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On matrix characterizations for P-property of the linear transformation in second-order cone linear complementarity problems

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The P-property of the linear transformation in second-order cone linear complementarity problems (SOCLCP) plays an important role in checking the globally uniquely solvable (GUS) property due to the work of Gowda et al. However, it is not easy to verify the P-property of the linear transformation, in general. In this paper, we provide matrix characterizations for checking the P-property, which is a new approach different from those in the literature. This is a do-able manipulation, which helps verifications of the P-property and globally uniquely solvable (GUS) property in second-order cone linear complementarity problems. Moreover, using an equivalence relation to the second-order cone linear complementarity problem, we study some sufficient and necessary conditions for the unique solution of the absolute value equations associated with second-order cone (SOCAVE).

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1. Introduction

Given a matrix $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the second-order cone linear complementarity problem, denoted by SOCLCP(M, q), is to find a vector $x \in \mathbb{R}^n$ satisfying the following conditions:

$$x \in \mathcal{K}^n, \quad Mx + q \in \mathcal{K}^n \quad \text{and} \quad \langle x, Mx + q\rangle = 0, \tag{1}$$

where $\langle \cdot, \cdot\rangle$ denotes the Euclidean inner product, and $\mathcal{K}^n$ represents a second-order cone which is defined as

$$\mathcal{K}^n := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \le x_1\}.$$

The SOCLCP(M, q) (1) belongs to the class of symmetric cone complementarity problems (SCCP for short), and has many applications in various fields, for example, in engineering, control, finance, robust optimization, and management science and so on. Moreover, the SOCLCP(M, q) also arises from the Karush-Kuhn-Tucker conditions of the second-order cone programming; and has been extensively studied from different aspects [1,2,4,6,7,11,12,20,30,31,39–41]. Moreover, the SOCLCP(M, q) (1) can be viewed as a generalization of the classical linear complementarity problem (LCP(M, q) for short), i.e., when $\mathcal{K}^n$ represents the nonnegative octant cone $\mathbb{R}^n_+$, the SOCLCP(M, q) reduces to the linear complementarity problem.

Roughly, there are two main research directions regarding the complementarity problem. One is on the theoretical side in which its corresponding solution properties are investigated, see [5,12,14–16,35,36,39]. The other one focuses on the algorithm for solving the complementarity problems, see [1,2,7,8,11,12,17,19,20,22,30,34,40,41]. From the theoretical aspect, there are many issues about solutions that have been studied. These include the existence and the uniqueness of solutions, the boundedness of solution set, and the convexity of solution set, etc. In particular, for asserting the uniqueness of the solution to the LCP(M, q), a popular approach is looking into the matrix M such that the solution of LCP(M, q) exists and is unique for any $q \in \mathbb{R}^n$. In the literature, it is well known that the following statements are equivalent ([5,12]):

(a) The LCP(M, q) has a unique solution for any $q \in \mathbb{R}^n$.
(b) M is a P-matrix, i.e., all principal minors of M are positive.
(c) $z_i(Mz)_i \le 0$ for all $i \Longrightarrow z = 0$.
(d) All real eigenvalues of M and its principal sub-matrices are positive.

Unfortunately, the favorable characterizations stated in (a)-(d) are not true any more for the linear complementarity problem over symmetric cones (SCLCP for short), including second-order cone linear complementarity problem and positive semidefinite linear complementarity problem. In [16], Gowda et al. propose the P-property, the cross commutative property and the GUS property to remedy them. In fact, via the Euclidean

Jordan algebra, Gowda et al. have concluded that the linear transformation possesses the GUS property for the SCLCP if and only if the linear transformation hold both the P-property and the cross commutative property. Nonetheless, there is difficulty to check the P-property of the linear transformation. To overcome it, Yang and Yuan [39] consider the relationship between the GUS property and the related linear algebra of the linear transformation in a special symmetric cone linear complementarity problem, i.e., the second-order cone linear complementarity problem. There still need four conditions to verify the GUS property of the SOCLCP(M, q) by Yang and Yuan's result. Besides, Chua and Yi in [9] claim an equivalent condition to the P-property of linear transformation in the SOCLCP(M, q). However, the result in [9] does not show the algebraic properties of the linear transformation M. Following all these directions, we study the relation between the P-property and the related properties of the linear transformation M in the SOCLCP(M, q) in this paper. More precisely, we provide matrix characterizations for checking the P-property, which is a new approach different from those in the literature. This is a suitable step, which helps verifications of the P-property and the GUS (globally uniquely solvable) property in SOCLCP(M, q). Our results indeed recover those conditions in [39]. Furthermore, as an application, we also study the unique solution of the absolute value equations associated with second-order cone (SOCAVE for short).

To close this section, we say a few words about notations and the organization of this paper. As usual, $\mathbb{R}_+$ denotes the nonnegative reals, and $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors. For any $x, y \in \mathbb{R}^n$, the Euclidean inner product are denoted $\langle x, y\rangle := x^T y$, and the Euclidean norm $\|x\|$ are denoted as $\|x\| := \sqrt{\langle x, x\rangle}$. The boundary and interior of the set C is denoted by $\mathrm{bd}(C)$ and $\mathrm{int}(C)$, respectively. $x \succeq (\preceq)\ 0$ means $x \in \mathcal{K}^n(-\mathcal{K}^n)$. This paper is organized as follows. In Section 2, we briefly recall some concepts and properties regarding second-order cone and the projection of x onto second-order cone. Besides, the Jordan product, the spectral decomposition and the Peirce decomposition for elements x and y in $\mathbb{R}^n$ associated with second-order cone are reviewed. In Section 3, we discuss the sufficient and necessary conditions for the linear transformation in SOCLCP(M, q) having the P-property, and give the equivalent conditions for the GUS property of the SOCLCP(M, q). In Section 4, in light of the matrix characterizations for P-property of the linear transformation established in Section 3, we explore the unique solvability of the SOCAVE via the GUS property of the SOCLCP.

2. Preliminaries

The second-order cone (SOC) in $\mathbb{R}^n$ $(n \geq 1)$, also called the Lorentz cone or ice-cream cone, is defined as

$$\mathcal{K}^n := \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x_2\| \leq x_1\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm. If $n=1$, then $\mathcal{K}^1$ is the set of nonnegative reals $\mathbb{R}_+$. In general, a general second-order cone $\mathcal{K}$ could be the Cartesian product of SOCs, i.e.,

$$\mathcal{K} := \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_r}.$$

For simplicity, we focus on the single SOC $\mathcal{K}^n$ because all the analysis can be carried over to the setting of Cartesian product. For any two vectors $x=(x_1,x_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$ and $y=(y_1,y_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, the *Jordan product* of x and y associated with $\mathcal{K}^n$ is defined as

$$x \circ y := \begin{bmatrix} x^T y \\ y_1 x_2 + x_1 y_2 \end{bmatrix}.$$

The Jordan product, unlike scalar or matrix multiplication, is not associative, which is a main source of complication in the analysis of optimization problems involved SOC, see [7,8,11] and references therein for more details. The identity element under this Jordan product is $e=(1,0,\cdots,0)^T \in \mathbb{R}^n$. With these definitions, x^2 means the Jordan product of x with itself, i.e., $x^2 := x\circ x$; and $\sqrt{x}$ with $x\in\mathcal{K}^n$ denotes the unique vector in $\mathcal{K}^n$ such that $\sqrt{x}\circ\sqrt{x}=x$. In light of this, the absolute value vector $|x|$ with respect to SOC is computed by

$$|x| := \sqrt{x\circ x}.$$

From the definition of $|x|$, it is not easy to write out the expression of $|x|$ explicitly. Fortunately, there is another way to reach $|x|$ via spectral decomposition and projection onto $\mathcal{K}^n$. We elaborate it as below. For $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, the *spectral decomposition* of x with respect to $\mathcal{K}^n$ is given by

$$x = \lambda_1 e_1 + \lambda_2 e_2, \tag{2}$$

where $\lambda_i = x_1 + (-1)^i \|x_2\|$ for $i=1,2$ and

$$e_i = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{x_2^T}{\|x_2\|}\right)^T & \text{if} \quad \|x_2\| \neq 0, \\ \frac{1}{2}\left(1, (-1)^i \omega^T\right)^T & \text{if} \quad \|x_2\| = 0, \end{cases} \tag{3}$$

with $\omega \in \mathbb{R}^{n-1}$ being any vector satisfying $\|\omega\|=1$. The two scalars λ_1 and λ_2 are called spectral values (or eigenvalues) of x; while the two vectors e_1 and e_2 are called the spectral vectors (or eigenvectors) of x, and the system $\{e_1,e_2\}$ is called as a Jordan frame associated with $\mathcal{K}^n$. Moreover, it is obvious that the spectral decomposition of $x\in\mathbb{R}^n$ is unique if $x_2\neq 0$.

We say that the elements x and y *operator commute* if L_x and L_y commute, i.e., $L_x L_y = L_y L_x$, where L_x denotes the Lyapunov transformation $L_x : V \to V$ defined by

$L_x(z) := x \circ z$. It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame. Moreover, in the setting of $\mathcal{K}^n$, the vectors $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ operator commute if and only if either y_2 is a multiple of x_2 or x_2 is a multiple of y_2.

In fact, there is another decomposition in Euclidean Jordan algebra $\mathbb{V}$, which is the Peirce decomposition. From [10, Theorem IV.2.1], the space $\mathbb{V}$ is the orthogonal direct sum of eigenspaces $\mathbb{V}_{ij} (i \leq j)$, i.e., fix a Jordan frame $\{e_1, e_2, \cdots, e_r\}$ in a Euclidean Jordan algebra $\mathbb{V}$. For any element $x \in \mathbb{V}$, we have

$$x = \sum_{i=1}^{r} \lambda_i e_i + \sum_{i<j} x_{ij}, \tag{4}$$

where $\lambda_i \in \mathbb{R}$ and $x_{ij} \in \mathbb{V}_{ij}$, and the eigenspaces are described as follows:

$$\mathbb{V}_{ii} := \{x \in \mathbb{V} \,|\, x \circ e_i = x\} = \mathbb{R} e_i$$

and

$$\mathbb{V}_{ij} := \left\{ x \in \mathbb{V} \,\Big|\, x \circ e_i = \frac{1}{2} x = x \circ e_j \right\} \quad \text{for} \quad i \neq j,$$

for $i, j \in \{1, 2, \cdots, r\}$. The expression $\sum_{i=1}^{r} \lambda_i e_i + \sum_{i<j} x_{ij}$ is called the *Peirce decomposition* of x. According to the Peirce decomposition and Jordan frame $\{e_1, e_2\}$ associate with the second-order cone, we have the Peirce decomposition of any vector $y \in \mathbb{R}^n$ associated with SOC as follows:

$$y = \mu_1 e_1 + \mu_2 e_2 + \mu_3 v_{12},$$

where $v_{12} \in \mathbb{V}_{12}$ is a unit vector.

Next, we consider the orthogonal projection onto the second-order cone. Let x^+ be the projection of x onto $\mathcal{K}^n$, while x^- be the projection of $-x$ onto its dual cone of $\mathcal{K}^n$. Since $\mathcal{K}^n$ is self-dual, the dual cone of $\mathcal{K}^n$ is itself, i.e., $(\mathcal{K}^n)^* = \mathcal{K}^n$. In fact, the explicit formula of projection of $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ onto $\mathcal{K}^n$ is characterized in [7,8,10–12] as below:

$$x^+ = \begin{cases} x & \text{if } x \in \mathcal{K}^n, \\ 0 & \text{if } x \in -\mathcal{K}^n, \\ u & \text{otherwise}, \end{cases} \qquad \text{where} \quad u = \begin{bmatrix} \frac{x_1 + \|x_2\|}{2} \\ \left(\frac{x_1 + \|x_2\|}{2}\right) \frac{x_2}{\|x_2\|} \end{bmatrix}.$$

Similarly, the expression of x^- is in the form of

$$x^- = \begin{cases} 0 & \text{if } x \in \mathcal{K}^n, \\ -x & \text{if } x \in -\mathcal{K}^n, \\ w & \text{otherwise}, \end{cases} \qquad \text{where} \quad w = \begin{bmatrix} -\frac{x_1 - \|x_2\|}{2} \\ \left(\frac{x_1 - \|x_2\|}{2}\right) \frac{x_2}{\|x_2\|} \end{bmatrix}.$$

Together with the spectral decomposition (2) of x, it can be verified that $x = x^+ - x^-$ and the expression of x^+ and x^- have the form:

$$\begin{aligned} x^+ &= (\lambda_1)_+ e_1 + (\lambda_2)_+ e_2, \\ x^- &= (-\lambda_1)_+ e_1 + (-\lambda_2)_+ e_2, \end{aligned}$$

where $(\alpha)_+ = \max\{0, \alpha\}$ for $\alpha \in \mathbb{R}$. Based on the definitions and expressions of x_+ and x_-, we introduce another expression of $|x|$ associated with $\mathcal{K}^n$. The alternative expression is obtained by the so-called SOC-function, which can be found in [3,4]. For any $x \in \mathbb{R}^n$, we define the *absolute value* $|x|$ of x with respect to SOC as $|x| := x^+ + x^-$. In fact, in the setting of SOC, the form $|x| = x^+ + x^-$ is equivalent to the form $|x| = \sqrt{x \circ x}$. Combining the above expression of x^+ and x^-, it is easy to see that the expression of the absolute value $|x|$ is in the form of

$$|x| = \big[(\lambda_1)_+ + (-\lambda_1)_+\big] e_1 + \big[(\lambda_2)_+ + (-\lambda_2)_+\big] e_2 = \big|\lambda_1\big| e_1 + \big|\lambda_2\big| e_2.$$

For subsequent analysis, we need the generalized Jacobian of the projection x^+. They are stated as below and can be found in [8,18,21,32].

Theorem 2.1. *The generalized Jacobian of the projection function $(\cdot)^+$ onto $\mathcal{K}^n$ is given as follows:*

(a) *Suppose that $z_2 = 0$. Then, $\partial(z^+) = \big\{ tI \,\big|\, t \in [0,1] \big\}$.*
(b) *Suppose that $z_2 \neq 0$.*
(1) If $z \in int(-\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| < 0$ and $z_1 - \|z_2\| < 0$, then

$$\partial(z^+) = \{\nabla(0)\} = \left\{ \begin{bmatrix} 0 & 0^T \\ 0 & O \end{bmatrix} \right\}.$$

(2) If $z \in int(\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| > 0$ and $z_1 - \|z_2\| > 0$, then

$$\partial(z^+) = \{\nabla(z)\} = \left\{ \begin{bmatrix} 1 & 0^T \\ 0 & I \end{bmatrix} \right\}.$$

(3) If $z \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| > 0$ and $z_1 - \|z_2\| < 0$, then

$$\partial(z^+) = \{\nabla(z^+)\} = \left\{ \frac{1}{2} \begin{bmatrix} 1 & \frac{z_2^T}{\|z_2\|} \\ \frac{z_2}{\|z_2\|} & I + \frac{z_1}{\|z_2\|}\left(I - \frac{z_2 z_2^T}{\|z_2\|^2}\right) \end{bmatrix} \right\}.$$

(4) If $z \in bd(-\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| = 0$ and $z_1 - \|z_2\| < 0$, then

$$\partial(z^+) = \left\{ \frac{t}{2} \begin{bmatrix} 1 & \frac{z_2^T}{\|z_2\|} \\ \frac{z_2}{\|z_2\|} & \frac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix} \,\middle|\, t \in [0,1] \right\}.$$

(5) If $z \in bd(\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| > 0$ and $z_1 - \|z_2\| = 0$, then

$$\partial(z^+) = \left\{ \frac{1}{2} \begin{bmatrix} 1+t & (1-t)\frac{z_2^T}{\|z_2\|} \\ (1-t)\frac{z_2}{\|z_2\|} & 2I - (1-t)\frac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix} \;\middle|\; t \in [0,1] \right\}.$$

Let

$$\Omega(z) := \{x \in \mathbb{R}^n \,|\, x \text{ and } z \text{ have the same Jordan frame}\}. \tag{5}$$

For any $x \in \Omega(z)$, we know that x and z have the same Jordan frame with respect to $\mathcal{K}^n$. Therefore, x has the spectral decomposition $x = \lambda_1 e_1 + \lambda_2 e_2$, where $e_i = \frac{1}{2}\left(1, (-1)^i \frac{z_2^T}{\|z_2\|}\right)^T$ if $\|z_2\| \neq 0$ and $e_i = \frac{1}{2}\left(1, (-1)^i \omega^T\right)^T$ if $\|z_2\| = 0$. Based on Theorem 2.1, we obtain the following result by a simple calculation. We omit the detailed proofs here.

Theorem 2.2. *For any $V \in \partial(z^+)$ and any $x \in \Omega(z)$, the following hold:*

(a) *When $z_2 = 0$, then $Vx = tx = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ with $t \in [0,1]$.*

(b) *When $z_2 \neq 0$,*

(1) If $z \in int(-\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| < 0$ and $z_1 - \|z_2\| < 0$, then

$$Vx = Ox = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

(2) If $z \in int(\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| > 0$ and $z_1 - \|z_2\| > 0$, then

$$Vx = x = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

(3) If $z \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| > 0$ and $z_1 - \|z_2\| < 0$, then

$$Vx = \begin{bmatrix} \frac{\lambda_2}{2} \\ \frac{\lambda_2}{2} \frac{z_2}{\|z_2\|} \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

(4) If $z \in bd(-\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| = 0$ and $z_1 - \|z_2\| < 0$, then

$$Vx = \begin{bmatrix} \frac{t}{2}\lambda_2 \\ \frac{t}{2}\lambda_2 \frac{z_2}{\|z_2\|} \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad \text{with } \; t \in [0,1].$$

(5) If $z \in bd(\mathcal{K}^n)$, i.e., $z_1 + \|z_2\| > 0$ and $z_1 - \|z_2\| = 0$, then

$$Vx = \begin{bmatrix} \frac{\lambda_2 + t\lambda_1}{2} \\ \left(\frac{\lambda_2 - t\lambda_1}{2}\right) \frac{z_2}{\|z_2\|} \end{bmatrix} = \begin{bmatrix} e_1 \ e_2 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad \textit{with} \ \ t \in [0,1].$$

3. The conditions of equivalence for M having P-property in SOCLCP(M, q)

In this section, we consider the sufficient and necessary conditions for M having the P-property in SOCLCP(M, q). Then, we present the conditions of equivalence for M having the GUS property in SOCLCP(M, q). Before presenting the main results, we state the definition of the P-property and the GUS property.

Definition 3.1. Consider a linear transformation $M : \mathbb{V} \to \mathbb{V}$ where $\mathbb{V}$ is a Euclidean Jordan algebra. We say that

(a) M has the P-property if

$$\begin{cases} x \text{ and } M(x) \text{ operator commute} \\ x \circ M(x) \preceq 0 \end{cases} \Longrightarrow x = 0.$$

(b) M has the GUS property if for all $q \in \mathbb{V}$, the SOCLCP(M, q) has a unique solution.

(c) elements x and y operator commutes if L_x and L_y commute, i.e., $L_x L_y = L_y L_x$. In the setting of SOC, vectors x and y operator commutes amounts to x and y having the same Jordan frame.

(d) M has the cross commutative property if for any $q \in \mathbb{V}$ and for any two solutions x and $\bar{x}$ of SOCLCP(M, q), x operator commutes with $\bar{y}$ and $\bar{x}$ operator commutes y where $y := M(x) + q$ and $\bar{y} := M(\bar{x}) + q$.

Choose and fix an arbitrary matrix M. By applying the spectral decomposition (2)-(3) and the Peirce decomposition (4) to x and Mx, respectively (here x is an arbitrary element of $\mathbb{R}^n$), we have

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad Mx = \mu_1 e_1 + \mu_2 e_2 + \mu_3 v,$$

where $v \in \mathbb{V}_{12}$ is a unit vector satisfying $v^T e_i = 0$ $(i = 1, 2)$. We compute

$$\begin{aligned} & (Me_1, Me_2, Mv) \\ := & (m_{11}e_1 + m_{21}e_2 + m_{31}v, m_{12}e_1 + m_{22}e_2 + m_{32}v, m_{13}e_1 + m_{23}e_2 + m_{33}v) \\ = & (e_1, e_2, v) \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}. \end{aligned}$$

Accordingly, it follows that

$$\begin{aligned} Mx &= \mu_1 e_1 + \mu_2 e_2 + \mu_3 v = (e_1, e_2, v) \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \\ &= \lambda_1 M e_1 + \lambda_2 M e_2 + 0 M v = (e_1, e_2, v) \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence, we obtain

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix}.$$

If x and Mx operator commute, x and Mx have the same Jordan frame. Then, the above formulas can be simplified as

$$Mx = (e_1, e_2, v) \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix}.$$

In fact, $m_{33} \in \mathbb{R}$ can take any value. For the sake of convenience, we let $m_{33} = 1$. In light of this, if x, Mx and z have the same Jordan frame, then for any $V \in \partial(z^+)$, it follows that

$$\begin{aligned} & (I - V + VM)x = x - Vx + VMx \\ = \ & (e_1, e_2, v) \\ & \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} \\ := \ & (e_1, e_2, v) \overline{N} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix}, \end{aligned}$$

where $\overline{N} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix}$ and the sub-matrix $\begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$ with $t_1, t_2 \in [0, 1]$ are defined as in Theorem 2.2. Moreover, for any $z \in \mathbb{R}^n$, if x, Mx and z have the same Jordan frame, by a direct calculation, we achieve that $(I - V + VM)x \neq 0$ for $V \in \partial(z^+)$ if and only if the matrix $\overline{N}$ is nonsingular. Accordingly, we may consider an equivalent condition of the P-property of M via the nonsingularity of the matrix $I - V + VM$ on $\Omega(z)$ as defined in (5). Here, the nonsingularity of the matrix $I - V + VM$ on $\Omega(z)$ means that $(I - V + VM)x \neq 0$ for any $0 \neq x \in \Omega(z)$.

Theorem 3.1. *For the SOC setting, consider the definition of the P-property given as in Definition 3.1. Let $\Omega(z)$ be defined as in (5). Then, the matrix M has the P-property if and only if the matrix $I - V + VM$ is nonsingular on the set $\Omega(z)$ for any $V \in \partial(z^+)$.*

Proof. "$\Rightarrow$" Suppose that M has the P-property. For any $x \in \Omega(z)$, let $x = \lambda_1 e_1 + \lambda_2 e_2$, where $e_i = \frac{1}{2}\left(1, (-1)^i \frac{z_2^T}{\|z_2\|}\right)^T$ if $\|z_2\| \neq 0$ and $e_i = \frac{1}{2}\left(1, (-1)^i \omega^T\right)^T$ if $\|z_2\| = 0$. To proceed, we discuss the following several cases:

Case 1 When $z_2 = 0$, we have $V = tI$ with $t \in [0, 1]$. If $t = 0$, it follows that $(I - V + VM)x = x$. It is clear that $(I - V + VM)x \neq 0$ for any $x \neq 0$. If $t = 1$, we have $(I - V + VM)x = Mx$. Since M has the P-property, by [16, Theorem 11], we know that the determinant of M is positive and the matrix M is invertible. Hence, there holds $(I - V + VM)x \neq 0$ for any $0 \neq x \in \Omega(z)$. If $t \in (0, 1)$, then $(I - V + VM)x = 0$ leads to $Mx = V^{-1}(V - I)x = \frac{t-1}{t}x$. This implies that x and Mx operator commute and

$$x \circ Mx = \frac{t-1}{t} x \circ x \preceq 0.$$

Since M has the P-property, it says $x = 0$. Thus, for this case of $z_2 = 0$, we have that $I - V + VM$ is nonsingular on the set $\Omega(z)$.

Case 2 When $z \in int(-\mathcal{K}^n)$, we have $V = O$. It is obvious that $(I - V + VM)x = x \neq 0$ for any $0 \neq x \in \mathbb{R}^n$. Hence, the matrix $I - V + VM$ is nonsingular on the set $\Omega(z)$.

Case 3 When $z \in int(\mathcal{K}^n)$, we have $V = I$. For any $x \in \mathbb{R}^n$, it follows that $(I - V + VM)x = Mx$. Using the P-property of M, we obtain that the matrix M is invertible. Hence, we assert that $I - V + VM$ is nonsingular on the set $\Omega(z)$.

Case 4 When $z \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we have

$$V = \frac{1}{2}\begin{bmatrix} 1 & \frac{z_2^T}{\|z_2\|} \\ \frac{z_2}{\|z_2\|} & I + \frac{z_1}{\|z_2\|}\left(I - \frac{z_2 z_2^T}{\|z_2\|^2}\right) \end{bmatrix}.$$

If $(I - V + VM)x = 0$ for any $x \in \Omega(z)$, it yields that $VMx = (V - I)x$. Then, applying Theorem 2.1 and Theorem 2.2 says that Mx also shares the same Jordan frame with x. By this, we let

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad Mx = \mu_1 e_1 + \mu_2 e_2,$$

where $e_i = \frac{1}{2}\left(1, (-1)^i \frac{z_2^T}{\|z_2\|}\right)^T$ if $\|z_2\| \neq 0$ and $e_i = \frac{1}{2}\left(1, (-1)^i \omega^T\right)^T$ if $\|z_2\| = 0$. Besides, by direct calculation, we conclude that

$$VMx = \frac{\mu_2}{2}\begin{pmatrix} 1 \\ \frac{z_2}{\|z_2\|} \end{pmatrix} \quad \text{and} \quad (V-I)x = \frac{\lambda_1}{2}\begin{pmatrix} -1 \\ \frac{z_2}{\|z_2\|} \end{pmatrix}.$$

Combining with $VMx = (V-I)x$, it implies that $\lambda_1 = \mu_2 = 0$. From this, we have

$$x \circ Mx = \lambda_1\mu_1 e_1 + \lambda_2\mu_2 e_2 = 0.$$

Again, since M has P-property, it says that $x = 0$. Therefore, under this case of $z \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we prove that $I - V + VM$ is nonsingular on the set $\Omega(z)$.

Case 5 When $z \in bd(-\mathcal{K}^n)$, we have

$$V = \frac{t}{2}\begin{bmatrix} 1 & \frac{z_2^T}{\|z_2\|} \\ \frac{z_2}{\|z_2\|} & \frac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix} \quad t \in [0,1].$$

If $t = 0$, it is clear that $(I - V + VM)x = x \neq 0$ for any $0 \neq x \in \mathbb{R}^n$. If $t \in (0,1]$, it is easy to see

$$VMx = \frac{t\mu_2}{2}\begin{pmatrix} 1 \\ \frac{z_2}{\|z_2\|} \end{pmatrix} \quad \text{and} \quad (V-I)x = \begin{pmatrix} \frac{t-1}{2}\lambda_2 - \frac{1}{2}\lambda_1 \\ (\frac{t-1}{2}\lambda_2 + \frac{1}{2}\lambda_1)\frac{z_2}{\|z_2\|} \end{pmatrix}.$$

Then, by $VMx = (V-I)x$, we obtain that

$$\lambda_1 = 0 \quad \text{and} \quad \mu_2 = \frac{t-1}{t}\lambda_2.$$

Hence, we have

$$x \circ Mx = \lambda_1\mu_1 e_1 + \lambda_2\mu_2 e_2 = \frac{t-1}{t}\lambda_2^2 e_2 \preceq 0.$$

Again, since M has the P-property, it follows that $x = 0$. Therefore, under this case of $z \in bd(-\mathcal{K}^n)$, we show that $I - V + VM$ is nonsingular on the set $\Omega(z)$.

Case 6 When $z \in bd(\mathcal{K}^n)$, we have

$$V = \frac{1}{2}\begin{bmatrix} 1+t & (1-t)\frac{z_2^T}{\|z_2\|} \\ (1-t)\frac{z_2}{\|z_2\|} & 2I - (1-t)\frac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix} \quad t \in [0,1].$$

If $t = 1$, we have $V = I$. Then, it is easy to verify that $(I - V + VM)x = Mx$. Using the P-property of M, we see that the matrix M is invertible. Thus, we conclude that $I - V + VM$ is nonsingular on the set $\Omega(z)$. If $t \in [0,1)$, it can be verified that

$$VMx = \frac{1}{2}\begin{pmatrix} t\mu_1 + \mu_2 \\ (-t\mu_1 + \mu_2)\frac{z_2}{\|z_2\|} \end{pmatrix} \quad \text{and} \quad (V - I)x = \frac{(t-1)\lambda_1}{2}\begin{pmatrix} 1 \\ -\frac{z_2}{\|z_2\|} \end{pmatrix}.$$

Applying $VMx = (V - I)x$ again yields that

$$\mu_2 = 0 \quad \text{and} \quad \mu_1 = \frac{t-1}{t}\lambda_1.$$

Hence, it follows that

$$x \circ Mx = \lambda_1\mu_1 e_1 + \lambda_2\mu_2 e_2 = \frac{t-1}{t}\lambda_1^2 e_1 \preceq 0.$$

Again, due to the P-property of M, it leads to $(I - V + VM)x \neq 0$ for any $0 \neq x \in \Omega(z)$. Thus, we prove that $I - V + VM$ is nonsingular on the set $\Omega(z)$.

All the aforementioned analysis indicate that $I - V + VM$ is nonsingular on the set $\Omega(z)$.

"$\Leftarrow$" Support that x and Mx operator commute and $x \circ Mx \preceq 0$. Because x and Mx have the same Jordan frame, we let

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad Mx = \mu_1 e_1 + \mu_2 e_2.$$

Therefore, $x \circ Mx = \lambda_1\mu_1 e_1 + \lambda_2\mu_2 e_2 \preceq 0$, which implies that $\lambda_1\mu_1 \leq 0$ and $\lambda_2\mu_2 \leq 0$. Moreover, following the same arguments as above, we know that for any $V \in \partial(z^+)$ and $x \in \Omega(z)$, $(I - V + VM)x \neq 0$ if and only if the matrix $\overline{N}$ is nonsingular. Applying [13, Theorem 4.3], it follows that matrix

$$\overline{M} := \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

is a P-matrix. From this, for any $0 \neq x \in \mathbb{R}^n$, i.e., the vector $\lambda := (\lambda_1, \lambda_2, 0)^T \neq 0$, there exists $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$ such that $\lambda_i(\overline{M}\lambda)_i > 0 \ \ (i = 1, 2)$. Since

$$Mx = \mu_1 e_1 + \mu_2 e_2 = (e_1, e_2, v)\begin{pmatrix} \mu_1 \\ \mu_2 \\ 0 \end{pmatrix} = (e_1, e_2, v)\overline{M}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} = (e_1, e_2, v)\overline{M}\lambda,$$

we compute that

$$\begin{aligned} x \circ Mx &= \lambda_1\mu_1 e_1 + \lambda_2\mu_2 e_2 \\ &= (e_1, e_2, v)\begin{pmatrix} \lambda_1\mu_1 \\ \lambda_2\mu_2 \\ 0 \end{pmatrix} = (e_1, e_2, v)\begin{pmatrix} \lambda_1(\overline{M}\lambda)_1 \\ \lambda_2(\overline{M}\lambda)_2 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence, there exists $\lambda_i(\overline{M}\lambda)_i > 0 \ \ (i = 1, 2)$, which contradicts $\lambda_i \mu_i \leq 0 \ \ (i = 1, 2)$. Thus, we have $x = 0$. To sum up, M has the P-property in the SOC setting. $\square$

Example 3.1. Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$M := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider $x := (0, x_2)^T$. Then, we see that x and $Mx = 0$ operator commute and $x \circ Mx \preceq 0$. However, $x \neq 0$. By the definition of the P-property, it follows that the matrix M does not have the P-property. In fact, we can conclude this result by applying Theorem 3.1. To see this, for the case of $z \in int(\mathcal{K}^2)$, we have $V = I$, which yields

$$I - V + VM = M.$$

Hence, we have $|I - V + VM| = 0$, which explains the matrix $I - V + VM$ is singular. To sum up, using the above Theorem 3.1, we assert that the matrix M does not have the P-property.

Example 3.2. Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$M := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The matrix M is a symmetric positive definite matrix. Hence, we have $x^T Mx > 0$ for any $x \neq 0$, i.e., the matrix M is strongly monotone. By Theorem 11 in [16], this implies that the matrix M has the P-property. Moreover, by a direct calculation, for any case of the six cases in Theorem 3.1, we have $|I - V + VM| \neq 0$, which explains that the matrix $I - V + VM$ is nonsingular. Then, from Theorem 3.1, it also follows that the matrix M has the P-property.

In fact, the P-property of the linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$ can also be expressed in another equivalent conclusion.

Theorem 3.2. *For the SOC setting, consider the definition of the P-property given as in Definition 3.1. Let $\Omega(z)$ be defined as in (5). Then, the matrix M has the P-property if and only if the matrix $M + V(I - M)$ is nonsingular on the set $\Omega(z)$ for any $V \in \partial(z^+)$.*

Proof. "⇒" Suppose that M has the P-property and $x \in \Omega(z)$. Let $x = \lambda_1 e_1 + \lambda_2 e_2$, where $e_i = \frac{1}{2}\left(1, (-1)^i \frac{z_2^T}{\|z_2\|}\right)^T$ if $\|z_2\| \neq 0$ and $e_i = \frac{1}{2}\left(1, (-1)^i \omega^T\right)^T$ if $\|z_2\| = 0$. Again, to proceed, we need to discuss several cases:

Case 1 When $z_2 = 0$, we have $V = tI$ with $t \in [0, 1]$. If $t = 1$, from $[M + V(I - M)]x = x$, it is obvious that $[M + V(I - M)]x \neq 0$ for any $x \neq 0$. If $t = 0$, we have $[M + V(I - M)]x = Mx$. It follows from the P-property of M that $[M + V(I - M)]x \neq 0$ for any $0 \neq x \in \Omega(z)$. If $t \in (0, 1)$, $[M + V(I - M)]x = 0$ can induce $Vx = (V - I)Mx$. By Theorem 2.1 and Theorem 2.2, we know that x and Mx operator commute. Since $V = tI$, we obtain that $Vx = tx$ and $(V - I)Mx = (t - 1)Mx$. Based on $Vx = (V - I)Mx$, this leads to $Mx = \frac{t}{t-1}x$ and

$$x \circ Mx = \frac{t}{t-1} x \circ x \preceq 0.$$

Since M has the P-property, we have $x = 0$. Hence, for this case of $z_2 = 0$, we prove that $M + V(I - M)$ is nonsingular on the set $\Omega(z)$.

Case 2 When $z \in int(-\mathcal{K}^n)$, we have $V = O$, which yields $[M + V(I - M)]x = Mx$. From the P-property of M, it is easy to check that $[M + V(I - M)]x \neq 0$ for any $0 \neq x \in \Omega(z)$. Hence, we prove that $M + V(I - M)$ is nonsingular on the set $\Omega(z)$.

Case 3 When $z \in int(\mathcal{K}^n)$, we have $V = I$. This implies that $[M + V(I - M)]x = x$. It is clear that $(I - V + VM)x \neq 0$ for any $0 \neq x \in \mathbb{R}^n$. Hence, we have that $M + V(I - M)$ is nonsingular on the set $\Omega(z)$.

Case 4 When $z \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we have

$$V = \frac{1}{2} \begin{bmatrix} 1 & \frac{z_2^T}{\|z_2\|} \\ \frac{z_2}{\|z_2\|} & I + \frac{z_1}{\|z_2\|}\left(I - \frac{z_2 z_2^T}{\|z_2\|^2}\right) \end{bmatrix}.$$

If $[M + V(I - M)]x = 0$ for $x \in \Omega(z)$, it follows from Theorem 2.1 and Theorem 2.2 that Mx has the same Jordan frame as x and z. In view of this, we let

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad Mx = \mu_1 e_1 + \mu_2 e_2,$$

where $e_i = \frac{1}{2}\left(1, (-1)^i \frac{z_2^T}{\|z_2\|}\right)^T$ if $\|z_2\| \neq 0$ and $e_i = \frac{1}{2}\left(1, (-1)^i \omega^T\right)^T$ if $\|z_2\| = 0$. By $[M + V(I - M)]x = 0$, we have $Vx = (V - I)Mx$. By a direct calculation gives

$$Vx = \frac{\lambda_2}{2} \begin{pmatrix} 1 \\ \frac{z_2}{\|z_2\|} \end{pmatrix} \quad \text{and} \quad (V - I)Mx = \frac{\mu_1}{2} \begin{pmatrix} -1 \\ \frac{z_2}{\|z_2\|} \end{pmatrix}.$$

Combining with $Vx = (V - I)Mx$, it implies that $\mu_1 = \lambda_2 = 0$. From this, we have

$$x \circ Mx = \lambda_1 \mu_1 e_1 + \lambda_2 \mu_2 e_2 = 0.$$

Since M has P-property, then we obtain $x = 0$. Therefore, under this case of $z \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we show that $M + V(I - M)$ is nonsingular on the set $\Omega(z)$.

Case 5 When $z \in bd(-\mathcal{K}^n)$, we have

$$V = \frac{t}{2}\begin{bmatrix} 1 & \frac{z_2^T}{\|z_2\|} \\ \frac{z_2}{\|z_2\|} & \frac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix} \quad t \in [0,1].$$

If $t = 0$, we have $V = O$ and $[M + V(I - M)]x = Mx$. If follows from the P-property of M that $[M + V(I - M)]x \neq 0$ for $0 \neq x \in \Omega(z)$. If $t \in (0,1]$, it follows that

$$Vx = \frac{t\lambda_2}{2}\begin{pmatrix} 1 \\ \frac{z_2}{\|z_2\|} \end{pmatrix} \quad \text{and} \quad (V - I)Mx = \begin{pmatrix} \frac{t-1}{2}\mu_2 - \frac{1}{2}\mu_1 \\ (\frac{t-1}{2}\mu_2 + \frac{1}{2}\mu_1)\frac{z_2}{\|z_2\|} \end{pmatrix}.$$

Then, by $Vx = (V - I)Mx$, we obtain that

$$\mu_1 = 0 \quad \text{and} \quad \lambda_2 = \frac{t-1}{t}\mu_2.$$

Hence, we have

$$x \circ Mx = \lambda_1\mu_1 e_1 + \lambda_2\mu_2 e_2 = \frac{t-1}{t}\mu_2^2 e_2 \preceq 0.$$

Since M has the P-property, then it follows that $x = 0$. Therefore, under this case of $z \in bd(-\mathcal{K}^n)$, we prove that $M + V(I - M)$ is nonsingular on the set $\Omega(z)$.

Case 6 When $z \in bd(\mathcal{K}^n)$, we have

$$V = \frac{1}{2}\begin{bmatrix} 1+t & (1-t)\frac{z_2^T}{\|z_2\|} \\ (1-t)\frac{z_2}{\|z_2\|} & 2I - (1-t)\frac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix} \quad t \in [0,1].$$

If $t = 1$, we have $V = I$. It is easy to check that $[M + V(I - M)]x = x \neq 0$ for any $0 \neq x \in \mathbb{R}^n$. If $t \in [0,1)$, it follows that

$$Vx = \frac{1}{2}\begin{pmatrix} t\lambda_1 + \lambda_2 \\ (-t\lambda_1 + \lambda_2)\frac{z_2}{\|z_2\|} \end{pmatrix} \quad \text{and} \quad (V - I)Mx = \frac{(t-1)\mu_1}{2}\begin{pmatrix} 1 \\ -\frac{z_2}{\|z_2\|} \end{pmatrix}.$$

Then, by $Vx = (V - I)Mx$, we conclude that

$$\lambda_2 = 0 \quad \text{and} \quad \lambda_1 = \frac{t-1}{t}\mu_1.$$

Hence, it yields that

$$x \circ Mx = \lambda_1\mu_1 e_1 + \lambda_2\mu_2 e_2 = \frac{t-1}{t}\mu_1^2 e_1 \preceq 0.$$

From the P-property of M, this leads to $[M + V(I - M)]x \neq 0$ for any $0 \neq x \in \Omega(z)$. Hence, we prove that $M + V(I - M)$ is nonsingular on the set $\Omega(z)$.

All the aforementioned analysis shows that $M+V(I-M)$ is nonsingular on the set $\Omega(z)$.

"$\Leftarrow$" Suppose that x and Mx operator commute and $x \circ Mx \preceq 0$. It follows that x and Mx have the same Jordan frame, which means

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad Mx = \mu_1 e_1 + \mu_2 e_2.$$

Then, $x \circ Mx = \lambda_1\mu_1 e_1 + \lambda_2\mu_2 e_2 \preceq 0$, which implies that $\lambda_1\mu_1 \le 0$ and $\lambda_2\mu_2 \le 0$. Moreover, similar to the arguments for Theorem 3.1, we know that for any $V \in \partial(z^+)$ and $0 \neq x \in \Omega(z)$, there holds

$$\begin{aligned}
&[M+V(I-M)]x = Vx + (I-V)Mx \\
&= (e_1, e_2, v)\left\{ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1-t_1 & 0 & 0 \\ 0 & 1-t_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} \\
&:= (e_1, e_2, v)\widetilde{N} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix},
\end{aligned}$$

where $\widetilde{N} := \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1-t_1 & 0 & 0 \\ 0 & 1-t_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix}$ and the sub-matrix $\begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$ with $t_1, t_2 \in [0,1]$ are defined as in Theorem 2.2. In addition, for $V \in \partial(z^+)$ and any $z \in \mathbb{R}^n$, by a direct calculation, we assert that $[M+V(I-M)]x \neq 0$ for $0 \neq x \in \Omega(z)$ if and only if the matrix $\widetilde{N}$ is nonsingular. By [13, Theorem 4.3] again, it follows that the matrix

$$\overline{M} := \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

is a P-matrix. The remaining arguments are quite similar to those in Theorem 3.1. Hence, the matrix M has the P-property. □

Remark 3.1. When the SOCLCP(M, q) reduces to the classical linear complementarity problem (LCP), Theorem 3.1 and Theorem 3.2 are exactly the equivalent conditions of P-matrix, see [12].

Based on the equivalent conditions of the linear transformation having the P-property for the SOCLCP(M, q), we study the conditions for M has the GUS property for the SOCLCP(M, q) as below.

Theorem 3.3. *Consider the SOCLCP(M, q). For the linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$, there holds*

$$\textit{GUS-property} = \textit{P-property} + \textit{Cross Commutative.}$$

Proof. This is an immediate consequence of [16, Theorem 14]. □

For any two solution x and $\bar{x}$ of SOCLCP(M, q), we know that x and $y = Mx + q$ have the same Jordan frame, and $\bar{x}$ and $\bar{y} = M\bar{x} + q$ have the same Jordan frame, respectively. By the definition of the Cross Commutative property again, we have that SOCLCP(M, q) has the Cross Commutative property if and only if x, $\bar{x}$, y and $\bar{y}$ all have the same Jordan frame. Hence, combining with Theorem 3.1 and Theorem 3.2, we state the sufficient and necessary condition of the GUS property of M for the SOCLCP(M, q).

Theorem 3.4. *Consider the SOCLCP(M, q) and let $\Omega(z)$ be defined as in (5). Then, the linear transformation M has the GUS property if and only if it satisfies the following conditions:*

- **(a)** *for any $V \in \partial(z^+)$, we have that the matrix $(I - V + VM)$ (or $M + V(I - M)$) is nonsingular on the set $\Omega(z)$.*
- **(b)** *for any two solutions x and $\bar{x}$ of SOCLCP(M, q), x, $\bar{x}$, $y = Mx + q$ and $\bar{y} = M\bar{x} + q$ all have the same Jordan frame.*

Proof. The results are clear by the definition of the Cross Commutative property, Theorem 3.1, Theorem 3.2, and Theorem 3.3. □

In light of Theorem 21 and Theorem 22 in [16] and our results, we have two other equivalent conditions of the GUS property.

Theorem 3.5. *Consider the SOCLCP(M, q) and let $\Omega(z)$ be defined as in (5). Then, the following statements hold:*

- **(a)** *If the matrix M is symmetric, then M has the GUS property if and only if for any $V \in \partial(z^+)$, the matrix $(I - V + VM)$ (or $M + V(I - M)$) is nonsingular on the set $\Omega(z)$.*
- **(b)** *If the matrix M is monotone, then M has the GUS property if and only if for any $V \in \partial(z^+)$, the matrix $(I - V + VM)$ (or $M + V(I - M)$) is nonsingular on the set $\Omega(z)$.*

Proof. These results are obvious by applying [16, Theorem 21 and Theorem 22]. □

Example 3.3. Let $M : \mathbb{R}^4 \to \mathbb{R}^4$ be defined as

$$M := \begin{bmatrix} 4 & 0.5 & 0 & 0 \\ 0.5 & 4 & 0.5 & 0 \\ 0 & 0.5 & 4 & 0.5 \\ 0 & 0 & 0.5 & 4 \end{bmatrix}.$$

By a direct calculation, we can verify that for any $0 \neq x \in \Omega(z)$ and $V \in \partial(z^+)$, there holds $[I - V + VM]x \neq 0$ or $[M + V(I - M)]x \neq 0$ for any case of those six cases in Theorem 3.1. This asserts that the matrix $I - V + VM$ is nonsingular on the set $\Omega(z)$. Then, by Theorem 3.1 and the symmetry of M, we have that M has the GUS property for the SOCLCP(M, q). On the other hand, we find that M is a symmetric positive definite matrix. Hence, the matrix M is strongly monotone. By [12, Proposition 2.3.11] or [16, Theorem 17] again, the SOCLCP(M, q) has a unique solution for any $q \in \mathbb{R}^n$.

4. Application

In this section, we apply the matrix characterizations for the P-property of the linear transformation established in Section 3 to investigate condition under which the unique solution of the below SOCAVE is guaranteed. All the results shown in this section are different from those in [28], which also studied the same issue.

The absolute value equation associated with second-order cone, abbreviated as SOCAVE, is in the form of

$$Ax - |x| = b. \tag{6}$$

Unlike the standard absolute value equation, here $|x|$ means the absolute value of x coming from the square root of the Jordan product "$\circ$" of x and x associated with $\mathcal{K}^n$, that is, $|x| := (x \circ x)^{1/2}$. More details about the classic absolute value equation (AVE for short) can be found in [23–27,33,37,38], whereas some references for SOCAVE are [18,28,29].

First, we claim that under mild conditions, the SOCAVE (6) is indeed equivalent to a special SOCLCP(M, q) as below:

$$z \in \mathcal{K}^n, \quad w = Mz + q \in \mathcal{K}^n \quad \text{and} \quad \langle z, w \rangle = 0,$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$.

Theorem 4.1. *Consider the SOCLCP(M, q) (1) and the SOCAVE (6).*

(a) *Suppose that 1 is not an eigenvalue of A, the SOCAVE (6) can be recast as the below SOCLCP(M, q):*

$$z \in \mathcal{K}^n, \quad w = Mz + q \in \mathcal{K}^n \quad \textit{and} \quad \langle z, w\rangle = 0,$$

where $M = (A+I)(A-I)^{-1}$, $q = 2(A-I)^{-1}b$ *and* $z = (A-I)x - b$.

(b) *Conversely, if 1 is not an eigenvalue of* M*, then the SOCLCP*(M, q) *(1) can be rewritten as the following special SOCAVE:*

$$Ax - |x| = b,$$

where $A = (M-I)^{-1}(M+I)$, $b = (M-I)^{-1}q$ *and* $x = \frac{1}{2}(M-I)^{-1}z + q$.

Proof. (a) Suppose that we are given the SOCAVE (6), i.e., $Ax - b = |x|$, which is further equivalent to the following conditions:

$$\begin{cases} x \text{ and } Ax - b \text{ have the same Jordan frame,} \\ |x| \preceq Ax - b, \\ |x| \succeq Ax - b. \end{cases} \tag{7}$$

Under the first condition in (7) that x and $Ax - b$ have the same Jordan frame, we obtain

$$|x| \preceq Ax - b \iff -Ax + b \preceq x \preceq Ax + b \iff \begin{cases} Ax - b - x \succeq 0, \\ Ax - b + x \succeq 0. \end{cases}$$

Using $Ax - b = |x|$ gives $\|Ax - b\| = \||x|\| = \|x\|$, which says

$$\langle Ax - b - x, Ax - b + x\rangle = 0.$$

Hence, we have

$$(A-I)x - b \in \mathcal{K}^n, \ \ (A+I)x - b \in \mathcal{K}^n \ \text{ and } \ \langle Ax - b - x, Ax - b + x\rangle = 0.$$

Let $z := (A-I)x - b$. In addition, from the assumption that 1 is not an eigenvalue of A, we know that the matrix $A - I$ is nonsingular. Thus, it follows that $x = (A-I)^{-1}(z+b)$. This implies that

$$\begin{aligned} (A+I)x - b &= (A+I)(A-I)^{-1}(z+b) - b \\ &= (A+I)(A-I)^{-1}z + 2(A-I)^{-1}b \\ &:= Mz + q, \end{aligned}$$

where $M := (A+I)(A-I)^{-1}$ and $q = 2(A-I)^{-1}b$.

(b) Since 1 is not an eigenvalue of M, we have that the matrix $M - I$ is invertible. Consider the SOCLCP(M, q):

$$z \in \mathcal{K}^n, \quad w = Mz + q \in \mathcal{K}^n \quad \text{and} \quad \langle z, w\rangle = 0,$$

with $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, let

$$A := (M-I)^{-1}(M+I), \quad b := (M-I)^{-1}q \;\text{ and }\; x := \frac{1}{2}[(M-I)z+q].$$

Then, we have

$$z = (A-I)x - b \in \mathcal{K}^n \;\text{ and }\; Mz+q = (A+I)x - b \in \mathcal{K}^n.$$

By the properties of the SOCLCP(M,q), it follows that $z = (A-I)x - b \succeq 0$ and $Mz+q = (A+I)x-b \succeq 0$ share the same Jordan frame. Moreover,

$$\langle z, Mz+q\rangle = \langle Ax-b-x, Ax-b+x\rangle = 0.$$

From the fact that $z = (A-I)x-b \succeq 0$ and $Mz+q = (A+I)x-b \succeq 0$ have the same Jordan frame, we know that $Ax-b$ and x have the same Jordan frame, and $-Ax+b \preceq x \preceq Ax+b$, i.e., $|x| \preceq Ax-b$. Therefore, we have $\||x|\| \le \|Ax-b\|$. In addition, it follows from $\langle z, Mz+q\rangle = \langle Ax-b-x, Ax-b+x\rangle = 0$ that $\|Ax-b\| = \|x\| = \||x|\|$. Combining with the property of $Ax-b$ and x having the same Jordan frame again, this leads to $Ax-b = |x|$. Thus, the proof is complete. □

Remark 4.1. We make a couple remarks regarding Theorem 4.1 as below.

- Hu, Huang and Zhang [18] have shown an equivalent expression for the SOCAVE (6). However, the form is not in the standard SOCLCP(M,q). Here, under the mild condition that 1 is not an eigenvalue of A or M, we provide the equivalence of the SOCAVE (6) to the standard SOCLCP(M,q).
- Note that the relation

$$|x| \preceq Ax-b \iff -Ax+b \preceq x \preceq Ax+b$$

does not hold without the condition of x and $Ax-b$ sharing the same Jordan frame. Indeed, the direction $|x| \preceq Ax-b \implies -Ax+b \preceq x \preceq Ax+b$ is always true, whereas the converse is not necessarily true without the condition of x and $Ax-b$ sharing the same Jordan frame. For instance, let

$$x := \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}, \quad A := \begin{bmatrix} 0 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad b := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

It can be verified that x and $y := Ax-b = (\frac{5}{4}, 1, \frac{1}{4})^T$ do not have the same Jordan frame. In addition, we compute that $|x| = (\frac{1}{4}, 0, 0)^T$. Then, it is easy to see that $-y \preceq x \preceq y$. But, $y - |x| \notin \mathcal{K}^3$ i.e., $|x| \preceq y$ is not true.

Theorem 4.2. *Consider the SOCAVE (6) and let $\Omega(z)$ be defined as in (5). The SOCAVE (6) has a unique solution for any $b \in \mathbb{R}^n$ if and only if the matrix A satisfies the following conditions:*

(a) *for any $z \in \mathbb{R}^n$ and $V \in \partial(z^+)$,*

$$A - I + 2V \quad or \quad A + I - 2V$$

is nonsingular on the set $\Omega(z)$.

(b) *for any two solution x and $\bar{x}$ of SOCAVE (6), $(A-I)x+b$, $(A-I)\bar{x}-b$, $y = (A+I)x-b$ and $\bar{y} = (A+I)\bar{x} - b$ all share the same Jordan frame.*

Proof. According to Theorem 4.1, when 1 is not an eigenvalue of A, the SOCAVE (6) can be recast as a SOCLCP(M, q):

$$(A - I)x - b \in \mathcal{K}^n, \quad (A + I)x - b \in \mathcal{K}^n \quad \text{and} \quad \langle Ax - b - x, Ax - b + x\rangle = 0.$$

Let $z := (A - I)x - b$. Then, we observe

$$w := (A + I)x - b = (A + I)(A - I)^{-1}z + 2(A - I)^{-1}b = Mz + q,$$

where $M := (A+I)(A-I)^{-1}$ and $q := 2(A-I)^{-1}b$. This says that, under the condition of 1 being not an eigenvalue of A, the unique solution to the SOCAVE (6) is exactly the unique solution of the below SOCLCP(M, q):

$$z \in \mathcal{K}^n, \quad w = Mz + q \in \mathcal{K}^n \quad \text{and} \quad \langle z, w\rangle = 0.$$

Moreover, we compute that

$$(I - V + VM) = \big[(I - V)(A - I) + V(A + I)\big](A - I)^{-1} = (A - I + 2V)(A - I)^{-1}$$

and

$$M + V(I - M) = (A + I)(A - I)^{-1} + V\big[I - (A + I)(A - I)^{-1}\big] = (A + I - 2V)(A - I)^{-1}.$$

Combining these with Theorem 3.4, it follows that the conditions for A are equivalent to the conditions (a) and (b). Thus, the proof is complete. □

By applying Theorem 3.5 and Theorem 4.2, we achieve two other equivalent conditions for the existence and uniqueness of solutions to the SOCAVE (6).

Theorem 4.3. *Consider the SOCAVE (6) and let $\Omega(z)$ be defined as in (5). Then, the following statements hold:*

(a) *If the matrix A satisfies the condition: $(A^T - I)^{-1}(A^T + I) = (A + I)(A - I)^{-1}$, then the SOCAVE (6) has a unique solution if and only if for any $z \in \mathbb{R}^n$ and $V \in \partial(z^+)$,*

$$A - I + 2V \quad \text{or} \quad A + I - 2V$$

is nonsingular on the set $\Omega(z)$.

(b) *If the matrix A satisfies the condition: $\|Az\| \geq \|z\|$ for any $z \in \mathbb{R}^n$, then the SOCAVE (6) has a unique solution if and only if for any $z \in \mathbb{R}^n$ and $V \in \partial(z^+)$,*

$$A - I + 2V \quad \text{or} \quad A + I - 2V$$

is nonsingular on the set $\Omega(z)$.

Proof. (a) In view of Theorem 3.5 and Theorem 4.2, in order to prove part (a), it suffices to verify that the matrix M is symmetric, i.e., $M^T = M$. Since $M = (A + I)(A - I)^{-1}$, it is clear that $M^T = (A^T - I)^{-1}(A^T + I) = (A + I)(A - I)^{-1} = M$. Hence, the desired result follows.

(b) By Theorem 4.1, we know that if 1 is not an eigenvalue of A, the SOCAVE (6) can be converted to a SOCLCP(M, q). Thus, it says that $A - I$ is invertible. Let $z = (A - I)^{-1}x$, i.e., $x = (A - I)z$. If follows from $M = (A + I)(A - I)^{-1}$ that

$$\begin{aligned} x^T Mx &= \langle (A - I)z, (A + I)z \rangle \\ &= \langle Az, Az \rangle - \langle z, z \rangle \\ &= \|Az\|^2 - \|z\|^2. \end{aligned}$$

Combining with the condition $\|Az\| \geq \|z\|$, we obtain that $x^T Mx \geq 0$. Besides, from the arbitrariness of $z \in \mathbb{R}^n$, we see that the element $x = (A - I)z$ is also arbitrary in $\mathbb{R}^n$, which implies the monotonicity of M in $\mathbb{R}^n$. Then, applying Theorem 3.5 and Theorem 4.2 again yields the desired result. □

Declaration of competing interest

None declared.

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