map $f_q(t)=t^q$, where $q>0$ is an odd integer. Applying the same map to our NCP function ϕ^p_θ, we obtain a discrete generalization as

$$(\phi^p_\theta)^q:=\|(a,b)\|^q_p-(\theta(b)a+\theta(a)b)^q, \tag{11}$$

where q is a positive odd integer. As mentioned above, a generalization can possibly yield NCP functions with different analytic properties. In the case of (11), it is easy to verify that $(\phi^p_\theta)^q$ is continuously differentiable on $\mathbb{R}^2$ whenever $q\ge p>1$, whereas the original function ϕ^p_θ is not differentiable at the origin.

Another discrete generalization of ϕ^p_θ can be obtained by applying the same map $f_q(t)=t^q$ to the equivalent form of ϕ^p_θ given by

$$\phi^p_\theta(a,b)=\phi^p_{\mathrm{FB}}(a,b)-\big[a(\theta(b)-1)+b(\theta(a)-1)\big]. \tag{12}$$

This yields another symmetric generalization

$$(\phi^p_\theta)^q_{\mathrm{FB}}(a,b)=[\phi^p_{\mathrm{FB}}(a,b)]^q-\big[(a(\theta(b)-1)+b(\theta(a)-1))\big]^q.$$

For $q=1$, note that Proposition 2.4 guarantees the semismoothness of ϕ^p_θ. Interestingly, the above generalization yields smooth NCP functions for any $p>1$ and odd integers $q\ge 3$. This can be easily verified and we omit proof. We summarize these results in Proposition 3.2. Note that the above generalizations are all symmetric. In general, the transformation given in (10) yields symmetric NCP functions when applied to our proposed NCP function ϕ^p_θ and its alternative form (12).

Proposition 3.2. *Suppose θ is continuously differentiable and satisfies the conditions of Proposition 2.1 if $p=1$, or Proposition 2.2 if $p>1$. Let $q\ge 1$ be an odd integer. Then,*

$$(\phi^p_\theta)^q(a,b):=\|(a,b)\|^q_p-\big(\theta(b)a+\theta(a)b\big)^q$$

is a discrete generalization of ϕ^p_θ, which is smooth if $q\ge p>1$. Additionally,

$$(\phi^p_\theta)^q_{\mathrm{FB}}(a,b):=[\phi^p_{\mathrm{FB}}(a,b)]^q-[(a(\theta(b)-1)+b(\theta(a)-1))]^q$$

is also a discrete generalizations of ϕ^p_θ, which is smooth if $q\ge 3$ and $p>1$

It is interesting to note that $f_q(t)=t^q$ with $q\ge 1$ an odd integer is one of the functions usually employed in order to improve numerical performance of algorithms. This is referred to as an "activation function" in the literature on neural network approach for optimization. Such a function is often utilized to improve convergence rate, and other examples are given as follows:

1. Bipolar Sigmoid Function [20,21]
$$f_q(t)=\frac{1-e^{-qt}}{1+e^{-qt}},\quad q>0.$$
2. Power-Sigmoid Function [20,21]
$$f_q(t)=\begin{cases}\frac{1+e^{-q_1}}{1-e^{-q_1}}\cdot\frac{1-e^{-q_1 t}}{1+e^{-q_1 t}} & \text{if } |t|<1\\ t^{q_2} & \text{if } |t|\ge 1\end{cases}$$
where $q=(q_1,q_2)$, $q_1>2$ and $q_2\ge 3$ is an odd integer.
3. Smooth Power-Sigmoid Function [20,21]
$$f_q(t)=\frac{1}{2}\cdot\frac{1+e^{-q_1}}{1-e^{-q_1}}\cdot\frac{1-e^{-q_1 t}}{1+e^{-q_1 t}}+\frac{1}{2}t^{q_2}$$
where $q=(q_1,q_2)$, $q_1>2$ and $q_2\ge 3$.
4. Sign-Bi-Power Function [14]
$$f_q(t)=\begin{cases}|t|^q+|t|^{\frac{1}{q}} & \text{if } t>0\\ 0 & \text{if } t=0\\ -|t|^q-|t|^{\frac{1}{q}} & \text{if } t<0\end{cases},\quad q>0.$$

These functions are all injective maps which can be employed to transform an NCP function of the form (9). However, none of these transformations lead to a generalization in the sense illustrated above. Indeed, a generalized version can only be obtained if there exists $\bar{q}$ such that $f_{\bar{q}}(t)\equiv t$. We do note, however, that $\frac{f_q}{2}$ yields a continuous generalization via the transformation (10) if f_q is the sign-bi-power function. In any case, an interesting research direction is to explore the applicability of the above injective functions in improving numerical efficiency of NCP functions-based solution methods, just as how these functions improve numerical performance in neural network approaches. In the case of the power function $f_q(t)=t^q$ and the generalized NR function, some numerical results are reported in [1]. Finally, we note that it is also worth considering in numerical implementations the composite map $f_q\circ\phi^p_\theta$. This is also an NCP function provided that f_q is injective with $f_q(0)=0$ such as the above four activation functions.

4. Concluding remarks

In this short paper, we proposed a new way to construct NCP functions. The family of generalized FB functions, in particular, can be generated from the proposed approach. We proved herein some basic properties of the newly discovered NCP function, which includes the growth behavior, nonconvexity and semismoothness of ϕ_θ^p. These are prerequisite to designing solution methods based on the new NCP function.

Observe that for a fixed θ, the NCP function ϕ_θ^p is parametrized by $p \geq 1$. Future research directions can explore the effects of tuning the parameter p in the performance of algorithms. This is worth exploring as it has been shown that for the case of the generalized FB and NR functions, better convergence rates of solution methods can be attained by controlling the values of p [1,4,6]. Numerical comparisons of these new NCP functions with popular NCP functions such as the FB and NR function are recommended. How to best choose the parameter p and the function θ are some topics that are worth venturing, as this could suggest alternative NCP functions that can work well with algorithms. Finally, it seems worthwhile to explore the effects of choosing different activation functions f_q such as the bipolar sigmoid function, power-sigmoid function, smooth power-sigmoid function, and the sign-bi-power function, in forming new NCP functions from old ones such as $(\phi_\theta^p)_{f_q}$ and $f_q \circ \phi_\theta^p$. We leave it for future research to study whether or not these transformations can be used to improve numerical performance of an NCP function-based algorithm. If these functions indeed improve some algorithms, it is recommended to determine which one of these will work best for the complementarity problem.

Acknowledgment

JS Chen is supported by Ministry of Science and Technology, Taiwan.

References

[1] J.H. Alcantara, J.-S. Chen, Neural networks based on three classes of NCP-functions for solving nonlinear complementarity problems, Neurocomputing 359 (2019) 102–113.

[2] Y.-L. Chang, J.-S. Chen, C.-Y. Yang, Symmetrization of generalized natural residual function for NCP, Oper. Res. Lett. 43 (2015) 354–358.

[3] J.-S. Chen, On some NCP-functions based on the generalized Fischer–Burmeister function, Asia-Pac. J. Oper. Res. 24 (2007) 401–420.

[4] J.-S. Chen, C.-H. Ko, S.-H. Pan, A neural network based on generalized Fischer–Burmeister function for nonlinear complementarity problems, Inform. Sci. 180 (2010) 697–711.

[5] J.-S. Chen, C.-H. Ko, X.-R. Wu, What is the generalization of natural residual function for NCP? Pac. J. Optim. 12 (2016) 19–27.

[6] J.-S. Chen, S.-H. Pan, A family of NCP functions and a descent method for the nonlinear complementarity problem, Comput. Optim. Appl. 40 (2008) 389–404.

[7] F. Facchinei, J.-S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. I, II, Springer-Verlag, New York, 2003.

[8] F. Facchinei, J. Soares, A new merit function for nonlinear complementarity problems and a related algorithm, SIAM J. Optim. 7 (1997) 225–247.

[9] M.C. Ferris, O.L. Mangasarian, J.-S. Pang (Eds.), Complementarity: Applications, Algorithms and Extensions, Kluwer Academic Publishers, Dordrecht, 2001.

[10] M.C. Ferris, J.-S. Pang, Engineering and economic applications of complementarity problems, SIAM Rev. 39 (1997) 669–713.

[11] A. Galantai, Properties and construction of NCP functions, Comput. Optim. Appl. 52 (2012) 805–824.

[12] C.-H. Huang, J.-S. Chen, J.E. Martinez-Legaz, Differentiability v.s. convexity of complementarity functions, Optim. Lett. 11 (1) (2017) 209–216.

[13] C. Kanzow, N. Yamashita, M. Fukushima, New NCP-functions and their properties, J. Optim. Theory Appl. 94 (1997) 115–135.

[14] S. Li, S. Chen, B. Liu, Accelerating a recurrent neural network to finite-time convergence for solving time-varying sylvester equation by using a sign-bi-power activation function, Neural Process Lett. 37 (2013) 189–205.

[15] Z.Q. Luo, P. Tseng, A new class of merit functions for the nonlinear complementarity problem, in: M.C. Ferris, J.S. Pang (Eds.), Complementarity and Variational Problems: State of the Art, SIAM, Philadelphia, 1997, pp. 204–225.

[16] O.L. Mangasarian, Equivalence of the complementarity problem to a system of nonlinear equations, SIAM J. Appl. Math. 31 (1976) 89–92.

[17] R. Mifflin, Semismooth and semiconvex function in constrained optimization, SIAM J. Control Optim. 15 (1977) 959–972.

[18] S.M. Miri, S. Effati, On generalized convexity of nonlinear complementarity functions, J. Optim. Theory Appl. 164 (2015) 723–730.

[19] L. Qi, J. Sun, A nonsmooth version of Newton's method, Math. Program. 58 (1993) 353–367.

[20] Y. Zhang, Z. Fan, Z. Li, Zhang neural network for online solution of time-varying sylvester equation, in: Proceedings of the 2nd International Conference on Advances in Computation and Intelligence, ISICA'07, Springer, Berlin, pp. 276–285.

[21] Y. Zhang, S.S. Ge, Design and analysis of a general recurrent neural network model for time-varying matrix inversion, IEEE Trans. Neural Netw. 16 (2005) 1477–1490.