Addendum to the proof of $\phi_8(\mu, x) \le \phi_4(\mu, x)$ on page 545

Neural network based on systematically generated smoothing functions for absolute value equation

Journal of Applied Mathematics and Computing, vol. 61, pp. 533–558, 2019.

The expression of $g'(y)$ on page 545 is wrong, there is a squared-term missing in the denominator of the first term. Accordingly, here is the revised proof of $\phi_8(\mu, x) \le \phi_4(\mu, x)$ in the Part(6) on page 545.

Part (6): $\phi_8(\mu, x) \le \phi_4(\mu, x)$. Consider

$$f(y) = \log\left(1 + e^{-y}\right) + \log\left(1 + e^{y}\right) - y\mathrm{erf}\left(\frac{y}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}e^{-\frac{y^2}{2}}.$$

Then, we have

$$\begin{aligned} f'(y) &= \frac{e^y - 1}{e^y + 1} - \mathrm{erf}\left(\frac{y}{\sqrt{2}}\right) \\ &= 1 - \left(\frac{2}{e^y + 1} + \mathrm{erf}\left(\frac{y}{\sqrt{2}}\right)\right) := 1 - g(y), \end{aligned}$$

which says that

$$\begin{aligned} g'(y) &= -\frac{2e^y}{(e^y + 1)^2} + \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{2}}e^{-\frac{y^2}{2}} \\ &= 2\left(-\frac{e^y}{(e^y + 1)^2} + \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\right) \\ &= 2\left(\frac{-\sqrt{2\pi}e^y + (1 + e^y)^2 e^{-\frac{y^2}{2}}}{\sqrt{2\pi}(1 + e^y)^2}\right) \\ &= 2\left(\frac{-\sqrt{2\pi}e^{y + \frac{y^2}{2}} + (1 + e^y)^2}{\sqrt{2\pi}(1 + e^y)^2 e^{\frac{y^2}{2}}}\right). \end{aligned}$$

Denote $h(y) = -\sqrt{2\pi}e^{y + \frac{y^2}{2}} + (1 + e^y)^2$. We consider two cases.

Case 1. $0 \le y \le 1$. We have

$$\begin{aligned} h(y) &= -\sqrt{2\pi}e^{y + \frac{y^2}{2}} + e^{2y} + 2e^y + 1 \\ &= \left(e^y - \frac{\sqrt{2\pi}}{2}e^{\frac{y^2}{2}}\right)^2 - \frac{\pi}{2}e^{y^2} + 2e^y + 1. \end{aligned}$$

Since $0 \le y \le 1$, $e^{y^2} \le e^y$. Then

$$h(y) \ge \left(e^y - \frac{\sqrt{2\pi}}{2} e^{\frac{y^2}{2}} \right)^2 + (2 - \frac{\pi}{2}) e^y + 1 > 0$$

which yields $g'(y) > 0$ for all $y \in [0,1]$. Therefore, $g(y) \ge g(0) = 1$ implies that $f'(y) \le 0$ for all $y \in [0,1]$.

Case 2. $y \ge 1$. We have

$$h'(y) = 2e^y\left(1+e^y\right) - \sqrt{2\pi}\left(1+y\right)e^{y+\frac{y^2}{2}} = \left(2\left(1+e^y\right) - \sqrt{2\pi}\left(1+y\right)e^{\frac{y^2}{2}} \right) e^y.$$

Set $k(y) = 2\left(1+e^y\right) - \sqrt{2\pi}\left(1+y\right)e^{\frac{y^2}{2}}$. If $y \ge 2$, then $e^y \le e^{\frac{y^2}{2}}$ which yields

$$k(y) \le 2\left(1 + e^{\frac{y^2}{2}}\right) - \sqrt{2\pi}(1+y)e^{\frac{y^2}{2}} \le 2 + \left(2 - \sqrt{2\pi}(1+y)\right) e^{\frac{y^2}{2}}.$$

On the other hand,

$$\left(2 - \sqrt{2\pi}(1+y)\right) e^{\frac{y^2}{2}} \le \left(2 - 3\sqrt{2\pi}\right) e^2 < -2 \quad \forall y \ge 2.$$

Hence, $k(y) < 0$ for all $y \ge 2$. If $1 \le y \le 2$, we consider

$$k'(y) = 2e^y - \sqrt{2\pi}(1+y+y^2)e^{\frac{y^2}{2}}.$$

$$k''(y) = 2e^y - \sqrt{2\pi}(1+3y+y^2+y^3)e^{\frac{y^2}{2}}.$$

Since $k''(y) \le 2e^2 - 6\sqrt{2e\pi} < 0$ for all $1 \le y \le 2$. This implies that $k'(y) \le k'(1) = 2e - 3\sqrt{2e\pi} < 0$ for all $1 \le y \le 2$. Hence, $k(y) \le k(1) = 2(1+e) - 2\sqrt{2e\pi} < 0$ for all $1 \le y \le 2$. Thus, $h'(y) \le 0$ for all $y \ge 1$ which leads to $h(y)$ is decreasing on $[1,\infty)$. Moreover, since $h(1) > 0$ and $h(3/2) < 0$, $h(y) = 0$ has a solution on $[1,\infty)$. This indicates that $g'(y) = 0$ has a solution on $[1,\infty)$. Assume that $y = \alpha$ ($\alpha > 1$) is a solution of $g'(y) = 0$, it follows that $g'(y) \ge 0$ on $[1,\alpha]$ and $g'(y) \le 0$ on $[\alpha,\infty]$. Then

$$\begin{cases} g(y) & \ge g(1) \\ g(y) & \ge \lim_{k\to\infty} g(y) \end{cases} \Longrightarrow g(y) \ge 1.$$

From the above two cases, we obtain that $f'(y) \le 0$ for all $y \ge 0$. On the other hand, we know that

$$\begin{aligned}
\lim_{y\to\infty} f(y) &= \lim_{y\to\infty} \log\left(1+e^{-y}\right) + \left[\log\left(1+e^y\right) - y\right] + y\left[1 - \operatorname{erf}\left(\frac{y}{\sqrt{2}}\right)\right] - \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} \\
&= \lim_{y\to\infty} \log\frac{e^y}{1+e^y} + y\left[1 - \operatorname{erf}\left(\frac{y}{\sqrt{2}}\right)\right] \\
&\ge \lim_{y\to\infty} y\left[1 - \sqrt{1 - e^{-y^2}}\right] = 0,
\end{aligned}$$

where the last inequality holds by applying Lemma 2.5 in [35] $\operatorname{erf}\left(\frac{y}{\sqrt{2}}\right) \le \left(1 - e^{-y^2}\right)^{\frac{1}{2}}$. Thus, $f(y) \ge \lim_{y\to\infty} f(y) = 0$ which shows that $\phi_8(\mu,x) \le \phi_4(\mu,x)$.