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COMPLEXITY ANALYSIS OF A PREDICTOR-CORRECTOR INTERIOR-POINT ALGORITHM FOR $P_*(\kappa)$-WEIGHTED LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. This paper aims at a predictor-corrector interior-point algorithm for solving weighted linear complementarity problem with $P_*(\kappa)$-matrices, which is a variant of weighted complementarity problem and has wide applications in science, engineering, and economics. We first apply the algebraic equivalent transformation technique, and then use the identity function to determine the new search directions. Under suitable conditions, the feasibility and convergence of the algorithm are established. Moreover, we show that the proposed algorithm has polynomial-time complexity. As far as we know, this is the first predictor-corrector interior-point algorithm for $P_*(\kappa)$-weighted linear complementarity problem based on the above-mentioned search directions. Preliminary numerical results demonstrate that our algorithm performs well and efficiently on the test problems.

1. **Introduction.** In this paper, we consider the weighted complementarity problem (WCP) which is to find a pair of vectors belonging to the intersection of a manifold with a cone such that their product in a certain algebra equals a given weight vector. On one hand, the motivation for introducing WCP is that a wide range of problems [33, 36] in engineering, science and economics can be formulated as WCP. For example, the Eisenberg-Gale markets [18] and the perfect price discrimination market model [15] can be phrased as WCPs. Even when an equilibrium problem can also be modeled as a complementarity problem (CP), it is shown that WCP model leads to a highly efficient method in some cases [26]. Besides, WCPs can be used to model equilibrium problems in atmospheric chemistry [20] and multibody dynamics [14]. On the other hand, the WCP is a very general complementarity system which significantly extends the CP [4, 5]. In addition, several interior-point

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algorithms (IPAs) for CP have been generalized as efficient algorithms for WCP. However, the existence of weight vector makes the theory and algorithms of WCP more complex than that of CP.

The roots of WCP came from [19], and the $P_*(\kappa)$-WLCP was introduced by Illés et al. [16] in 1997. The results in [16] with detailed proofs were published in Nagy's Ph.D. thesis [25]. However, it has received great attention in recent years until Potra [26] gave the concept of WCP. Potra [26] proved that the Fisher market equilibrium problem in economics could be a skew-symmetric WLCP and proposed two IPAs for solving WLCP. Subsequently, Potra [27] introduced the concept of sufficient WLCP, in which he extended not only the concept of monotone WLCP, but also the definition of sufficient linear complementarity problem (LCP) given by Cottle et al. [6]. Potra [27] also presented a corrector-predictor algorithm for sufficient WLCP and showed that the complexity bound of the algorithm is proportional to $1 + \kappa$ under certain conditions. Chi et al. [7] gave some existence and uniqueness results for weighted horizontal linear complementarity problem under Euclidean Jordan algebras. Especially, Asadi et al. [1] extended the full-Newton step IPA initiated by Roos et al. [29] for linear optimization (LO) to monotone WLCP (MWLCP), which has the best-known iteration complexity. In [8, 9], a full-Newton step IPA was designed to solve WLCP.

In recent years, a few Newton methods have been proposed for solving WCPs. For instance, by using a one-parametric class of smoothing functions involving the weight vector, Tang and Zhang [31] proposed a smoothing Newton algorithm with non-monotone line search for WCP. Lately, Tang and Zhou [32] presented a modified damped Gauss-Newton method for non-monotone WLCP. Compared to Newton algorithms, IPAs can find an approximate solution in polynomial time. In practice, predictor-corrector interior-point algorithm (PC IPA) is a class of the most effective IPAs. The first PC IPA for LO was provided by Mehrotra [22] and Sonnevend et al. [30]. Nonetheless, the PC IPAs usually perform more corrector steps after each predictor step in the main iteration. Then, Mizuno, Todd and Ye [21] introduced the first Mizuno-Todd-Ye (MTY) PC IPA for LO, which uses only one corrector step in a main iteration. MTY PC IPA was the first method having both $\mathrm{O}(\sqrt{n}L)$ iteration complexity and superlinear convergence. Although, Miao [23] extended MTY IPA to $P_*(\kappa)$-LCP, the handicap κ of $P_*(\kappa)$-matrix is not sometimes easy to compute. Miao's algorithm is not suitable for general $P_*(\kappa)$-LCPs. Recently, PC IPAs have received renewed attention [10, 11], there are few available results on PC IPAs for $P_*(\kappa)$-WLCP.

The determination of search directions is crucial for IPAs, which could be defined by barrier functions. Peng et al. [28] introduced the concept of a self-regular function as well as a class of search directions based on self-regular proximity functions. Darvay [12] proposed the algebraic equivalent transformation (AET) technique to obtain the search directions for LO. He [12] applied a function $\varphi(t) = \sqrt{t}$ to both sides of a system of equations that defined the central path. By applying Newton's method to the transformed system, a search direction is obtained. Later, Achache [2] and Wang [34] extended the AET technique to convex quadratic programming and monotone LCPs over symmetric cones, respectively. Zhang and Xu [37] used a simple univariate function to formulate the search direction for LO that employs only one kind of search step.

According to Darvay's work, the case of the identity map $\varphi(t) = t$ can be regarded as a trivial and special case of the AET approach, which implies that IPAs do not use any transformation of the central path. Roos et al. [29] and Wang et al. [35] used $\varphi(t) = t$ for constructing search directions for LO and $P_*(\kappa)$-LCP, respectively. Using the same function $\varphi(t) = t$, Achache and Tabchouche [3] gave a full-Newton step feasible IPA for monotone horizontal LCP. Later, based on $\varphi(t) = t$, Asadi et al. [1] studied IPA for MWLCP and gave its computational complexity. Darvay et al. [11] introduced PC IPAs for $P_*(\kappa)$-LCP that are based on the AET technique and they used function $\varphi(t) = t^2$.

TABLE 1. Analysis of IPAs with $\varphi(t) = t$

Problem	Algorithm	Variance vector	Complexity
LO [29]	IPA	$v = \sqrt{\frac{xs}{\mu}}$	$O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$
LCP [3]	IPA	$v = \sqrt{\frac{xs}{\mu}}$	$O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$
$P_*(\kappa)$-LCP [35]	IPA	$v = \sqrt{\frac{xs}{\mu}}$	$O\left((1+4\kappa)\sqrt{n}\log\frac{n}{\varepsilon}\right)$
MWLCP[1]	IPA	$v = \sqrt{\frac{xs}{\mu}}$	$O\left(\frac{5\left(\Vert\omega - x^0s^0\Vert + \min x^0s^0\right)}{\min x^0s^0}\log\frac{\frac{\min x^0s^0}{2} + \Vert\omega - x^0s^0\Vert}{\varepsilon}\right)$
$P_*(\kappa)$-WLCP	PC IPA	$v = \sqrt{\frac{xs}{\omega(t)}}$	$O\left((1+4\kappa')\sqrt{n}\log\frac{\frac{5}{4(1+4\kappa')}\max x^0s^0 + \Vert x^0s^0 - \omega\Vert}{\varepsilon}\right)$

The aim of our paper is to propose a PC IPA for $P_*(\kappa)$-WLCP by using the identity function. In summary, we use the AET technique with function $\varphi(t) = t$ to determine the search directions for $P_*(\kappa)$-WLCP. We compare some existing IPAs in [1, 3, 29, 35] with our PC IPA in Table 1. The main contributions of this article are described below.

1. Based on function $\varphi(t) = t$, we extend IPA in [1, 3, 29, 35] to $P_*(\kappa)$-WLCP.
2. This is the first PC IPA for $P_*(\kappa)$-WLCP based on the function $\varphi(t) = t$.
3. The analysis of the PC IPA is more complex than that of the algorithms in [1, 3, 29, 35], due to the existence of non-negative weight vectors.
4. Under some mild assumptions, all the iterates generated by the PC IPA are feasible and always lie in the local convergence neighborhood.
5. We give an iteration bound for $P_*(\kappa)$-WLCP that coincides with the best-known iteration bound for these types of problems. Moreover, numerical results indicate the efficiency of the proposed algorithm.

The paper is organized as follows. In Section 2, we describe some concepts related to the $P_*(\kappa)$-WLCP and give a PC IPA for $P_*(\kappa)$-WLCP. Section 3 is dedicated to the complexity analysis of the IPA. Section 4 contains numerical results to demonstrate the efficiency of this IPA. Finally, concluding remarks are provided in Section 5. To close this section, we say a few words about the notations used in this paper. $\mathbb{R}^n$ represents the space of all n dimensional vectors. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ mean the nonnegative orthant and positive orthant of $\mathbb{R}^n$, respectively. We denote by $\varphi(x)$ the vector with components $\varphi(x_i)$, i.e., $\varphi(x) = [\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)]^{\mathrm{T}}$. Given $x, s \in \mathbb{R}^n_+$, the inner product of vectors x and s is defined as $x^{\mathrm{T}}s = \sum_{i=1}^{n} x_is_i$, whereas $xs = (x_is_i)_{1\le i\le n}$ is the componentwise product and the same as for the vectors $x/s = (x_i/s_i)_{1\le i\le n}$ $(s_i \neq 0)$, $\sqrt{x} = (\sqrt{x_i})_{1\le i\le n}$. We denote by $\Vert x\Vert$ the 2-norm of x and by $\Vert x\Vert_\infty$ the infinity norm of x. $X = \mathrm{diag}(x)$ is a diagonal matrix with the elements of the vector x as diagonal entries and e represents the vector of all ones.

2. **PC IPA for $P_*(\kappa)$-WLCP.**

For subsequent needs, we recall basic concepts regarding $P_*(\kappa)$-WLCP and then present a PC IPA (predictor-corrector interior-point algorithm).

2.1. **The central path for $P_*(\kappa)$-WLCP.**

The $P_*(\kappa)$-WLCP seeks for a pair of vectors $(x,s) \in \mathbb{R}^{2n}$ satisfying

$$\begin{aligned} -Mx + s = q, &\quad x \geq 0, \\ xs = \omega, &\quad s \geq 0, \end{aligned} \tag{1}$$

where vector $q \in \mathbb{R}^n$, weight vector $\omega \in \mathbb{R}^n_{++}$ and $M \in \mathbb{R}^{n\times n}$. A matrix M is called a $P_*(\kappa)$-matrix [19] if there exists a nonnegative number κ such that

$$(1+4\kappa) \sum_{i \in I_+(x)} x_i(Mx)_i + \sum_{i \in I_-(x)} x_i(Mx)_i \geq 0, \quad \forall x \in \mathbb{R}^n,$$

where $I_+(x) = \{i : x_i(Mx)_i > 0\}$ and $I_-(x) = \{i : x_i(Mx)_i < 0\}$. For the case $\kappa = 0$, we have

$$\sum_{i \in I_+(x)} x_i(Mx)_i + \sum_{i \in I_-(x)} x_i(Mx)_i = x^{\mathrm{T}}Mx \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Then $P_*(0)$-matrix reduces to the positive semidefinite matrix. The infimum value of $\kappa \geq 0$ such that M is $P_*(\kappa)$-matrix is called the handicap of matrix M. For notational convenience, we denote by

$$\mathcal{F} := \left\{(x,s) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : -Mx + s = q\right\}$$

the feasible region of WLCP, and by

$$\mathcal{F}^* := \{(x,s) \in \mathcal{F} : xs = \omega\}$$

the solution set of WLCP. The relative interior of WLCP is described by

$$\mathcal{F}^0 := \left\{(x,s) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} : -Mx + s = q\right\}.$$

Throughout the paper, we assume $\mathcal{F}^0 \neq \emptyset$, i.e., there is a strictly feasible initial point $(x^0, s^0) \in \mathcal{F}^0$. Now, define

$$\omega(t) = (1-t)\omega + t x^0 s^0, \quad t \in (0, t_0], \quad t_0 = 1. \tag{2}$$

In order to find an approximate solution of $P_*(\kappa)$-WLCP, we consider system

$$\begin{aligned} -Mx + s = q, &\qquad x \geq 0, \\ xs = \omega(t), &\qquad s \geq 0. \end{aligned} \tag{3}$$

It is known that if $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$-matrix and $\mathcal{F}^0 \neq \emptyset$, then the system of equations (3) has a unique solution $(x(t), s(t))$ for any $t \in (0, t_0]$ [27]. The set of all points $\{x(t), s(t) \mid t \in (0, t_0]\}$ forms the central path of $P_*(\kappa)$-WLCP (1). From the definition of $\mathcal{F}^*$, we can obtain a solution of system (1) as $t \to 0$.

2.2. **The search directions and proximity for $P_*(\kappa)$-WLCP.** Motivated by the way of extending the AET for LO in [12], we describe our search directions for $P_*(\kappa)$-WLCP(1) as below.

First, consider a function

$$\varphi \in C^1, \quad \varphi : \mathbb{R}_+ \to \mathbb{R}_+,$$

and assume that its inverse function φ^{-1} exists. Then, system (3) that defines the central path can be equivalently reformulated as

$$\begin{aligned} -Mx + s &= q, \qquad x \geq 0, \\ \varphi\left(\frac{xs}{\omega(t)}\right) &= \varphi(e), \qquad s \geq 0. \end{aligned} \tag{4}$$

Applying Newton's method to system (4), we have the following system with the search directions Δx and Δs:

$$\begin{aligned} -M\left(x + \Delta x\right) + \left(s + \Delta s\right) &= q, \qquad x \geq 0, \\ \varphi\left(\frac{xs + x\Delta s + s\Delta x + \Delta x \Delta s)}{\omega(t)}\right) &= \varphi(e), \qquad s \geq 0. \end{aligned}$$

For (small) displacements Δx and Δs in the $x-$ and $s-$space, respectively, we obtain

$$\begin{aligned} -M\Delta x + \Delta s &= 0, \qquad x \geq 0, \\ s\Delta x + x\Delta s &= a_\varphi, \qquad s \geq 0, \end{aligned} \tag{5}$$

where

$$a_\varphi = \omega(t)\left(\varphi'\left(\frac{xs}{\omega(t)}\right)\right)^{-1}\left(\varphi(e) - \varphi\left(\frac{xs}{\omega(t)}\right)\right).$$

By defining

$$v = \sqrt{\frac{xs}{\omega(t)}}, \quad \Delta x = \frac{x}{v}d_x, \quad \Delta s = \frac{s}{v}d_s, \quad d = \sqrt{\frac{x}{s}}, \tag{6}$$

it yields

$$\omega(t)d_x d_s = \Delta x \Delta s, \quad x\Delta s + s\Delta x = \omega(t)v\left(d_x + d_s\right). \tag{7}$$

Furthermore, in light of (7), system (5) can be reformulated as

$$\begin{aligned} -\overline{M}d_x + d_s &= 0, \\ d_x + d_s &= p_v, \end{aligned}$$

where $W(t) = \operatorname{diag}(\omega(t))$, $D = \operatorname{diag}(d)$, $\overline{M} = \sqrt{W^{-1}(t)}DMD\sqrt{W(t)}$ and

$$p_v = \frac{\varphi(e) - \varphi\left(v^2\right)}{v\varphi'\left(v^2\right)}.$$

Note that M and $\overline{M}$ are the $n \times n$ $P_*(\kappa)$-matrix and $P_*(\kappa')$-matrix, respectively. For sake of computation and analysis, we take $\varphi(t) = t$ and get

$$p_v = v^{-1} - v. \tag{8}$$

The right-hand side $v^{-1} - v$ of (8) is equal to the steepest descent direction of function $\Psi(v) := \sum_{i=1}^{n}\left(\frac{v_i^2 - 1}{2} - \log v_i\right)$. Consequently, we have $d_x + d_s = -\nabla\Psi(v)$, where $\Psi(v)$ is the classical logarithmic barrier function. To the best of our knowledge, this is the first time that function $\varphi(t) = t$ is used to determine the search directions for $P_*(\kappa)$-WLCP. Now, we introduce a proximity measure

$$\delta(v) := \delta(x, s; \omega(t)) = \frac{1}{2}\|d_x + d_s\| = \frac{1}{2}\|v^{-1} - v\|. \tag{9}$$

Note that for $(x, s) \in \mathcal{F}^0$, we have

$$\delta(v) = 0 \Longleftrightarrow v = e \Longleftrightarrow xs = \omega(t).$$

The value of $\delta(v)$ is 0 when the given pair (x, s) coincides with the corresponding t-center $(x(t), s(t))$, and is positive otherwise. Hence, $\delta(v)$ measures the distance from the iteration point (x, s) to t-center $(x(t), s(t))$.

2.3. **New PC IPA for $P_*(\kappa)$-WLCP.** In this subsection, we introduce the scaled systems in PC IPA for $P_*(\kappa)$-WLCP. To see this, we give the scaled corrector system

$$\begin{aligned} -\overline{M}d_x + d_s &= 0, \\ d_x + d_s &= v^{-1} - v. \end{aligned} \tag{10}$$

Using (6), the search direction $(\Delta x, \Delta s)$ can be calculated. Then, we update the corrector iterate as follows

$$x^+ = x + \Delta x, \quad s^+ = s + \Delta s.$$

To obtain the scaled predictor system, we decompose a_φ in the system (5) into the form of

$$a_\varphi = f(x, s, t) + g(x, s), \tag{11}$$

where $f : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{+} \to \mathbb{R}^n$ with $f(x, s, 0) = 0$ and $g : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}^n$. We would like to make as greedy predictor steps as possible, therefore we set $t = 0$ in (11). Consequently, we have

$$a_\varphi = g(x, s) = -xs. \tag{12}$$

Using (6), (7) and (12) yields

$$d_x + d_s = \frac{g(x, s)}{\omega(t)v} = \frac{vg(x, s)}{xs} = -v.$$

To proceed, we define

$$v^+ = \sqrt{\frac{x^+s^+}{\omega(t)}}, \quad d^+ = \sqrt{\frac{x^+}{s^+}}, \quad D^+ = \operatorname{diag}(d^+),$$

and

$$\overline{M}_+ = \sqrt{W^{-1}(t)}D^+MD^+\sqrt{W(t)}. \tag{13}$$

Then, the scaled predictor system becomes

$$\begin{aligned} -\overline{M}_+d_x^p + d_s^p &= 0, \\ d_x^p + d_s^p &= -v^+, \end{aligned} \tag{14}$$

where d_x^p and d_s^p are search directions in predictor step. Similarly to Δx and Δs, we can easily calculate the search direction $(\Delta^p x, \Delta^p s)$ by

$$\Delta^p x = \frac{x^+}{v^+}d_x^p, \quad \Delta^p s = \frac{s^+}{v^+}d_s^p. \tag{15}$$

Besides, it is noted that

$$\omega(t)d_x^pd_s^p = \Delta^p x\Delta^p s, \quad x^+\Delta^p s + s^+\Delta^p x = \omega(t)v^+\left(d_x^p + d_s^p\right). \tag{16}$$

Then, the new iterate point after a predictor step is described as

$$x^p = x^+ + \theta t \Delta^p x, \quad s^p = s^+ + \theta t \Delta^p s,$$

where the update parameter $\theta \in (0, 1)$.

Now, based on the AET technique and function $\varphi(t) = t$, we propose a predictor-corrector interior-point algorithm for solving $P_*(\kappa)$-WLCP. Given a weight vector $\omega > 0$, we suppose that there exists an initial point $(x^0, s^0) \in \mathcal{F}^0$ such that $x^0 s^0 \geq \omega$ and $\delta(x^0, s^0, \omega(t_0)) \leq \frac{t_0}{2(1+4\kappa')}$ with $t_0 = 1$. Our algorithm performs a corrector step and a predictor step in a main iteration. In corrector step, we compute the search direction $(\Delta x, \Delta s)$ for $P_*(\kappa)$-WLCP (1) by solving the linear system (10) and using (6). Then, we take a full-Newton step along this search direction, which is the corrector iterate $(x^+, s^+) = (x + \Delta x, s + \Delta s)$. Next, we calculate the predictor search direction $(\Delta^p x, \Delta^p s)$ by solving the system of equations (14) and using (15). So we perform the predictor iterate with $(x^p, s^p) = (x^+ + \theta t \Delta^p x, s^+ + \theta t \Delta^p s)$, where $\theta \in (0, 1)$ and $t \in (0, 1]$. The next section will show that the new iteration point is feasible and convergent by appropriately selecting parameters. We repeat the process until an iteration point satisfying the stopping criterion $\left\| x^k s^k - \omega \right\| \leq \varepsilon$. In the sequel, the PC IPA for $P_*(\kappa)$-WLCP (1) is depicted in Figure 1.

Algorithm 1 The PC IPA for $P_*(\kappa)$-WLCP

Input:
An accuracy parameter $\varepsilon > 0$;
A barrier update parameter $\theta \in (0, 1)$;
Choose $(x^0, s^0) \in \mathcal{F}^0$ such that $x^0 s^0 \geq \omega$ and $\delta(x^0, s^0; \omega(t_0)) \leq \frac{t_0}{2(1+4\kappa')}$ with $t_0 = 1$;
begin
$k := 0$;
while $\|x^k s^k - \omega\| > \varepsilon$ **do**
begin
(Corrector step)
obtain $(\Delta x^k, \Delta s^k)$ by solving (10) and using (6);
let $(x^+)^k := x^k + \Delta x^k, \quad (s^+)^k := s^k + \Delta s^k$;
(Predictor step)
obtain $(\Delta^p x^k, \Delta^p s^k)$ by solving (14) and using (15);
let $(x^p)^k := (x^+)^k + \theta t_k \Delta^p x^k, \quad (s^p)^k := (s^+)^k + \theta t_k \Delta^p s^k$;
(Update of the parameters and iterates)
$(\omega^p(t))^k := (1 - \theta t_k)\omega(t_k)$, $(t_p)^k := (1 - \theta) t_k$;
set $x^{k+1} := (x^p)^k, \quad s^{k+1} := (s^p)^k$;
$\omega(t_{k+1}) := (\omega^p(t))^k$, $t_{k+1} := (t_p)^k$;
$k := k + 1$;
end
end

FIGURE 1. Algorithm 1

3. **Analysis of the PC IPA.** This section provides an analysis of the corrector and predictor steps respectively. We also derive the polynomial iteration complexity of the PC IPA. Note that the corrector step performed by the algorithm is a classical small-update step of IPAs.

3.1. **The corrector step.** First, we give an upper bound on the product of the scaled search directions d_x and d_s.

Lemma 3.1. *Let $x^0 s^0 \geq \omega$ and (d_x, d_s) be a solution to system (10) with $\delta := \delta(x, s; \omega(t))$. One has*

$$\|d_x d_s\|_\infty \leq (1 + 4\kappa')\,\delta^2, \quad \|d_x d_s\| \leq \sqrt{1 + (1 + 4\kappa')^2}\delta^2,$$

where the parameters κ and κ' of $P_*(\kappa)$-matrix M and $P_*(\kappa')$-matrix $\overline{M}$ satisfy $\dfrac{1 + 4\kappa'}{1 + 4\kappa} = \dfrac{\max x^0 s^0}{\min \omega}$.

Proof. Please refer to Theorem 3.5 in [19], and Lemma 3.2 and Lemma 3.3 in [35] for detailed arguments. □

The following lemma gives a condition for the strict feasibility of the corrector step.

Lemma 3.2. *Let $x^0 s^0 \geq \omega$. Then, the corrector iterate $(x^+, s^+) = (x + \Delta x, s + \Delta s)$ is strictly feasible if $\delta < \dfrac{1}{\sqrt{1 + 4\kappa'}}$.*

Proof. Let $x(\alpha) = x + \alpha \Delta x$ and $s(\alpha) = s + \alpha \Delta s$, for $0 \leq \alpha \leq 1$. Using (6) we have

$$xs = \omega(t) v^2, \quad x(\alpha) = \frac{x}{v}(v + \alpha d_x), \quad s(\alpha) = \frac{s}{v}(v + \alpha d_s).$$

From the second equation of system (10) we obtain

$$\begin{aligned} x(\alpha)s(\alpha) &= \frac{xs}{v^2}(v + \alpha d_x)(v + \alpha d_s) \\ &= \omega(t)\left[v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s\right] \\ &= \omega(t)\left[v^2 + \alpha v(v^{-1} - v) + \alpha^2 d_x d_s\right] \\ &= \omega(t)\left[(1 - \alpha)v^2 + \alpha\,(e + \alpha d_x d_s)\right]. \end{aligned} \tag{17}$$

Using (17) and Lemma 3.1 leads to

$$\begin{aligned} x(\alpha)s(\alpha) &\geq \omega(t)\left[(1 - \alpha)v^2 + \alpha\,(e - \|d_x d_s\|_\infty\, e)\right] \\ &\geq \omega(t)\left[(1 - \alpha)v^2 + \alpha\left(1 - (1 + 4\kappa')\,\delta^2\right) e\right]. \end{aligned}$$

If $\delta < \dfrac{1}{\sqrt{1 + 4\kappa'}}$, then $x(\alpha)s(\alpha) > 0$ for $0 \leq \alpha \leq 1$. This indicates that $x(\alpha)$ and $s(\alpha)$ do not change the sign when $0 \leq \alpha \leq 1$. Since $x(0) = x^0 > 0$ and $s(0) = s^0 > 0$, it follows that $s(\alpha) > 0$ and $x(\alpha) > 0$ for any $0 \leq \alpha \leq 1$. Especially, for $\alpha = 1$, $x(1) = x^+ > 0$ and $s(1) = s^+ > 0$. Then, the proof is complete. □

The next lemma provides a lower bound for the minimum value of the components of v^+, where $v^+ = \sqrt{\dfrac{x^+ s^+}{\omega(t)}}$.

Lemma 3.3. *Let $x^0 s^0 \geq \omega$ and $\delta < \dfrac{1}{\sqrt{1 + 4\kappa'}}$. Then, we have*

$$\min v^+ \geq \sqrt{1 - (1 + 4\kappa')\delta^2}.$$

Proof. From (17) with $\alpha = 1$, we have

$$(v^+)^2 = e + d_x d_s. \tag{18}$$

Using Lemma 3.1 and (18), we deduce that

$$\min(v^+)^2 \geq 1 - \|d_x d_s\|_\infty \geq 1 - (1+4\kappa')\delta^2.$$

This completes the proof. □

In the next lemma, we estimate the upper bound of the proximity measure $\delta\left(x^+, s^+; \omega(t)\right)$ after a full-Newton step.

Lemma 3.4. *Let $x^0 s^0 \geq \omega$ and $\delta < \frac{1}{\sqrt{1+4\kappa'}}$. Then, we have*

$$\delta\left(x^+, s^+; \omega(t)\right) \leq \frac{\sqrt{1+(1+4\kappa')^2\delta^2}}{2\sqrt{1-(1+4\kappa')\delta^2}}.$$

Proof. It follows from (9), (18), Lemma 3.1 and Lemma 3.3 that

$$\begin{aligned}\delta\left(x^+, s^+; \omega(t))\right) &= \frac{1}{2}\|(v^+)^{-1} - v^+\| = \frac{1}{2}\left\|\frac{e-(v^+)^2}{v^+}\right\| \\ &\leq \frac{\|d_x d_s\|}{2\min v^+} \leq \frac{\sqrt{1+(1+4\kappa')^2\delta^2}}{2\sqrt{1-(1+4\kappa')\delta^2}}.\end{aligned}$$

□

Corollary 3.5. *Let $x^0 s^0 \geq \omega$. If $\delta < \frac{1}{2\sqrt{1+4\kappa'}}$, then the corrector step is strictly feasible and*

$$\delta\left(x^+, s^+; \omega(t)\right) \leq \sqrt{\frac{2}{3}}\left(1+4\kappa'\right)\delta^2.$$

Proof. From Lemma 3.4, it follows that

$$\delta\left(x^+, s^+; \omega(t)\right) \leq \frac{\sqrt{2}\left(1+4\kappa'\right)\delta^2}{2\sqrt{1-\left(1+4\kappa'\right)\frac{1}{4(1+4\kappa')}}} = \sqrt{\frac{2}{3}}\left(1+4\kappa'\right)\delta^2.$$

□

3.2. **Some technical results.** Since M is a $P_*(\kappa)$-matrix and $-M\Delta^p x + \Delta^p s = 0$, we obtain

$$\sum_{i\in I_+} \Delta^p x_i \Delta^p s_i + \sum_{i\in I_-} \Delta^p x_i \Delta^p s_i \geq -4\kappa \sum_{i\in I_+} \Delta^p x_i \Delta^p s_i, \tag{19}$$

where $I_+ = \{i : \Delta^p x_i \Delta^p s_i > 0\}$ and $I_- = \{i : \Delta^p x_i \Delta^p s_i < 0\}$. From Theorem 3.5 in [19], (13) and (14), we know

$$\sum_{i\in I_+} d^p_{x_i} d^p_{s_i} + \sum_{i\in I_-} d^p_{x_i} d^p_{s_i} \geq -4\kappa' \sum_{i\in I_+} d^p_{x_i} d^p_{s_i}. \tag{20}$$

We point out that the relationship between κ in (19) and κ' in (20) can be found in Lemma 3.1.

Lemma 3.6 provides an upper bound on $\|d^p_x d^p_s\|$, which will be used in the subsequent analysis of the predictor step.

Lemma 3.6. *Let $x^0 s^0 \geq \omega$. Then, there holds*

$$\|d^p_x d^p_s\| \leq \frac{n(1+2\kappa')\left[1+\left(1+4\kappa'\right)\delta^2\right]}{2}.$$

Proof. It follows from (14) that

$$4\sum_{i\in I_+} d^p_{x_i} d^p_{s_i} \le \sum_{i\in I_+}\left(d^p_{x_i}+d^p_{s_i}\right)^2 \le \sum_{i=1}^n \left(d^p_{x_i}+d^p_{s_i}\right)^2 = \|d^p_x+d^p_s\|^2 = \left\|v^+\right\|^2. \quad (21)$$

Using (20) and (21) gives

$$\begin{aligned}\left\|v^+\right\|^2 = \|d^p_x+d^p_s\|^2 &= \|d^p_x\|^2+\|d^p_s\|^2 + 2\left(\sum_{i\in I_+} d^p_{x_i}d^p_{s_i} + \sum_{i\in I_-} d^p_{x_i}d^p_{s_i}\right)\\ &\ge \|d^p_x\|^2+\|d^p_s\|^2 - 8\kappa' \sum_{i\in I_+} d^p_{x_i}d^p_{s_i}\\ &\ge \|d^p_x\|^2+\|d^p_s\|^2 - 2\kappa'\left\|v^+\right\|^2,\end{aligned}$$

which says

$$\|d^p_x\|^2+\|d^p_s\|^2 \le (1+2\kappa')\left\|v^+\right\|^2. \quad (22)$$

Now, we derive an upper bound for $\|v^+\|^2$. By applying (18) and Lemma 3.1, we have

$$\begin{aligned}\left\|v^+\right\|^2 = \sum_{i=1}^n \left(v^+\right)_i^2 &= \sum_{i=1}^n (1+(d_xd_s)_i)\\ &\le n\left[1+\|d_xd_s\|_\infty\right]\\ &\le n\left[1+(1+4\kappa')\,\delta^2\right].\end{aligned} \quad (23)$$

Therefore it follows from (22) and (23) that

$$\begin{aligned}\|d^p_xd^p_s\| \le \|d^p_x\|\,\|d^p_s\| \le \frac{1}{2}\left(\|d^p_x\|^2+\|d^p_s\|^2\right) &\le \frac{1}{2}(1+2\kappa')\left\|v^+\right\|^2\\ &\le \frac{n(1+2\kappa')\left[1+(1+4\kappa')\,\delta^2\right]}{2}.\end{aligned}$$

□

3.3. **The predictor step.** The next lemma shows the feasibility of the predictor step.

Lemma 3.7. *Let* $x^0s^0 \ge \omega$, $(x^+,s^+)>0$ *and* $\theta < \dfrac{1}{\sqrt{2n\,(1+2\kappa')}}$ *with* $n\ge 2$. *Let*

$$h(\delta,\theta,n) = 1-(1+4\kappa')\delta^2 - \frac{n\,(1+2\kappa')\,\theta^2t^2}{2(1-\theta t)}\left[1+(1+4\kappa')\,\delta^2\right].$$

If $\delta < \dfrac{1}{2\sqrt{1+4\kappa'}}$, *then the predictor iterate* $(x^p,s^p) = (x^+ + \theta t\Delta^p x, s^+ + \theta t\Delta^p s)$ *is strictly feasible.*

Proof. For $0\le\alpha\le 1$, denote

$$x^p(\alpha) = x^+ + \alpha\theta t\Delta^p x, \quad s^p(\alpha) = s^+ + \alpha\theta t\Delta^p s.$$

From (16) and the second equation of (14), we have

$$\begin{aligned}x^p(\alpha)s^p(\alpha) &= x^+s^+ + \alpha\theta t\left(x^+\Delta^p s + s^+\Delta^p x\right) + \alpha^2\theta^2t^2\Delta^p x\Delta^p s\\ &= \omega(t)\left[\left(v^+\right)^2 + \alpha\theta t v^+\left(d^p_x+d^p_s\right) + \alpha^2\theta^2t^2 d^p_xd^p_s\right]\\ &= \omega(t)\left[(1-\alpha\theta t)\left(v^+\right)^2 + \alpha^2\theta^2t^2 d^p_xd^p_s\right].\end{aligned} \quad (24)$$

Since $f(\alpha) = \dfrac{\alpha^2\nu^2}{1-\alpha\nu}$ is monotonically increasing with respect to $\alpha \in [0,1]$, it follows from Lemma 3.3 and Lemma 3.6 that

$$\begin{aligned}
\min\left(\frac{x^p(\alpha)s^p(\alpha)}{\omega(t)(1-\alpha\theta t)}\right) &= \min\left(\left(v^+\right)^2 + \frac{\alpha^2\theta^2t^2}{1-\alpha\theta t}d_x^p d_s^p\right)\\
&\geq \min(v^+)^2 - \frac{\alpha^2\theta^2t^2}{1-\alpha\theta t}\left\|d_x^p d_s^p\right\|_\infty\\
&\geq \min(v^+)^2 - \frac{\theta^2t^2}{1-\theta t}\left\|d_x^p d_s^p\right\| \qquad (25)\\
&\geq 1-(1+4\kappa')\delta^2 - \frac{n\left(1+2\kappa'\right)\theta^2t^2}{2(1-\theta t)}\left[1+\left(1+4\kappa'\right)\delta^2\right]\\
&= h(\delta,\theta,n).
\end{aligned}$$

Moreover, let $\delta < \dfrac{1}{2\sqrt{1+4\kappa'}}$ and $\theta < \dfrac{1}{\sqrt{2n\left(1+2\kappa'\right)}}$ with $n \geq 2$. Then, we see that

$$\begin{aligned}
h\left(\delta,\theta,n\right) &> 1-\frac{1}{4}-\frac{n\left(1+2\kappa'\right)\theta^2t^2}{2(1-\theta t)}\left(1+\frac{1}{4}\right)\\
&> \frac{3}{4}-\frac{5t^2}{16\left(1-\theta t\right)}\\
&\geq \frac{3}{4}-\frac{5}{16\left(1-\theta\right)}\\
&> \frac{3}{4}-\frac{5\sqrt{2n\left(1+2\kappa'\right)}}{16\left(\sqrt{2n\left(1+2\kappa'\right)}-1\right)}\\
&\geq \frac{3}{4}-\frac{5}{8}\\
&= \frac{1}{8}\\
&> 0.
\end{aligned}$$

Therefore, $x^p(\alpha)s^p(\alpha) > 0$ for $0 \leq \alpha \leq 1$. Since $x^p(\alpha)$ and $s^p(\alpha)$ are linear functions of α, $x^p(0) = x^+ > 0$ and $s^p(0) = s^+ > 0$, we conclude that $x^p(1) = x^p > 0$ and $s^p(1) = s^p > 0$. Then, the proof is complete. □

To proceed, we define

$$v^p = \sqrt{\frac{x^p s^p}{\omega(t_p)}}, \quad \omega(t_p) = (1-\theta t)\,\omega(t), \quad t_p = (1-\theta)t. \qquad (26)$$

From (24) and (25) with $\alpha = 1$, it is clear that

$$\left(v^p\right)^2 = \left(v^+\right)^2 + \frac{\theta^2t^2}{1-\theta t}d_x^p d_s^p, \qquad (27)$$

and

$$\min v^p \geq \sqrt{h(\delta,\theta,n)}. \qquad (28)$$

Next, we investigate the effect on the proximity measure $\delta(v^p) := \delta(x^p, s^p; \omega(t_p))$ after a predictor step and the update of the parameter t.

Lemma 3.8. *Let $x^0 s^0 \geq \omega$, $\delta < \frac{1}{2\sqrt{1+4\kappa'}}$ and $\theta < \frac{1}{\sqrt{2n(1+2\kappa')}}$ with $n \geq 2$. Then, there holds*

$$\delta(v^p) \leq \frac{\sqrt{1+(1+4\kappa')^2\delta^2} + \frac{n(1+2\kappa')\theta^2 t^2}{2(1-\theta t)}\left[1+(1+4\kappa')\delta^2\right]}{2\sqrt{h(\delta,\theta,n)}}.$$

Proof. Using (9), we have

$$\delta(v^p) = \frac{1}{2}\left\|(v^p)^{-1} - v^p\right\| \leq \frac{\left\|e-(v^p)^2\right\|}{2\min v^p}. \tag{29}$$

Applying (18), (27), Lemma 3.1 and Lemma 3.6, we obtain

$$\begin{aligned} \left\|e-(v^p)^2\right\| &= \left\|e-\left(v^+\right)^2 - \frac{\theta^2 t^2}{1-\theta t} d_x^p d_s^p\right\| \\ &\leq \left\|e-\left(v^+\right)^2\right\| + \frac{\theta^2 t^2}{1-\theta t}\left\|d_x^p d_s^p\right\| \\ &\leq \|d_x d_s\| + \frac{\theta^2 t^2}{1-\theta t}\left\|d_x^p d_s^p\right\| \\ &\leq \sqrt{1+(1+4\kappa')^2\delta^2} + \frac{n(1+2\kappa')\theta^2 t^2}{2(1-\theta t)}\left[1+(1+4\kappa')\delta^2\right]. \end{aligned} \tag{30}$$

Then, plugging (28) and (30) into (29) yields

$$\begin{aligned} \delta(v^p) &\leq \frac{\left\|e-(v^p)^2\right\|}{2\min v^p} \\ &\leq \frac{\sqrt{1+(1+4\kappa')^2\delta^2} + \frac{n(1+2\kappa')\theta^2 t^2}{2(1-\theta t)}\left[1+(1+4\kappa')\delta^2\right]}{2\sqrt{h(\delta,\theta,n)}} \end{aligned}$$

which is the desired result. □

Lemma 3.9. *Let $x^0 s^0 \geq \omega$ and $\theta \leq \frac{1+t}{4(1+4\kappa')\sqrt{n}}$ with $n \geq 2$. If $\delta \leq \frac{t}{2(1+4\kappa')}$, one has $\delta(v^p) \leq \frac{t_p}{2(1+4\kappa')}$.*

Proof. Note that our goal is to keep

$$\delta(v^p) \leq \frac{t_p}{2(1+4\kappa')}. \tag{31}$$

From Lemma 3.8, it follows that (31) holds if

$$\begin{aligned} F := &\sqrt{1+(1+4\kappa')^2\delta^2} + \frac{n(1+2\kappa')\theta^2 t^2}{2(1-\theta t)}\left[1+(1+4\kappa')\delta^2\right] \\ &\leq \frac{t_p}{1+4\kappa'}\sqrt{h(\delta,\theta,n)} := G. \end{aligned} \tag{32}$$

Since $\delta \leq \dfrac{t}{2\left(1+4\kappa'\right)}$ and $\sqrt{1+\left(1+4\kappa'\right)^2} < 2\left(1+2\kappa'\right)$, we have

$$
\begin{aligned}
\frac{F^2}{t^2} - \frac{G^2}{t^2} < & \frac{\left(1+2\kappa'\right)^2 t^2}{4\left(1+4\kappa'\right)^4} + \frac{n^2\left(1+2\kappa'\right)^2 \theta^4 t^2}{4(1-\theta t)^2}\left[1+\frac{t^2}{4\left(1+4\kappa'\right)}\right]^2 \\
& + \frac{n\left(1+2\kappa'\right)^2 \theta^2 t^2}{2\left(1+4\kappa'\right)^2(1-\theta t)}\left[1+\frac{t^2}{4\left(1+4\kappa'\right)}\right] - \frac{(1-\theta)^2}{\left(1+4\kappa'\right)^2} \\
& + \frac{(1-\theta)^2 t^2}{4\left(1+4\kappa'\right)^3} + \frac{n\left(1+2\kappa'\right)(1-\theta)^2\theta^2 t^2}{2\left(1+4\kappa'\right)^2(1-\theta t)}\left[1+\frac{t^2}{4\left(1+4\kappa'\right)}\right] \\
:= & H.
\end{aligned} \tag{33}
$$

In light of (32) and (33), to guarantee the inequality (31) holds, it suffices to show that

$$H \leq 0.$$

Since H is monotonically increasing for $t \in (0,1]$, we obtain

$$
\begin{aligned}
H < & \frac{\left(1+2\kappa'\right)^2}{4\left(1+4\kappa'\right)^4} + \frac{n^2\left(1+2\kappa'\right)^2 \theta^4}{4(1-\theta)^2}\left[1+\frac{1}{4\left(1+4\kappa'\right)}\right]^2 \\
& + \frac{n\left(1+2\kappa'\right)^2 \theta^2}{2\left(1+4\kappa'\right)^2(1-\theta)}\left[1+\frac{1}{4\left(1+4\kappa'\right)}\right] - \frac{(1-\theta)^2}{\left(1+4\kappa'\right)^2} \\
& + \frac{(1-\theta)^2}{4\left(1+4\kappa'\right)^3} + \frac{n\left(1+2\kappa'\right)(1-\theta)^2\theta^2}{2\left(1+4\kappa'\right)^2(1-\theta)}\left[1+\frac{1}{4\left(1+4\kappa'\right)}\right].
\end{aligned} \tag{34}
$$

Multiplying both sides of (34) by $4\left(1+4\kappa'\right)^4(1-\theta)^2$ yields

$$
\begin{aligned}
& 4\left(1+4\kappa'\right)^4(1-\theta)^2 H \\
& < \left(1+2\kappa'\right)^2(1-\theta)^2 - 4\left(1+4\kappa'\right)^2(1-\theta)^4 + \left(1+4\kappa'\right)(1-\theta)^4 \\
& \quad + \left(1+4\kappa'\right)^4\left(1+2\kappa'\right)^2 n^2\theta^4 + \frac{\left(1+4\kappa'\right)^2\left(1+2\kappa'\right)^2 n^2\theta^4}{16} \\
& \quad + \frac{\left(1+4\kappa'\right)^3\left(1+2\kappa'\right)^2 n^2\theta^4}{2} + 2\left(1+4\kappa'\right)^2\left(1+2\kappa'\right) n(1-\theta)^3\theta^2 \\
& \quad + \frac{\left(1+2\kappa'\right)^2\left(1+4\kappa'\right)\left(5+16\kappa'\right) n(1-\theta)\theta^2}{2} \\
& \quad + \frac{\left(1+4\kappa'\right)\left(1+2\kappa'\right) n(1-\theta)^3\theta^2}{2} \\
& := J.
\end{aligned} \tag{35}
$$

Moreover, if $\theta \leq \dfrac{1+t}{4\left(1+4\kappa'\right)\sqrt{n}}$, it follows from (35) that

$$
\begin{aligned}
J < & \left(1+2\kappa'\right)^2 + (1-\theta)^2\left[1+4\kappa' - 4\left(1+4\kappa'\right)^2\right] + \frac{(1+t)^4\left(1+2\kappa'\right)^2}{256} \\
& + \frac{(1+t)^4\left(1+2\kappa'\right)^2}{256\times 16\left(1+4\kappa'\right)^2} + \frac{(1+t)^4\left(1+2\kappa'\right)^2}{256\times 2\left(1+4\kappa'\right)} + \frac{(1+t)^2\left(1+2\kappa'\right)^2\left(5+16\kappa'\right)}{32\left(1+4\kappa'\right)} \\
& + \frac{(1+t)^2\left(1+2\kappa'\right)}{8} + \frac{(1+t)^2\left(1+2\kappa'\right)}{32\left(1+4\kappa'\right)} \\
< & \left(-60\left(\kappa'\right)^2 - 24\kappa' - 2\right) + \frac{\left(1+2\kappa'\right)^2}{16} + \frac{\left(1+2\kappa'\right)^2}{256\left(1+4\kappa'\right)^2} + \frac{\left(1+2\kappa'\right)^2}{32\left(1+4\kappa'\right)} \\
& + \frac{\left(1+2\kappa'\right)^2\left(5+16\kappa'\right)}{8\left(1+4\kappa'\right)} + \frac{1+2\kappa'}{2} + \frac{1+2\kappa'}{8\left(1+4\kappa'\right)}.
\end{aligned} \tag{36}
$$

Then, using $\frac{1+2\kappa'}{1+4\kappa'} < 1$ and (36) leads to

$$256J < -14272(\kappa')^2 - 4976\kappa' - 167. \tag{37}$$

Now, from (35) and (37), we compute that

$$\begin{aligned} 4 \times 256\,(1+4\kappa')^4\,(1-\theta)^2 H &< 256J \\ &< -14272(\kappa')^2 - 4976\kappa' - 167 \\ &< 0, \end{aligned}$$

which indicates that $H < 0$. Thus, the proof is complete. □

3.4. **Iteration bound for the PC IPA.** In this section, we provide an upper bound for $\|x^p s^p - \omega\|$ after a main iteration, and then derive the complexity of the algorithm.

Lemma 3.10. *Let $x^0 s^0 \geq \omega$. If $\delta \leq \frac{t}{2\,(1+4\kappa')}$ and $\theta \leq \frac{1+t}{4\,(1+4\kappa')\,\sqrt{n}}$ with $n \geq 2$, then*

$$\|x^p s^p - \omega\| \leq \left[\frac{5}{4\,(1+4\kappa')}\gamma + \left\|x^0 s^0 - \omega\right\|\right] t,$$

where $\gamma = \max x^0 s^0$.

Proof. First, from (2), we see that

$$\omega(t) = (1-t)\omega + t x^0 s^0 \leq \max\{x^0 s^0, \omega\} e := \gamma e. \tag{38}$$

Combining (2) and (26) yields

$$\begin{aligned} \|x^p s^p - \omega\| &\leq \|x^p s^p - \omega(t_p)\| + \|\omega(t_p) - \omega(t)\| + \|\omega(t) - \omega\| \\ &\leq \left\|e - (v^p)^2\right\| \|\omega(t_p)\|_\infty + \theta t \|\omega(t)\| + \left\|x^0 s^0 - \omega\right\| t. \end{aligned} \tag{39}$$

Next,

$$\begin{aligned} &\left\|e - (v^p)^2\right\| \|\omega(t_p)\|_\infty + \theta t\|\omega(t)\| \\ &\leq \left\{\sqrt{1 + (1+4\kappa')^2}\delta^2 + \frac{n(1+2\kappa')\theta^2 t^2}{2(1-\theta t)}\left[1 + \left(1+4\kappa'\right)\delta^2\right] + \sqrt{n}\theta t\right\} \|\omega(t)\|_\infty \\ &\leq \left\{\frac{(1+2\kappa')\,t}{2\,(1+4\kappa')^2} + \frac{n(1+2\kappa')\theta^2 t}{2(1-\theta t)}\left[1 + \frac{t^2}{4\,(1+4\kappa')}\right] + \sqrt{n}\theta\right\}\gamma\, t \\ &\leq \left\{\frac{(1+2\kappa')}{2\,(1+4\kappa')^2} + \frac{n(1+2\kappa')}{4n\,(1+4\kappa')^2}\frac{\sqrt{2}}{2\sqrt{2}-1}\left[1 + \frac{1}{4\,(1+4\kappa')}\right] + \frac{1}{2\,(1+4\kappa')}\right\}\gamma\, t \\ &< \left[\frac{1}{2\,(1+4\kappa')} + \frac{1}{4\,(1+4\kappa')} \times \frac{4}{5} \times \frac{5}{4} + \frac{1}{2\,(1+4\kappa')}\right]\gamma\, t \\ &= \frac{5}{4\,(1+4\kappa')}\gamma\, t, \end{aligned} \tag{40}$$

where the first inequality is due to (26) and (30). Taking into account of the facts that (38), $\delta \leq \frac{t}{2\,(1+4\kappa')}$ and $\sqrt{1 + (1+4\kappa')^2} < 2\,(1+2\kappa')$, we obtain the second inequality of (40). Since $\theta \leq \frac{1+t}{4\,(1+4\kappa')\,\sqrt{n}}$ with $0 < t \leq 1$, θ gets its maximum value $\theta_{\max}$, i.e.,

$$\theta_{\max} = \frac{1}{2\,(1+4\kappa')\,\sqrt{n}} \leq \frac{1}{2\sqrt{2}}.$$

Thus, the third inequality of (40) holds.

Then, substituting (40) into (39) gives

$$\|x^p s^p - \omega\| < \left[\frac{5}{4(1+4\kappa')}\gamma + \left\|x^0 s^0 - \omega\right\|\right] t$$

which is the desired result. □

Theorem 3.11. *Let $x^0 s^0 \geq \omega$, $(x^0, s^0) \in \mathcal{F}^0$ and $\theta = \dfrac{1+t}{4(1+4\kappa')\sqrt{n}}$ with $n \geq 2$. Then, the algorithm requires at most*

$$\left\lceil (1+4\kappa')\sqrt{n}\log \frac{\frac{5}{4(1+4\kappa')}\max x^0 s^0 + \left\|x^0 s^0 - \omega\right\|}{\varepsilon}\right\rceil + 1$$

iterations to achieve an ε-approximate solution of $P_*(\kappa)$-WLCP (1).

Proof. After k iterations, it follows from Lemma 3.10 that

$$\begin{aligned}\left\|x^k s^k - \omega\right\| &\leq \left[\frac{5}{4(1+4\kappa')}\gamma + \left\|x^0 s^0 - \omega\right\|\right] t_{k-1} \\ &\leq \left[\frac{5}{4(1+4\kappa')}\gamma + \left\|x^0 s^0 - \omega\right\|\right] (1-\theta_{\min})^{k-1},\end{aligned}$$

where $\theta_{\min}$ is the minimum value of θ. Thus, we see that $\left\|x^k s^k - \omega\right\| \leq \varepsilon$ holds, if

$$\left[\frac{5}{4(1+4\kappa')}\gamma + \left\|x^0 s^0 - \omega\right\|\right] (1-\theta_{\min})^{k-1} \leq \varepsilon.$$

By taking logarithms and using $-\log(1-\theta) \geq \theta$ with $\theta \in (0,1)$, the above inequality holds if

$$k \geq \frac{1}{\theta_{\min}} \log \frac{\frac{5}{4(1+4\kappa')}\gamma + \left\|x^0 s^0 - \omega\right\|}{\varepsilon} + 1.$$

Since $\theta \leq \dfrac{1+t}{4(1+4\kappa')\sqrt{n}}$ with $0 < t \leq 1$ and $\gamma = \max x^0 s^0$, the algorithm requires at most

$$\left\lceil (1+4\kappa')\sqrt{n}\log \frac{\frac{5}{4(1+4\kappa')}\max x^0 s^0 + \left\|x^0 s^0 - \omega\right\|}{\varepsilon}\right\rceil + 1$$

iterations to find an ε-approximate solution (x, s) satisfying $\|xs - \omega\| \leq \varepsilon$. □

4. **Numerical results.** In this section, we conduct some numerical experiments to demonstrate that Algorithm 1 is efficient. We implement the simulations on MATLAB R2022a with 12th Gen Intel(R) Core(TM) i5-12500 3.00 GHz processor and 16.0 GB RAM. The number of iterations (Iter) and the running time (CPU) in seconds are calculated when the algorithm terminates. Besides, the termination condition of Algorithm 1 is $\|xs - \omega\| \leq \varepsilon$. Note that the duality gap (Gap) and proximity measure $\delta(v)$ are the values of $\|xs - \omega\|$ and $\frac{1}{2}\|v^{-1} - v\|$, respectively. The accuracy parameter is set as $\varepsilon = 10^{-5}$.

Problem 1. [17] Consider the $P_*(\kappa)$-WLCP (1), where

$$M = \begin{pmatrix} 1 & -2 & -3 & -1 & 1 & -1 & 2 \\ 1 & 1 & 5 & -1 & 1 & -1 & -1 \\ 3 & -3 & 0 & -3 & 3 & -3 & 3 \\ -1 & 2 & 3 & 1 & -1 & 1 & -2 \\ 2 & -4 & -6 & -2 & 2 & -2 & 4 \\ -1 & 2 & 3 & 1 & -1 & 1 & -2 \\ -1 & -1 & -5 & 1 & -1 & 1 & 10 \end{pmatrix},$$

$q = (4\ -4\ 1\ -2\ 7\ -2\ -3)^{\mathrm{T}}$, and $\omega = rand(7,1)$. We take $x^0 = s^0 = e$ as the strictly feasible initial point and choose the update parameter $\theta = 0.2$ for Problem 1. As a result, it takes 0.0521432 seconds and 55 iterations to achieve an ε-approximate solution as below:

$$x^* = (0.0400475\ 0.8990355\ 0.9055148\ 0.0819724\ 0.8008115\ 0.9564732\ 0.8776278)^{\mathrm{T}},$$

$$s^* = (1.0430535\ 0.3513950\ 0.3430169\ 0.9569464\ 1.0861070\ 0.9569464\ 0.5472554)^{\mathrm{T}}.$$

Problem 2. [24] Consider the $P_*(\kappa)$-WLCP (1) with the following $P_*(\kappa)$-matrix

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad \omega = rand(n,1).$$

The initial points are $x^0 = s^0 = e$ and let $q = s^0 - Mx^0$. Set the update parameters as $\theta \in \{0.1, 0.2\}$. We perform Algorithm 1 on Problem 2 with the dimensions $n = 20, 50, 150, 400, 600, 800, 1100$. The numerical results are summarized in Table 2.

TABLE 2. Numerical results of Problem 2

n	$\theta = 0.1$			$\theta = 0.2$		
	CPU	Gap	Iter	CPU	Gap	Iter
20	0.0045	9.4076e-06	119	0.0022	9.2338e-06	57
50	0.0195	9.4954e-06	123	0.0090	8.8642e-06	59
150	0.1873	9.4244e-06	129	0.0899	9.4189e-06	61
400	2.4229	9.4926e-06	133	1.1419	9.5502e-06	63
600	6.2758	9.5998e-06	135	3.0406	9.6510e-06	64
800	12.7843	9.6426e-06	136	6.1306	8.9644e-06	65
1100	27.6153	9.4409e-06	138	13.1285	8.4376e-06	66

Problem 3. [13] Consider the $P_*(\kappa)$-WLCP (1), where

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad \omega = rand(n,1) \in \mathbb{R}^n.$$

Choose $x^0 = s^0 = e$ to be the starting point and $q = s^0 - Mx^0$. Moreover, we set the update parameter as $\theta = 0.25$ and the dimension of the problem as $n \in \{10, 50, 100, 300, 600, 900, 1300\}$. We generate 10 problem instances for each size n. Table 3 lists the average numerical results of Algorithm 1 for solving Problem 3 with different dimension.

TABLE 3. Numerical results of Problem 3

n	CPU	Gap	$\delta(v)$	Iter
10	0.0009	8.7421e-06	7.4332e-11	43
50	0.0063	8.6282e-06	3.1546e-10	46
100	0.0268	8.8371e-06	2.8767e-09	47
300	0.4783	8.8438e-06	1.0170e-07	49
600	2.4864	9.1619e-06	1.7849e-09	50
900	6.3176	8.5291e-06	2.7819e-09	51
1300	16.541	7.7105e-06	1.4474e-09	52

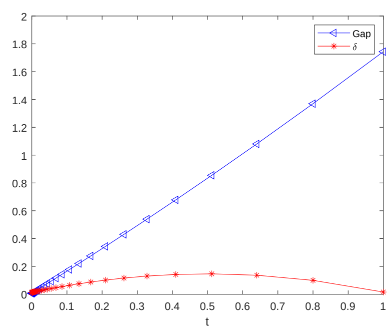

FIGURE 2. The duality gap and proximity measure for Problem 1

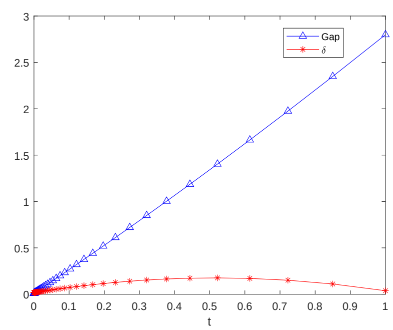

FIGURE 3. The duality gap and proximity measure for Problem 2

In Figure 2 and Figure 3, the duality gap and proximity measure of Problem 1, Problem 2 are depicted. For Problem 3, the duality gap and proximity measure with various n are shown in Figure 4 and Figure 5, respectively.

From all the tables and figures, we summarize our numerical findings as below.

1. From Table 2 and Table 3, the CPU and the Iter of the algorithm depend on the values of n and θ.
2. When θ gets larger, Algorithm 1 takes less time and fewer iterations to solve Problem 2 with the same dimension.
3. Given a fixed value of θ, the number of iterations required by the algorithm does not increase significantly as the size goes up.
4. The duality gap and proximity measure gradually decrease to 0 as t tends to 0.

To sum up, the numerical simulations indicate that the proposed algorithm works well and is effective and efficient.

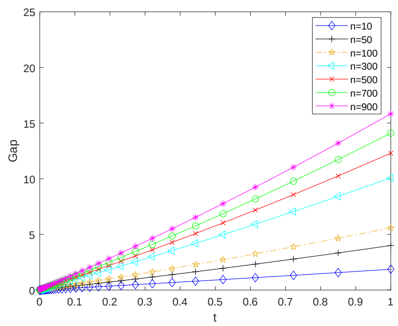

FIGURE 4. The Gap for Problem 3 with various n

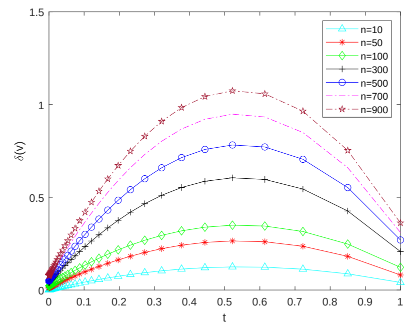

FIGURE 5. $\delta(v)$ for Problem 3 with various n

5. **Conclusions.** In this paper, by using the function $\varphi(t) = t$, we propose a predictor-corrector interior-point algorithm (PC IPA) for solving $P_*(\kappa)$-WLCP. Since there is a nonnegative weight vector in $P_*(\kappa)$-WLCP, the theory and algorithms of $P_*(\kappa)$-WLCP are more complicated than its counterpart $P_*(\kappa)$-LCP. We analyze its complexity and prove that the iteration complexity of this IPA is polynomial, namely,

$$\left\lceil (1+4\kappa')\sqrt{n}\log\frac{\frac{5}{4(1+4\kappa')}\max x^0s^0 + \left\|x^0s^0-\omega\right\|}{\varepsilon}\right\rceil + 1.$$

Moreover, numerical experiments are implemented to verify the practical efficiency of the proposed method. Future research directions include exploring new PC IPAs for $P_*(\kappa)$-WLCP based on other possible functions (like $\varphi(t) = t - \sqrt{t}$, $\varphi(t) = t^2 - t + \sqrt{t}$, $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$, etc.), accompanied by numerical and theoretical comparison.

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Data availability. All data generated or analyzed during this study are included in this article.

Declarations. Conflict of interest The authors declare that they have no conflicts of interest. All authors have reviewed and approved the final manuscript and its submission to the journal.

REFERENCES

[1] S. Asadi, Z. Darvay, G. Lesaja, N. Mahdavi-Amiri and F. Potra, A full-Newton step interior-point method for monotone weighted linear complementarity problems, *J. Optim. Theory Appl.*, **186** (2020), 864-878.

[2] M. Achache, A new primal-dual path-following method for convex quadratic programming, *Comput. Appl. Math.*, **25** (2006), 97-110.

[3] M. Achache, N. Tabchouche, A full-Newton step feasible interior-point algorithm for monotone horizontal linear complementarity problems, *Optim. Lett.*, **13** (2019), 1039-1057.

[4] J.-S. Chen, *SOC Functions and Their Applications*, Springer Optimization and Its Applications, 143. Springer, Singapore, 2019.

[5] J.-S. Chen, J. J. Ye, J. Zhang and J. C. Zhou, Exact formula for the second-order tangent set of the second-order cone complementarity set, *SIAM J. Optim.*, **29** (2019), 2986-3011.

[6] R. W. Cottle, J.-S. Pang and V. Venkateswaran, Sufficient matrices and the linear complementarity problem, *Linear Algebra Appl.*, **114** (1989), 231-249.

[7] X. N. Chi, M. S. Gowda and J. Y. Tao, The weighted horizontal linear complementarity problem on a Euclidean Jordan algebra, *J. Glob. Optim.*, **73** (2019), 153-169.

[8] X. N. Chi, Z. P. Wan and Z. J. Hao, A full-modified-Newton step $O(n)$ infeasible interior-point method for the special weighted linear complementarity problem, *J. Ind. Manag. Optim.*, **18** (2022), 2579-2598.

[9] X. N. Chi, G. Q. Wang and G. Lesaja, Kernel-based full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-weighted linear complementarity problem, *J. Optim. Theory Appl.*, (2023), 1-25.

[10] Z. Darvay, T. Illés, J. Povh and P. R. Rigó, Feasible corrector-predictor interior-point algorithm for $P_*(\kappa)$-linear complementarity problems based on a new search direction, *SIAM J. Optim.*, **30** (2020), 2628-2658.

[11] Z. Darvay, T. Illés and P. R. Rigó, Predictor-corrector interior-point algorithm for $P_*(\kappa)$-linear complementarity problems based on a new type of algebraic equivalent transformation technique, *Eur. J. Oper. Res.*, **298** (2022), 25-35.

[12] Z. Darvay, New interior-point algorithm in linear programming, *Adv. Model. Optim.*, **5** (2003), 51-92.

[13] Y. Fathi, Computational complexity of LCPs associated with positive definite symmetric matrices, *Math. Program.*, **17** (1979), 335-344.

[14] P. Flores, R. Leine and C. Glocker, Modeling and analysis of planar rigid multibody systems with translational clearance joints based on the non-smooth dynamics approach, *Multibody Syst. Dyn.*, **23** (2010), 165-190.

[15] G. Goel and V. V. Vazirani, A perfect price discrimination market model with production, and a rational convex program for it, *Math. Oper. Res.*, **36** (2011), 762-782.

[16] T. Illés, C. Roos and T. Terlaky, *General Linear Complementarity Problems*, Unpublished work, 1997.

[17] T. Illés and S. Morapitiye, Generating sufficient matrices, *Friedler, F. (ed.) Short Papers of the 8th VOCAL Optimization Conference: Advanced Algorithms, Pázmány Péter Catholic University, Budapest*, (2018), 56-61.

[18] K. Jain and V. V. Vazirani, Eisenberg-Gale markets: algorithms and game-theoretic properties, *Games Econom. Behav.*, **70** (2010), 84-106.

[19] M. Kojima, N. Megiddo, T. Noma and A. Yoshise, *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*, Lecture Notes in Computer Science, vol. 538. Springer, New York, 1991.

[20] C. Landry, A. Caboussat and E. Hairer, Solving optimization-constrained differential equations with discontinuity points, with application to atmospheric chemistry, *SIAM J. Sci. Comput.*, **31** (2009), 3806-3826.

[21] S. Mizuno, M. J. Todd and Y. Y. Ye, On adaptive-step primal-dual interior-point algorithms for linear programming, *Math. Oper. Res.*, **18** (1993), 964-981.

[22] S. Mehrotra, On the implementation of a primal-dual interior point method, *SIAM J. Optim.*, **2** (1992), 575-601.

[23] J. Miao, A quadratically convergent $O\left((\kappa+1)\sqrt{n}L\right)$-iteration algorithm for the $P_*(\kappa)$-matrix linear complementarity problem, *Math. Program.*, **69** (1995), 355-368.

[24] K. G. Murty and F. T. Yu, *Linear Complementarity, Linear and Nonlinear Programming*, Berlin: Heldermann, 1998.

[25] M. Nagy, *Interior Point Algorithms for General Linear Complementarity Problems*, PhD thesis, Budapest, Hungary, 2009.

[26] F. A. Potra, Weighted complementarity problems-a new paradigm for computing equilibria, *SIAM J. Optim.*, **22** (2012), 1634-1654.

[27] F. A. Potra, Sufficient weighted complementarity problems, *Comput. Optim. Appl.*, **64** (2016), 467-488.

[28] J. Peng, C. Roos and T. Terlaky, Primal-dual interior-point methods for second-order conic optimization based on self-regular proximities, *SIAM J. Optim.*, **13** (2002), 179-203.

[29] C. Roos, T. Terlaky and J.-P. Vial, *Theory and Algorithm for Linear Optimization-An Interior Point Approach*, New York: John Wiley & Sons, 1997.

[30] G. Sonnevend, J. Stoer and G. Zhao, On the complexity of following the central path by linear extrapolation. II. Interior point methods for linear programming: theory and practice, *Math. Program.*, **52** (1991), 527-553.

[31] J. Y. Tang and H. C. Zhang, A nonmonotone smoothing Newton algorithm for weighted complementarity problem, *J. Optim. Theory Appl.*, **189** (2021), 679-715.

[32] J. Y. Tang and J. C. Zhou, A modified damped Gauss-Newton method for non-monotone weighted linear complementarity problems, *Optim. Methods Softw.*, **37** (2022), 1145-1164.

[33] V. V. Vazirani, The notion of a rational convex program, and an algorithm for the Arrow-Debreu Nash bargaining game, *J. ACM.*, **59** (2012), Art. 7, 36 pp.

[34] G. Q. Wang, A new polynomial interior-point algorithm for the monotone linear complementarity problem over symmetric cones with full NT-steps, *Asia Pac. J. Oper. Res.*, **29** (2012), 1250015, 20 pp.

[35] G. Q. Wang, C. J. Yu and K. L. Teo, A full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-linear complementarity problems, *J. Glob. Optim.*, **59** (2014), 81-99.

[36] Y. Y. Ye, A path to the Arrow-Debreu competitive market equilibrium, *Math. Program.*, **111** (2008), 315-348.

[37] L. P. Zhang and Y. H. Xu, A full-Newton step infeasible interior point algorithm and its parameters analysis, *J. Ind. Manag. Optim.*, **20** (2024), 1916-1933.

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