Optimality and KKT conditions for interval valued optimization problems on Hadamard manifolds

Le Tram Nguyen [1]
Faculty of Mathemactics,
The University of Da Nang - University of Science and Education
and
Department of Mathematics
National Taiwan Normal University
Taipei 116059, Taiwan

Yu-Lin Chang [2]
Department of Mathematics
National Taiwan Normal University
Taipei 116059, Taiwan

Chu-Chin Hu [3]
Department of Mathematics
National Taiwan Normal University
Taipei 116059, Taiwan

Jein-Shan Chen [4]
Department of Mathematics
National Taiwan Normal University
Taipei 116059, Taiwan

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Abstract. Recently, a new type of optimization problems, the so-called interval optimization problems on Hadamard manifolds, is introduced by the authors in [30]. In this follow-up, we further offer the algorithmic bricks for these problems. More specifically, we characterize the optimality and KKT conditions for the interval valued optimization problems on Hadamard manifolds. For unconstrained problems, the existence of efficient points and the steepest descent algorithm are investigated. To the contrast, the KKT conditions and exact penalty approach are explored in the ones involving inequality constraints. These results pave the foundations for the solvability of interval valued optimization problems on Hadamard manifolds.

[1]E-mail: letram07st@gmail.com.
[2]E-mail: ylchang@math.ntnu.edu.tw.
[3]E-mail: cchu@ntnu.edu.tw.
[4]Corresponding author. E-mail: jschen@math.ntnu.edu.tw.

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1 Introduction

It is well known that optimization problems have a lot applications in various research fields. Roughly, due to the features of objective functions, we categorize them as different types of problems like, deterministic problems, stochastic problems, or interval problems. The last type of problem mean the value of objective functions are closed interval in $\mathbb{R}$. In addition, because the variation bound of the uncertain variables can be obtained only through small amount of uncertainly information, the interval programming can easily handle some optimization problems. Nowadays, the uncertainty handling optimization techniques are most powerful to increase the productivity. In general, fuzzy, stochastic and grey optimization techniques are some approaches to tackle these problems. Each of these methods has some strengths and limitations. While formulating mathematical models from the available data, one may replace those by intervals. It could be customer's age, monthly electricity consumption or kiln temperature, etc., see [13, 25] for more information.

There are plenty of optimization problems, which cannot be described on Euclidean space and require the Riemannian manifolds structure. For instance, there exist some real problems in engineering [3] and in control themornuclear fusion research [13], which reflect this demand. The generalization of optimization algorithms from Euclidean space to Riemannian manifold, particularly to Hadamard manifolds, possesses some important advantages [1, 6, 8–10, 19]. Roughly, a Riemannian manifold has no linear structure, nonetheless, it is locally identified with Euclidean space. In this setting, the Euclidean metric is replaced by Riemannian metric and the line segments are replaced by minimal geodesics. Then, the Riemannian optimization problem is, at least locally, equivalent to the smoothly constrained optimization problem on Euclidean space. Consequently, under some circumstance, solving nonconvex constrained problem on $\mathbb{R}^n$ may be equivalent to solving a convex unconstrained problem on Riemannian manifold. All the above provides good motivation to investigate Riemannian optimization problems.

Since the set of all intervals are not linearly ordered, the frequently algorithms for solving optimization problems cannot be easily applied on handling interval valued optimization problems (IOPs). For readers' reference, we do quick literature review as below. Ishibuchi et al. [20] studied the IOPs with linear objective by using the multi-objective programming. Bhurjee and Panda [7] had given a notion of efficient solutions of interval optimization problems, which is similar to the Pareto optimality concept in multi-objective optimization problems. Based on this idea, some authors investigated the optimality condition for IOPs [21, 24]. Gosh [17] applied the Newton method and quasi-Newton method with rank-two to obtain efficient point of the IOPs, which exploit the parametric representation technique. However, all of them were studied on the structure of $\mathbb{R}^n$. To our best knowledge, there is very limited studying on Riemannian interval optimization problems (RIOPs) in the literature.

In this paper, we will generalize the IOPs onto Hadamard manifolds. In [30], Nguyen et al. already established some background materials for RIOPs, which are on the concepts of gH-directional and gH-Gâteaux differentiabilities. Following these results, the existence of solutions to the RIOPs will be explored. For unconstrained case, the steepest descent method for gH-Fréchet diffirentiable problems will be studied, and the partial convergence was obtained, which using Amijo's rule. For the traditional optimization problem, the norm of the gradient of objective function is often used in a stopping criterion. But, in [30], it is noted that, the necessary condition for efficient point of gH-Gâteaux diffirentiable RIOP is

$$0 \in f_G(x)(v), \quad \forall v \in T_x\mathcal{M}.$$

Then, it is impossible to use the approximative gradient in a stopping criterion. We will apply the idea coming from [32], which introduced a great function satisfying the necessary conditions above. On the other hand, in the case of constrained RIOPs, the KKT conditions will be studied. These are more general concepts than those in [10] since the constraint is defined by the interval valued functions. The exact penalty approach will be established to convert the constrained RIOPs to be an unconstrained RIOPs.

This paper is organized as follows. At first, some basic notions and notations about Riemannian manifolds, together with the interval analysis, the Riemannian interval valued functions (RIVF), and their properties are introduced. In the next sections main results, including unconstrained and constrained interval valued optimization in Hadamard manifolds are demonstrated. In Section 3, we study about the existence of efficient point to RIOPs. We also build up the steepest descent algorithm for solving unconstrained RIOPs, together with partial convergence. In Section 4, we consider the constrained Riemannian interval valued optimization problems (CRIOPs) and the Karush–Kuhn–Tucker (KKT) conditions. In addition, we study the exact penalty approach for solving the CRIOPs. At the end, we provide the summary and conclusions in Section 5.

2 Premilinaries

2.1 Interval analysis and Riemannian interval valued functions

In this section, we recall some background materials including interval analysis and Riemannian interval value functions (RIVF). Their properties are important, which will be presented in the subsequent sections.

In light of the traditional notations in most textbooks, see [13] and references therein, we denote $\mathcal{I}(\mathbb{R})$ be the set of all closed bounded intervals in $\mathbb{R}$, i.e.,

$$\mathcal{I}(\mathbb{R}) = \{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R},\ \underline{a} \leq \overline{a}\}.$$

The well known Hausdorff metric d_H on $\mathcal{I}(\mathbb{R})$ is defined by

$$d_H(A, B) = \max\left\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\right\}, \quad \forall A = [\underline{a}, \overline{a}],\ B = [\underline{b}, \overline{b}] \in \mathcal{I}(\mathbb{R}).$$

Then, $(\mathcal{I}(\mathbb{R}), d_H)$ is a complete metric space [26]. In addition, the Minkowski sum and scalar multiplications is described, respectively, by

$$A + B = [\underline{a} + \underline{b}, \overline{a} + \overline{b}],$$
$$\lambda A = \begin{cases} [\lambda \underline{a}, \lambda \overline{a}] & \text{if } \lambda \geq 0, \\ [\lambda \overline{a}, \lambda \underline{a}] & \text{if } \lambda < 0, \end{cases}$$

where $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$. Note that $A - A = A + (-1)A \neq \mathbf{0}$, in general.

A crucial concept in achieving a useful working definition of derivative for interval valued functions is trying to derive a suitable difference between two intervals.

Definition 2.1. [33, Definition 1] Let $A, B \in \mathcal{I}(\mathbb{R})$. The gH-difference between A and B is defined as the interval C such that

$$C = A -_{gH} B \quad \Longleftrightarrow \quad \begin{cases} A = B + C \\ \text{or} \\ B = A - C. \end{cases}$$

Proposition 2.1. *[33, Proposition 4] For any two intervals $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, the gH-difference $C = A -_{gH} B$ always exists and is expressed as*

$$C = \left[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\},\ \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\right].$$

Notice that, for all $A \in \mathcal{I}(\mathbb{R})$, we define $||A||_H := d_H(A, \mathbf{0})$. It is clear that $||A||_H$ is a norm on $\mathcal{I}(\mathbb{R})$ and $d_H(A, B) = ||A -_{gH} B||_H$, (see[29]).

Lemma 2.1. *[18, Lemma 2.2, 2.3] For all $A, B, C \in \mathcal{I}(\mathbb{R})$, the following hold.*

(a) $||A||_H - ||B||_H \leq ||A -_{gH} B||_H$.

(b) $||A -_{gH} B||_H \leq ||(A -_{gH} C) + (C -_{gH} B)||_H$.

Definition 2.2. [34, Definition 3.5] For $A, A_n \in \mathcal{I}(\mathbb{R}), n = 1, 2, ...,$ if $||A_n -_{gH} A||_H = 0$ as $n \to \infty$, then $\{A_n\}$ is said to be convergent to A, for which we denote it as $\lim\limits_{n\to\infty} A_n = A$.

Follow above definition, it is easily to see that if $A_n = [\underline{a_n}, \overline{a_n}], n = 1, 2, ...$ and $A = [\underline{a}, \overline{a}]$, then

$$\lim_{n\to\infty} A_n = A \Longleftrightarrow \begin{cases} \lim\limits_{n\to\infty} \underline{a_n} & = \underline{a} \\ \lim\limits_{n\to\infty} \overline{a_n} & = \overline{a} \end{cases}.$$

Lemma 2.2. *Let $\{A_n\}, \{B_n\}$ be sequences in $\mathcal{I}(\mathbb{R})$. If $\lim_{n\to\infty} A_n$ and $\lim_{n\to\infty} B_n$ exist, then, we have*

$$\lim_{n\to\infty} (A_n -_{gH} B_n) = \boldsymbol{0} \quad \Longrightarrow \quad \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n.$$

Proof. Assume that $A_n = [\underline{a_n}, \overline{a_n}]$ and $B_n = [\underline{b_n}, \overline{b_n}]$. If

$$\lim_{n\to\infty} (A_n -_{gH} B_n) = \mathbf{0},$$

it indicates that for all $\epsilon > 0$, there exists $N > 0$ such that

$$||A_n -_{gH} B_n||_H < \epsilon \quad \Longrightarrow \quad \max\{|\underline{a_n} - \underline{b_n}|, |\overline{a_n} - \overline{b_n}|\} < \epsilon, \text{ for all } n \geq N.$$

This implies

$$\lim_{n\to\infty} (\underline{a_n} - \underline{b_n}) = \lim_{n\to\infty} (\overline{a_n} - \overline{b_n}) = 0 \quad \text{or} \quad \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n,$$

which is the desired result. $\square$

Remark 2.1. We point out that, for any $A, B, C \in \mathcal{I}(\mathbb{R})$,

$$A -_{gH} B = C \quad \not\Rightarrow \quad A = B + C.$$

Please see [30] for more details. Consequently, Lemma 2.2 cannot be extended to the general case, i.e.

$$\lim_{n\to\infty} (A_n -_{gH} B_n) = C \quad \not\Rightarrow \quad \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n + C.$$

There is no natural ordering on $\mathcal{I}(\mathbb{R})$, therefore we need to define it. The following definition is based on the concept in [39].

Definition 2.3. Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two elements of $\mathcal{I}(\mathbb{R})$.

(i) We say A is dominated by B if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$. In this case, we write $A \preceq_{LU} B$.

(ii) We write $A \prec_{LU} B$ if $A \preceq_{LU} B$ and $A \neq B$. Equivalently, $A \prec_{LU} B$ if and only if one of the following cases holds:

1. $\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$.
2. $\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$.

3. $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

(iii) We write $A \prec_{LUst} B$ if $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

(iv) Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of closed intervals. We write $\mathcal{A} \preceq_{LU} \mathcal{B}$ if and only if $A \preceq_{LU} B$ for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Lemma 2.3. *For any elements $A, B, C, D, A_i, B_i, i = 1, ..., n$ of $\mathcal{I}(\mathbb{R})$, there hold*

(i) $A \preceq_{LU} B \iff A -_{gH} B \preceq_{LU} \boldsymbol{0}$.

(ii) $A \not\prec_{LU} B \iff A -_{gH} B \not\prec_{LU} \boldsymbol{0}$.

(iii) $A \preceq_{LU} B \implies A -_{gH} C \preceq_{LU} B -_{gH} C$,

(iv) $A \preceq_{LU} B -_{gH} C \implies B \not\prec_{LU} A + C$.

(v) $\boldsymbol{0} \preceq_{LU} \sum\limits_{i=1}^{n}(A_i -_{gH} B_i) \Longrightarrow \boldsymbol{0} \preceq_{LU} \sum\limits_{i=1}^{n} A_i -_{gH} \sum\limits_{i=1}^{n} B_i$.

Proof. The proofs of (i), (ii), (iii) and (iv) can be found in [30, Lemma 2.10] and we only verify part(v). Assume that $A_i = [\underline{a_i}, \overline{a_i}], B_i = [\underline{b_i}, \overline{b_i}]$, we have

$$
\begin{aligned}
\mathbf{0} \preceq_{LU} \sum_{i=1}^{n}(A_i -_{gH} B_i) &\Longrightarrow \sum_{i=1}^{n} \min\{\underline{a_i} - \underline{b_i}, \overline{a_i} - \overline{b_i}\} \geq 0 \\
&\Longrightarrow \begin{cases} \sum\limits_{i=1}^{n}(\underline{a_i} - \underline{b_i}) & \geq 0 \\ \sum\limits_{i=1}^{n}(\overline{a_i} - \overline{b_i}) & \geq 0 \end{cases} \\
&\Longrightarrow \begin{cases} \sum\limits_{i=1}^{n} \underline{a_i} & \geq \sum\limits_{i=1}^{n} \underline{b_i} \\ \sum\limits_{i=1}^{n} \overline{a_i} & \geq \sum\limits_{i=1}^{n} \overline{b_i} \end{cases} \\
&\Longleftrightarrow \sum_{i=1}^{n} B_i \preceq_{LU} \sum_{i=1}^{n} A_i \\
&\Longleftrightarrow \mathbf{0} \preceq_{LU} \sum_{i=1}^{n} A_i -_{gH} \sum_{i=1}^{n} B_i.
\end{aligned}
$$

Then, the proof is complete. □

Before proceeding to Riemannian interval valued functions (RIVF), we need some notations about Riemannian manifold, which can be found in some textbooks about Riemannian geometry, such as [14, 23, 31]. Let $\mathcal{M}$ be a Riemannian manifold, we denote by $T_x\mathcal{M}$ the tangent space of $\mathcal{M}$ at $x \in \mathcal{M}$, and the tangent bundle of $\mathcal{M}$ is denoted by $T\mathcal{M} = \cup_{x\in\mathcal{M}} T_x\mathcal{M}$. For any $x, y \in \mathcal{M}$, the Riemannian distance $d(x, y)$ on $\mathcal{M}$ is defined by the minimal length over the set of all piecewise smooth curves joining x to y. Let ∇ be the Levi-Civita connection on Riemannian manifold $\mathcal{M}$, $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathcal{M}$ is a smooth curve on $\mathcal{M}$, a vector field X is called parallel along γ if $\nabla_{\gamma'} X = 0$. We say that γ is a geodesic if γ' is parallel along itself, in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1, \gamma$ is said to be normalized. A geodesic joining x and y in $\mathcal{M}$ is called minimal if its length equals $d(x, y)$.

For any $x \in \mathcal{M}$, let U be a neighborhood of $0_x \in T_x\mathcal{M}$, the exponential mapping $\exp_x : U \longrightarrow \mathcal{M}$ is defined by $\exp_x(v) = \gamma(1)$ where γ is the geodesic at $\gamma(0) = x$ such that $\gamma'(0) = v$. It is known that exponential mapping is the special case of retraction mapping [1], and the derivative of $\exp_x$ at $0_x \in T_x\mathcal{M}$ is the identity map; furthermore, by the Inverse Theorem, it is a local diffeomorphism. The inverse map of $\exp_x$ is denoted by $\exp_x^{-1}$. A Riemannian manifold is complete if for any $x \in \mathcal{M}$, the exponential map $\exp_x$ is defined on $T_x\mathcal{M}$. A simply connected, complete Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. If $\mathcal{M}$ is a Hadamard manifold, for all $x, y \in \mathcal{M}$, by the Hopf-Rinow Theorem and Cartan-Hadamard Theorem [23], $\exp_x$ is a diffeomorphism and there exists a unique normalized geodesic joining x and y, which is indeed a minimal geodesic. From now, in this paper, when we mention $\mathcal{M}$, it means that $\mathcal{M}$ is a Hadamard manifold.

Lemma 2.4. *[27, Lemma 2.4] Suppose that $x_0 \in \mathcal{M}$ and $\{x_k\}$ be a sequence in $\mathcal{M}$ with $x_k \to x_0$ as $k \to \infty$. Then, the following hold.*

(i) *For any $y \in \mathcal{M}$, $\exp_{x_k}^{-1}(y) \longrightarrow \exp_{x_0}^{-1}(y)$ and $\exp_y^{-1}(x_k) \longrightarrow \exp_y^{-1}(x_0)$ as $k \to \infty$.*

(ii) *If $\{v_k\}$ is a sequence such that $v_k \in T_{x_k}\mathcal{M}, k = 1, 2, ...$ and $v_k \longrightarrow v_0$ as $k \to \infty$, then $v_0 \in T_{x_0}\mathcal{M}$.*

(iii) *Given the sequences $\{u_k\}, \{v_k\}$ with $u_k, v_k \in T_{x_k}\mathcal{M}, k = 1, 2, ...,$ if $u_k \longrightarrow u_0$ and $v_k \longrightarrow v_0$ as $k \to \infty$, then*

$$\langle u_k, v_k\rangle_{x_k} \longrightarrow \langle u_0, v_0\rangle_{x_0},$$

where $\langle\cdot,\cdot\rangle_{x_k}$ and $\langle\cdot,\cdot\rangle_{x_0}$ are inner products on $T_{x_k}\mathcal{M}$ and $T_{x_0}\mathcal{M}$, respectively.

Lemma 2.5. *[11, Proposition 2.9] Suppose that $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$. Define the function $g : \mathcal{M} \longrightarrow \mathbb{R}$ by*

$$g(y) = \langle v, \exp_x^{-1}(y)\rangle_x.$$

Then, g is affine, in other words, g and $-g$ are geodesically convex functions.

Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty set, a mapping $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is called a Riemannian interval valued function (RIVF). We write $f(x) = [\underline{f}(x), \overline{f}(x)]$, where $\underline{f}, \overline{f}$ are real valued functions satisfy $\underline{f}(x) \leq \overline{f}(x)$, for all $x \in \mathcal{D}$. Since $\mathbb{R}^n$ is a Hadamard manifold, an interval valued function (IVF) $f : U \subseteq \mathbb{R}^n \longrightarrow \mathcal{I}(\mathbb{R})$ is also a RIVF. Furthermore, since $\mathbb{R} \subset \mathcal{I}(\mathbb{R})$, then a Riemannian real valued function $f : \mathcal{D} \longrightarrow \mathbb{R}$ is also a RIVF. For gH-continuity and geodesically convexity of RIVF, see [30].

Definition 2.4. [18, Definition 4.1] Let $\mathcal{V}$ be a normed linear space. The IVF (interval valued function) $F : \mathcal{V} \longrightarrow \mathcal{I}(\mathbb{R})$ is said to be generalized linear (g-linear) if

(i) $F(\lambda v) = \lambda F(v)$, for all $v \in \mathcal{V}$, $\lambda \in \mathbb{R}$; and

(ii) for all $v, w \in \mathcal{V}$, either $F(v)+F(w) = F(v+w)$ or none of $F(v)+F(w)$ and $F(v+w)$ dominates the other.

Definition 2.5. [18, Definition 4.2] Let $(\mathcal{V}, ||\cdot||)$ be a normed linear space. The g-linear IVF $F : \mathcal{V} \longrightarrow \mathcal{I}(\mathbb{R})$ is said to be a bounded g-linear operator if there exists $K > 0$ such that

$$||F(v)||_H \leq K||v||, \quad \forall v \in \mathcal{V}.$$

Lemma 2.6. *[18, Lemma 4.2] Let $\mathcal{V}$ be a normed linear space. If the g-linear IVF $F : \mathcal{V} \longrightarrow \mathcal{I}(\mathbb{R})$ is gH-continuous at $0 \in \mathcal{V}$, then F is a bounded g-linear operator.*

In [30], the authors build up the concepts gH-directional and gH-Gâteaux differentiability of RIVF. They also shown that gH-Gâteaux differentiable does not imply the gH-continuity of interval valued function. We will introduce a stronger concept of the differentiability for RIVF, from which, the gH-continuity is implied.

Definition 2.6 (gH-Fréchet differentiable)**.** Let $\mathcal{D} \subseteq \mathcal{M}$ be a non empty open set and $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a RIVF. For $x_0 \in \mathcal{D}$, if there exists a gH-continuous g-linear IVF $G : T_{x_0}\mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ such that

$$f(\exp_{x_0}(v)) = f(x) + G(v) + R(v),$$

where $R : T_{x_0}\mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ is an IVF and

$$\lim_{||v||_{x_0} \to 0} \frac{||R(v)||_H}{||v||_{x_0}} = 0,$$

then f is said gH-Fréchet differentiable at x_0, and we write $G = f_F(x_0)$, the gH-Fréchet derivative of f at x_0. The RIVF f is called gH-Fréchet differentiable on $\mathcal{D}$ if f is gH-Fréchet differentiable at every $x \in \mathcal{D}$.

Example 2.1. Let $A, B \in \mathcal{I}(\mathbb{R})$ and $f : \mathbb{R}^2 \longrightarrow \mathcal{I}(\mathbb{R})$ such that $f(x) = x_1^2 A + x_2 B$, where $x = (x_1, x_2)$. Consider $x = (0,0)$ and for any $v = (v_1, v_2) \in T_{(0,0)}\mathbb{R}^2 = \mathbb{R}^2$, we have

$$\begin{aligned} f_G(0,0)(v) &= \lim_{t \to 0^+} \frac{1}{t}(f((0,0) + tv) -_{gH} f(0,0)) \\ &= \lim_{t \to 0^+} \frac{1}{t}((tv_1)^2 A + (tv_2)B) \\ &= v_2 B. \end{aligned}$$

It is clear to see that $f_G(0,0)(\cdot)$ is gH-continuous and g-linear IVF. Let $G(\cdot) = f_G(0,0)(\cdot)$, then

$$\begin{aligned} \lim_{h \to 0} \frac{||f(h) -_{gH} f(0,0) -_{gH} G(h)||_H}{||h||} &= \lim_{h \to 0} \frac{||h_1^2 A + h_2 B -_{gH} \mathbf{0} -_{gH} h_2 B||_H}{\sqrt{h_1^2 + h_2^2}} \\ &= \lim_{h \to 0} \frac{||h_1^2 A||_H}{\sqrt{h_1^2 + h_2^2}} \\ &= \mathbf{0}. \end{aligned}$$

Hence, f is gH-Fréchet differentiable at $(0,0)$ and $f_F(0,0)(v) = f_G(0,0)(v) = v_2 B$.

Theorem 2.1. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty open set and letf $: \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a RIVF. If f is gH-Fréchet differentiable at some $x_0 \in \mathcal{D}$, then f is gH-Gâteaux differentiable at x_0 and both the derivetives are coincide.*

Proof. Let $f_F(x_0)$ be the gH-Fréchet derivetive of f at x_0. Then for all $v \in T_{x_0}\mathcal{M} \backslash \{0_{x_0}\}$, we have

$$f(\exp_{x_0}(tv)) -_{gH} f(x_0) -_{gH} f_F(x_0)(tv) = R(tv),$$

then

$$\lim_{t \to 0^+} \frac{||f(\exp_{x_0}(tv)) -_{gH} f(x_0) -_{gH} f_F(x_0)(tv)||_H}{||tv||_{x_0}} = \lim_{t \to 0^+} \frac{||R(tv)||_H}{||tv||_{x_0}} = 0,$$

which says

$$\lim_{t \to 0^+} \frac{1}{t}(f(\exp_{x_0}(tv)) -_{gH} f(x_0) -_{gH} f_F(x_0)(tv)) = \mathbf{0}. \tag{1}$$

Since $f_F(x_0)$ is g-linear, we have $f_F(x_0)(tv) = t f_F(x_0)(v)$. Hence, (1) leads to

$$\lim_{t \to 0^+} \frac{1}{t}(f(\exp_{x_0}(tv)) -_{gH} f(x_0) -_{gH} t f_F(x_0)(v)) = \mathbf{0},$$

This together with Lemma 2.2 yields

$$\lim_{t \to 0^+} \frac{1}{t}(f(\exp_{x_0}(tv)) -_{gH} f(x_0)) = f_F(x_0)(v),$$

which indicates f is gH-Gâteaux differentiable at x_0 and $f_F(x_0) = f_G(x_0)$. $\square$

Theorem 2.2. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty open set. If the RIVF $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is gH-Fréchet differentiable at $x_0 \in \mathcal{D}$, then f is gH-continuous at x_0.*

Proof. Let $f_F(x_0)$ denote the gH-Fréchet derivative of f at x_0. Then, $f_F(x_0)$ is a gH-continuous and g-linear IVF. By Lemma 2.6, there exists $K > 0$ such that for all $v \in T_{x_0}\mathcal{M}$, there holds

$$||f_F(x_0)(v)||_H \leq K||v||_{x_0}.$$

Due to the RIVF f being gH-Fréchet differentiable at x_0, for $\epsilon > 0$ and $v \in T_{x_0}\mathcal{M}$ such that $\exp_{x_0}(v) \in B(x_0, \epsilon)$ ($B(x_0, \epsilon)$ means the geodesic ball with the center at x_0 and radius ϵ in manifold $\mathcal{M}$), we have

$$||f(\exp_{x_0}(v)) -_{gH} f(x_0) -_{gH} f_F(x_0)(v)||_H \leq \epsilon||v||_{x_0}.$$

Thus, together with Lemma 2.1, for all $v \in T_{x_0}\mathcal{M}$ such that $\exp_{x_0}(v) \in B(x_0, \epsilon)$, we see that

$$\begin{aligned}
||f(\exp_{x_0}(v)) -_{gH} f(x_0)||_H &= ||f(\exp_{x_0}(v)) -_{gH} f(x_0)||_H - ||f_F(x_0)(v)||_H + ||f_F(x_0)(v)||_H \\
&\leq ||f(\exp_{x_0}(v)) -_{gH} f(x_0) -_{gH} f_F(x_0)(v)||_H + ||f_F(x_0)(v)||_H \\
&\leq \epsilon||v||_{x_0} + K||v||_{x_0} \\
&= (\epsilon + K)||v||_{x_0}.
\end{aligned}$$

This implies

$$\lim_{||v|| \to 0} (f(\exp_{x_0}(v)) -_{gH} f(x_0)) = \mathbf{0},$$

and hence the RIVF f is gH-continuous at x_0. □

There still exist some RIVFs, which are gH-Gâteaux differentiable, but not gH-Fréchet differentiable. For example, in [30], the authors consider the RIVF

$$\begin{aligned}
f : \mathcal{M} &\longrightarrow \mathcal{I}(\mathbb{R}) \\
(x_1, x_2) &\longmapsto \begin{cases} \dfrac{x_1 x_2^2}{x_1^4 + x_2^2}[1, 2] & \text{if } (x_1, x_2) \neq (0, 0), \\ \mathbf{0} & \text{otherwise,} \end{cases}
\end{aligned}$$

where $\mathcal{M}$ is flat manifold $\mathbb{R}^2$. Then, the function f is gH-Gâteaux differentiable at $(0, 0)$, but f is not gH-continuous at $(0, 0)$. And, by Theorem 2.2, f is not gH-Fréchet differentiable at $(0, 0)$.

3 Unconstraint interval valued problem on Hadamard manifolds

3.1 Existence of solution

Consider the Riemannian interval optimization problem (RIOP):

$$\min_{x\in\mathcal{M}} f(x) \tag{2}$$

where $f : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ is a RIVF. Since the objective function $f(x) = [\underline{f}(x), \overline{f}(x)]$ in the RIOP (2) is an interval-valued function, we can consider two corresponding scalar problems for (2) as follows:

$$\min_{x\in\mathcal{M}} \underline{f}(x) \tag{3}$$

and

$$\min_{x\in\mathcal{M}} \overline{f}(x) \tag{4}$$

Definition 3.1. [30, Definition 4.1] An element $x_0 \in \mathcal{M}$ is said to be an efficient point to the RIOP (2) if

$$f(x) \not\prec_{LU} f(x_0), \forall x \in \mathcal{M}.$$

In this case, $f(x_0)$ is called an efficient objective value of the RIOP (2).

In [30], the authors already proved the Characterization I and II of efficient point to the RIOP (2) in light of the gH-directional differentiability and gH-Gâteaux differentiability, respectively. Based on these results, we further consider the concept of critical point of RIOP (2) as below.

Definition 3.2. Consider the RIOP (2) and let $x_0 \in \mathcal{M}$. We call that x_0 is a critical point of the RIOP (2) if

$$0 \in f'(x_0, v), \quad \forall v \in T_{x_0}\mathcal{M}. \tag{5}$$

Remark 3.1. The necessary condition (5) is equivalent to

$$\overline{f'}(x_0, v) \geq 0, \quad \forall v \in T_{x_0}\mathcal{M}. \tag{6}$$

In fact, it is easy to see that (5) implies (6). Conversely, suppose that there exist $v \in T_{x_0}\mathcal{M}$ such that $\underline{f'}(x_0, v) > 0$, then $\overline{f'}(x_0, -v) = -\underline{f'}(x_0, v) < 0$. This is a contradiction.

Lemma 3.1. *[11, Lemma 3.1] Let $G : \mathcal{M} \longrightarrow \mathcal{M}$ be the set valued mapping such that for each $x \in \mathcal{M}, G(x)$ is closed. Suppose that*

(i) *there exists $x_0 \in \mathcal{M}$ such that $G(x_0)$ is compact;*

(ii) *for any* $x_1, x_2, ..., x_m \in \mathcal{M}$,

$$\text{conv}(\{x_1, x_2, ..., x_m\}) \subset \bigcup_{i=1}^{m} G(x_i).$$

Then, there holds

$$\bigcap_{x\in\mathcal{M}} G(x) \neq \emptyset.$$

Theorem 3.1. *Suppose that* f *is strictly geodesically convex RIVF and* $x_0 \in \mathcal{M}$ *is an optimal solution to problems (3) and (4), simultaneously. Then,* x_0 *is the unique efficient point to the RIOP (2).*

Proof. By [30, Theorem 4.2], x_0 is an efficient point to the RIOP (2). Suppose that there exists $x_1 \in \mathcal{M}\backslash\{x_0\}$ is another efficient point of the RIOP (2). Since f is strictly geodesically convex RIVF, the function $\overline{f}$ or $\underline{f}$ is strictly geodesically convex real function. Without loss of generality, assume that $\overline{f}$ is strictly geodesically convex real function, then x_0 is the unique optimal solution to problem (4). Hence, we conclude that

$$\overline{f}(x_0) < \overline{f}(x_1).$$

Otherwise, x_0 is an optimal solution to (3), which says

$$\underline{f}(x_0) \leq \underline{f}(x_1).$$

Therefore,

$$f(x_0) \prec_{LU} f(x_1),$$

which is a contradiction. $\square$

Theorem 3.2. *Suppose that* $\underline{f}$ *and* $\overline{f}$ *are strictly geodesically convex real functions. Furthermore, assume that* x_0*,* x_1 *are the optimal solutions of (3) and (4), respectively. Let* $\gamma : [0,1] \longrightarrow \mathcal{M}$ *be the minimal geodesic joining* x_0 *with* x_1*. Then for all* $x = \gamma(t), t \in [0,1], x$ *is an efficient point to the RIOP (2).*

Proof. Since $\underline{f}$ and $\overline{f}$ are strictly geodesically convex real functions, f is a strictly geodesically convex RIVF. For all $x = \gamma(t), t \in [0,1]$, we have

$$f(\gamma(t)) \prec_{LU} (1-t)f(x_0) + tf(x_1). \tag{7}$$

Otherwise, by [30, Theorem 4.2] and strictly convexity of $\underline{f}$ and $\overline{f}$, we obtain x_0, x_1 are the efficient points to the RIOP (2). For all $y \in \mathcal{M}$, we have

$$\begin{cases} f(y) & \not\prec_{LU} f(x_0) \\ f(y) & \not\prec_{LU} f(x_1) \end{cases} \Longrightarrow f(y) = (1-t)f(y) + tf(y) \not\prec_{LU} (1-t)f(x_0) + tf(x_1). \tag{8}$$

From (7) and (8), we see

$$f(y) \not\prec_{LU} f(\gamma(t)),$$

which indicates that $x = \gamma(t)$ is an efficient point to the RIOP (2). □

Example 3.1. Let $\mathcal{M} = \mathbb{R}$ with standard metric and consider interval valued problem as below

$$\min \left[\frac{1}{4}x^2, (x-1)^2 + 1 \right]. \tag{9}$$

It is easy to see that

$$0 = \operatorname{argmin} \frac{1}{4}x^2 \quad \text{and} \quad 1 = \operatorname{argmin}[(x-1)^2 + 1].$$

Then, $\forall x \in [0, 1]$, x is an efficient point of problem (9), see Figure 1.

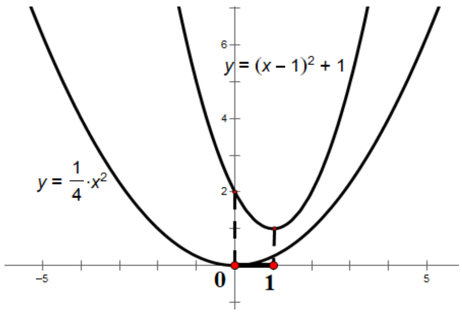

Figure 1: Illustration of Example 3.1

Theorem 3.3. *Suppose that $f : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ be geodesically convex on $\mathcal{M}$ and $f'(x, \cdot)$ is gH-continuous IVF. Assume that, for some $y \in \mathcal{M}$, the set*

$$\left\{ x \in \mathcal{M} | f'(x, \exp_x^{-1}(y)) \not\prec_{LU} \mathbf{0} \right\}$$

is compact. Then, the RIOP (2) reaches an efficient point x^*.

Proof. According to [30, Theorem 4.4], it suffices to prove that the Riemannian interval valued inequality

$$f'(x, \exp_x^{-1}(y)) \not\prec_{LU} \mathbf{0}, \quad \forall y \in \mathcal{M}$$

has a solution x^*. To verify it, for any given $y \in \mathcal{M}$, define

$$G(y) = \left\{ x \in \mathcal{M} | f'(x, \exp_x^{-1}(y)) \not\prec_{LU} \mathbf{0} \right\}.$$

For any $x_1, x_2, ..., x_m \in \mathcal{M}$ and $t_1, t_2, .., t_m \geq 0$ such that $\sum_{i=1}^m t_i = 1$, we will show that the Assumption (ii) of Lemma 3.1 is held. We suppose, by contradiction, there exist x_0 such that

$$x_0 \in \operatorname{conv}(\{x_1, x_2, ..., x_m\}) \setminus \bigcup_{i=1}^m G(x_i),$$

which implies that for any $i = 1, ..., m$, $f'(x_0, \exp_{x_0}^{-1}(x_i)) \prec_{LU} \mathbf{0}$. Hence, we have

$$x_i \in P := \{y \in \mathcal{M} | f'(x_0, \exp_{x_0}^{-1}(y)) \prec_{LU} \mathbf{0}\} \quad \forall i = 1, ..., m.$$

To proceed, we check that P is a geodesically convex set. Let $y_1, y_2 \in P$ and $\gamma : [0, 1] \longrightarrow \mathcal{M}$ be the minimal geodesic joining y_1 and y_2. It follows from Lemma 2.5 that $(\underline{f})'(x_0, \cdot), (\overline{f})'(x_0, \cdot)$ are geodesically convex. Then,

$$\begin{aligned}
f'(x_0, \exp_{x_0}^{-1}(\gamma(t))) = &[\min\{(\underline{f})'(x_0, \exp_{x_0}^{-1}(\gamma(t))), (\overline{f})'(x_0, \exp_{x_0}^{-1}(\gamma(t)))\}, \\
&\max\{(\underline{f})'(x_0, \exp_{x_0}^{-1}(\gamma(t))), (\overline{f})'(x_0, \exp_{x_0}^{-1}(\gamma(t)))\}] \\
&\preceq_{LU} [\min\{t(\underline{f})'(x_0, \exp_{x_0}^{-1}(y_1)) + (1-t)(\underline{f})'(x_0, \exp_{x_0}^{-1}(y_2)), \\
&t(\underline{f})'(x_0, \exp_{x_0}^{-1}(y_1)) + (1-t)(\underline{f})'(x_0, \exp_{x_0}^{-1}(y_2))\}, \\
&\max\{t(\underline{f})'(x_0, \exp_{x_0}^{-1}(y_1)) + (1-t)(\underline{f})'(x_0, \exp_{x_0}^{-1}(y_2), \\
&t(\underline{f})'(x_0, \exp_{x_0}^{-1}(y_1)) + (1-t)(\underline{f})'(x_0, \exp_{x_0}^{-1}(y_2))\}],
\end{aligned} \tag{10}$$

Since $y_1, y_2 \in P$, we have

$$f'(x_0, \exp_{x_0}^{-1}(y_1)) \prec_{LU} \mathbf{0}, \quad f'(x_0, \exp_{x_0}^{-1}(y_2)) \prec_{LU} \mathbf{0}.$$

Consequently, for $i = 1, 2$

$$\begin{cases} \min\{\underline{f}'(x_0, \exp_{x_0}^{-1}(y_i)), \overline{f}'(x_0, \exp_{x_0}^{-1}(y_i))\} < 0 \\ \max\{\underline{f}'(x_0, \exp_{x_0}^{-1}(y_i)), \overline{f}'(x_0, \exp_{x_0}^{-1}(y_i))\} \leq 0. \end{cases} \tag{11}$$

Combining (10) and (11) yields

$$f'(x_0, \exp_{x_0}^{-1}(\gamma(t))) \prec_{LU} \mathbf{0},$$

which says P is a geodesically convex set. Therefore, we obtain that

$$x_0 \in \text{conv}(\{x_1, x_2, ..., x_m\}) \subseteq P,$$

and hence

$$\mathbf{0} = f'(x_0, \exp_{x_0}^{-1}(x_0)) \prec_{LU} \mathbf{0}.$$

This is a contradiction. Thus, $G(\cdot)$ satisfies the Assumption (ii) of Lemma 3.1. Now, by Lemma 3.1, we only need prove that for any $y \in \mathcal{M}, G(y)$ is closed. Consider $x \in \mathcal{M}$ and $\{x_k\}$ be a sequence of $G(y)$ with $x_k \to x$ as $k \to \infty$, we know

$$\lim_{k \to +\infty} || \exp_x^{-1}(x_k)||_x = 0.$$

Since $f'(x, \cdot)$ is gH-continuous on $T_x\mathcal{M}$, it is clear that

$$\lim_{k \to +\infty} f'(x, \exp_x^{-1}(x_k)) = f'(x, 0)$$

Then, it follows from Lemma 2.4 that

$$\lim_{k\to+\infty} f'(x_k, \exp_{x_k}^{-1}(y)) = f'(x, \exp_x^{-1}(y)) \quad \Longrightarrow \quad f'(x, \exp_x^{-1}(y)) \not\prec_{LU} \mathbf{0} \quad \Longrightarrow \quad x \in G(y),$$

which says $G(y)$ is closed. Next, applying Lemma 3.1 gives

$$\bigcap_{y\in\mathcal{M}} G(y) \neq \emptyset.$$

Let $x^* \in \bigcap_{y\in\mathcal{M}} G(y)$, by the definition of $G(\cdot)$, x^* is a solution of Riemannian interval valued variational inequality

$$f'(x^*, \exp_{x^*}^{-1}(y)) \not\prec_{LU} \mathbf{0}, \forall y \in \mathcal{M},$$

which means that the RIOP (2) reaches an efficient point x^*. $\square$

Example 3.2. Let $\mathcal{M} = \mathbb{R}_{++} := \{x \in \mathbb{R} \,|\, x > 0\}$ be endowed with the Riemannian metric given by

$$\langle u, v\rangle_x = \frac{1}{x^2}uv, \quad \forall u, v \in T_xM \equiv \mathbb{R}.$$

Then, it is known that $\mathcal{M}$ is a Hadamard manifold. For all $x \in \mathcal{M}$, $v \in T_x\mathcal{M}$, the geodesic $\gamma : \mathbb{R} \longrightarrow \mathcal{M}$ such that $\gamma(0) = x, \gamma'(0) = v$ is described by

$$\gamma(t) = \exp_x(tv) = xe^{(v/x)t} \quad \text{and} \quad \exp_x^{-1} y = x \ln\frac{y}{x}, \quad \forall y \in \mathcal{M}.$$

We consider the RIOP $\min_{x\in\mathcal{M}} f(x)$ with $f : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ being defined by

$$f(x) = \left[x, x + \frac{1}{x}\right], \quad \forall x \in \mathcal{M}.$$

For all $x, y \in \mathcal{M}$ and $v = \exp_x^{-1} y$, we compute

$$\begin{aligned}
f'(x, v) &= \lim_{t\longrightarrow 0^+} \frac{1}{t}(f(\exp_x(tv)) -_{gH} f(x))) \\
&= \lim_{t\longrightarrow 0^+} \frac{1}{t}\left[\min\left\{x(e^{(v/x)t} - 1), x(e^{(v/x)t} - 1) + \frac{1}{x}(e^{(v/x)t} - 1)\right\},\right. \\
&\quad \left.\max\left\{x(e^{(v/x)t} - 1), x(e^{(v/x)t} - 1) + \frac{1}{x}(e^{(v/x)t} - 1)\right\}\right] \\
&= \left[\min\left\{v, v - \frac{1}{x^2}v\right\}, \max\left\{v, v - \frac{1}{x^2}v\right\}\right] \\
&= v\left[1 - \frac{1}{x^2}, 1\right] \\
&= \ln\frac{y}{x}\left[x - \frac{1}{x}, x\right],
\end{aligned}$$

which says that f is gH-directional differentiable on $\mathcal{M}$. It can be easily verified that $f'(x,\cdot)$ is gH-continuous. In addition, $\forall x, y \in \mathcal{M}$, we have

$$f'(x, \exp_x^{-1} y) \not\prec \mathbf{0} \Longleftrightarrow \ln\frac{y}{x}\left[x - \frac{1}{x}, x\right] \not\prec \mathbf{0}$$
$$\Longleftrightarrow \begin{cases} x \in (0, y] \text{ if } y \geq 1 \\ x \in (0,1) \text{ if } y < 1 \end{cases}.$$

Therefore, for any $y \in \mathcal{M}$, the set

$$\left\{x \in \mathcal{M} | f'(x, \exp_x^{-1}(y)) \not\prec_{LU} \mathbf{0}\right\}$$

is not compact.

On the other hand, by the Cauchy-Schwarz inequality, for all $x > 0$, we have

$$x + \frac{1}{x} \geq 2, \text{ and } x + \frac{1}{x} = 2 \Leftrightarrow x = 1.$$

Consequently,

$$\left[x, x + \frac{1}{x}\right] \not\prec [1,2], \forall x > 0,$$

or $x = 1$ is an efficient point of this RIOP. It means that the converse of Theorem 3.3 is not true.

3.2 Steepest descent method

Consider the RIOP (2) with f is gH-Fréchet differentiable on $\mathcal{M}$. We shall build up an algorithm for solving the interval valued problem on Hadamard manifolds. By Definition 3.2, we know that the gradient of f may be not vanished at the critical point. Thus, we consider the unconstrained minimized subproblem:

$$\min_{v \in T_x\mathcal{M}} \left(h_x(v) := \overline{f'}(x, v) + \frac{1}{2}||v||_x^2\right). \tag{12}$$

For any given point $x \in \mathcal{M}$, the functions $\overline{f'}(x,\cdot)$ is convex (as the maximum of linear functions) and homogeneous. Then, the objective function of problem (12) is proper, closed and strongly convex, it always has a (unique) solution.

Proposition 3.1. *Let $v(x)$ and $h(x)$ be the solution and the optimal value of problem (12), respectively. Then, the following hold.*

(i) *If x is a critical point of RIOP (2), then $v(x) = 0_x$ and $h(x) = 0$.*

(ii) *If x is not a critical point of RIOP (2), then $h(x) < 0$.*

(iii) *Consider the mappings*

$$\begin{aligned} v : \mathcal{M} &\longrightarrow TM \\ x &\longmapsto v(x) \in T_x\mathcal{M} \end{aligned}$$

where $v(x)$ is the optimal solution of problem (12), and

$$\begin{aligned} h : &\mathcal{M} \longrightarrow \mathbb{R} \\ &x \longmapsto h(x) \end{aligned} \tag{13}$$

where $h(x)$ be the optimal value of problem (12). Then v, h are continuous mappings.

Proof. It is easy to see part (i).

For part (ii), since x is not a critical point to the RIOP (2), there exist $v \in T_x\mathcal{M}$ such that $\overline{f'}(x, v) = a < 0$. Let $b := ||v||_x^2$, and for all $t \in \mathbb{R}$, we have

$$h_x(tv) = ta + \frac{t^2}{2}b.$$

This implies $h_x(tv) < 0$ for all $t \in (0, \frac{-a}{b})$. Therefore, $h(x) < 0$.

For part (iii), please see [5, Lemma 5.1]. $\square$

Assumption 3.1. The function f is bounded from below and the level set

$$\Omega_{x_0} := \{x \in \mathcal{M} | f(x) \preceq_{LU} f(x_0)\}$$

is a bounded set.

Algorithm 3.1. [Interval valued Riemannian steepest descent method (IRSD)]

Require: Initial iterate $x_0, \beta \in (0, 1)$;

1. **for** $k = 0, ...$ **do**

2. If x_k is a critical point of the RIOP, stop. Otherwise, define

$$\eta_{v_k} = \arg \min_{v \in T_{x_k}\mathcal{M}} \left(\overline{f'}(x_k, v) + \frac{1}{2}||v||_{x_k}^2 \right) ;$$

3. Compute steplength $t_k \in [0, 1]$ as the maximum of

$$\left\{ t = \frac{1}{2^i} | i \in \mathbb{N}, f(\exp_{x_k}(tv_k)) \preceq_{LU} f(x) + \beta t f'(x_k, v_k) \right\} ;$$

4. $x_{k+1} = \exp_{x_k}(t_k v_k)$;

5. **end for**

Lemma 3.2. *For any $x \in \mathcal{M}$, if there exist $v \in T_x\mathcal{M}$ such that*

$$\overline{f'}(x, v) < 0,$$

then there exist some $\epsilon > 0$ such that

$$f(\exp_x(tv)) \prec_{LUst} f(x) + \beta t f'(x, v),$$

for any $t \in (0, \epsilon]$. In other words, the Step 3 is well defined.

Proof. Because f is gH-Fréchet differentiable, we see that

$$f(\exp_x(v)) = f(x) + f'(x_0, v) + R(v),$$

where $R : T_x\mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ is an IVF such that

$$\lim_{||v|| \to 0} \frac{||R(v)||_H}{||v||_x} = 0.$$

Observe that $v \neq 0_x$ and $\beta < 1$, there exists $\epsilon_1 > 0$ such that

$$0 < t \leq \epsilon_1 \quad \Longrightarrow \quad \frac{\max\{|\underline{R}(tv)|, |\overline{R}(tv)|\}}{||tv||_x} < \frac{(1-\beta)|\overline{f'}(x, v)|}{||v||_x}.$$

Thus, for $0 < t \leq \epsilon_1$, we have

$$\max\left\{|\underline{R}(tv)|, |\overline{R}(tv)|\right\} < t(1-\beta)|\overline{f'}(x, v)|.$$

This together with $\overline{f'}(x, v) < 0$ yields

$$R(tv) \prec_{LUst} -t(1-\beta)f'(x, v).$$

Therefore, for $0 < t \leq \epsilon_1$, we obtain

$$\begin{aligned} f(\exp_x(tv)) &= f(x) + tf'(x, v) + R(tv) \\ &\prec_{LUst} f(x) + tf'(x, v) - t(1-\beta)f'(x, v). \end{aligned} \tag{14}$$

Letting $A = [\underline{a}, \overline{a}] \in \mathcal{I}(\mathbb{R})$ and $t, \beta \in (0, 1)$ give

$$tA - t(1-\beta)A = [t\underline{a} - t\overline{a} + t\beta\overline{a}, t\overline{a} - t\underline{a} + t\beta\underline{a}], \quad t\beta A = [t\beta\underline{a}, t\beta\overline{a}].$$

Consequently,

$$\begin{cases} t\underline{a} - t\overline{a} + t\beta\overline{a} - t\beta\underline{a} = t(\underline{a} - \overline{a})(1-\beta) \leq 0 \\ t\overline{a} - t\underline{a} + t\beta\underline{a} - t\beta\overline{a} = t(\overline{a} - \underline{a})(1-\beta) \geq 0 \end{cases}$$

and

$$t\beta A \subseteq tA - t(1-\beta)A. \tag{15}$$

From (14) and (15), we have

$$\underline{f}(\exp_x(v)) < \underline{f}(x) + t\beta \underline{f'}(x,v), \quad \forall t \in (0, \epsilon_1]. \tag{16}$$

On the other hand, $\overline{f}$ is a differentiable real function, we have

$$\overline{f}(\exp_x(tv)) = \overline{f}(x) + (\overline{f})'(x,v) + r(v),$$

where $r : T_x\mathcal{M} \longrightarrow \mathbb{R}$ is a real valued function such that

$$\lim_{||v||\to 0} \frac{|r(v)|}{||v||_x} = 0.$$

Since $\overline{f'}(x,v) < 0$ and $\beta < 1$, there exists ϵ_2 such that

$$0 < t \leq \epsilon_2 \quad \Longrightarrow \quad \frac{|r(tv)|}{||tv||_x} < \frac{(1-\beta)|\overline{f'}(x,v)|}{||v||_x}.$$

Hence, for $0 < t \leq \epsilon_2$, we achieve

$$r(tv) < -t(1-\beta)\overline{f'}(x).$$

Therefore, for $0 < t \leq \epsilon_2$, we have

$$\begin{aligned} \overline{f}(\exp_x(tv)) &= \overline{f}(x) + t(\overline{f})'(x,v) + r(tv) \\ &< \overline{f}(x) + t\overline{f'}(x,v) - t(1-\beta)\overline{f'}(x,v) \\ &= \overline{f}(x) + t\beta\overline{f'}(x,v). \end{aligned} \tag{17}$$

From (16) and (17), we conclude

$$f(\exp_x(tv)) \prec_{LUst} f(x) + \beta t f'(x,v),$$

for any $t \in (0, \epsilon]$ with $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. $\square$

Theorem 3.4. *Let $\{x_k\}$ be the sequence generated by Algorithm 3.1. Suppose Assumption 3.1 holds, then the sequence $\{f(x_k)\}$ is decreasing and $\{x_k\}$ has at least once accumulation point. Furthermore, every accumulation point of the sequence $\{x_k\}$ is an efficient point to the RIOP (2).*

Proof. The decreasing property of $\{f(x_k)\}$ is directly inferred from Lemma 3.2. By Assumption 3.1, Ω_{x_0} is bounded and f is a gH-continuous RIVF, the Hopf-Rinow's theorem indicates that Ω_{x_0} is a compact set. Now, by Lemma 3.2, $\{x_k\} \subset \Omega_{x_0}$, which says $\{x_k\}$ is bounded. Hence, $\{x_k\}$ has at least once accumulation point.

Let x^* be an accumulation point of the sequence $\{x_k\}$ and let $v(x^*)$ and $h(x^*)$ be the solution and the optimum value of problem (12) at x^*, respectively. In view of Remark 3.1, we need to show that $h(x^*) = 0$.

Consider that $\{x_{k_r}\}$ is a subsequence of $\{x_k\}$ such that $\lim_{r\to+\infty} x_{k_r} = x^*$ from the gH-continuity of f, we have

$$\lim_{r\to+\infty} f(x_{k_r}) = f(x^*).$$

Hence, we obtain

$$\lim_{r\to+\infty} ||f(x_{k_{r+1}}) -_{gH} f(x_{k_r})||_H = 0.$$

On the other hand, we know

$$f(x_{k_{r+1}}) -_{gH} f(x_{k_r}) \preceq_{LU} t_{k_r}\beta f'(x_{k_r}, v_{k_r}) \preceq_{LU} \mathbf{0},$$

which implies

$$\lim_{r\to+\infty} t_{k_r} f'(x_{k_r}, v_{k_r}) = \mathbf{0}. \tag{18}$$

To proceed, we discuss two possibilities: (i) $\limsup_{r\to+\infty} t_{k_r} > 0$; (ii) $\limsup_{r\to+\infty} t_{k_r} = 0$.

Case (i). Let subsequence $\{x_{k_u}\}$ of $\{x_{k_r}\}$ satisfies

$$\lim_{u\to+\infty} t_{k_u} = t_0 > 0.$$

From (18), we have $\lim_{r\to+\infty} f'(x_{k_u}, v_{k_u}) = \mathbf{0}$, which gives $\lim_{u\to+\infty} h(x_{k_u}) = 0$. By Proposition 3.1 (iii), the mapping $x \longmapsto h(x)$ is continuous. This together with $\lim_{u\to+\infty} x_{k_u} = x^*$ concludes that $h(x^*) = 0$, so x^* is an efficient point to the RIOP (2).

Case (ii). Since x_k is not an efficient point to the RIOP (2), from Proposition 3.1 (ii), it says $h(x_k) < 0$ and consequently

$$\overline{f'}(x_k, v_k) + \frac{1}{2}||v_k||^2_{x_k} < 0 \Longrightarrow \overline{f'}(x_k, v_k) < -\frac{1}{2}||v_k||^2_{x_k} < 0.$$

Thus, the sequence $\{v_k\}$ is bounded, which implies the subsequence $\{v_{k_r}\}$ is also bounded. Therefore, we can take a subsequence $\{v_{k_l}\}$ of $\{v_{k_r}\}$, which converges to some $v^* \in T_{x^*}\mathcal{M}$. For all l, we have

$$\overline{f'}(x_{k_l}, v_{k_l}) \leq h(x_{k_l}) < 0,$$

passing onto the limit $l \to +\infty$, we achieve

$$\overline{f'}(x^*, v^*) \leq h(x^*) \leq 0. \tag{19}$$

Take some $n \in \mathbb{N}$, for l large enough $t_{k_l} > 1/2^n$, which means that Armijo condition is not satisfied for $t = 1/2^n$, i.e.,

$$f(\exp_{x_{k_l}}(tv_{k_l})) \npreceq_{LU} f(x_{k_l}) + t\beta f'(x_{k_l}, v_{k_l}), \quad t = 1/2^n,$$

for l large enough. Passing onto limit $l \to +\infty$ yields

$$f(\exp_{x^*}(tv^*)) \not\prec_{LU} f(x^*) + t\beta f'(x *' v^*), t = 1/2^n.$$

Then, from Lemma 3.2, it follow that

$$\overline{f'}(x^*, v^*) \geq 0. \tag{20}$$

Combining (19) and (20) leads to $h(x^*) = 0$. Thus, x^* is an efficient point to the RIOP (2). $\square$

In practice, an algorithm is hoped to terminate in finite time. In view of this, the stopping criterion used in Step 2 of Algorithm 3.1, cannot be employed. For the traditional optimization problem, the norm of the gradient of objective function is often used in a stopping criterion. However, it is impossible with interval valued functions, because the gradient of objective functions now is an IVF, which may be not vanished at a critical point. Nonetheless, since the direction provided by (12) generalizes the steepest descent direction, the optimal value of problem (12) can be used as a stopping criterion for Algorithm 3.1. In particular, we can rewrite the Algorithm 3.1 as below.

Algorithm 3.2. Require: Initial iterate x_0, $\beta \in (0, 1), \alpha > 0$;

1. **for** $k = 0, ...$ **do**
2. Define
$$v_{x_k} = \arg \min_{v \in T_{x_k}\mathcal{M}} h_{x_k}(v) := \overline{f'}(x_k, v) + \frac{1}{2}||v||^2_{x_k};$$
3. If $h_{x_k}(v_k) > -\alpha$, stop. Otherwise, go to Step 4;
4. Compute steplength $t_k \in [0, 1]$ as the maximum of
$$\left\{ t = \frac{1}{2^i} | i \in \mathbb{N}, f(\exp_{x_k}(tv_k)) \preceq_{LU} f(x) + \beta t f'(x_k, v_k) \right\}$$
5. $x_{k+1} = \exp_{x_k}(t_k v_k)$;
6. **end for**

On the other hand, it is interesting for algorithms to deal problem (12) with inexact solution. Suppose that x is not a critical point of the RIOP (2), we say that v is an approximative of problem (12) with tolerance $\delta \in (0,1]$ if

$$\overline{f'}(x,v) + \frac{1}{2}||v||_x^2 \leq \delta h(x).$$

Note that, for $\delta = 1, v$ is the exact solution to problem (12). Therefore, we can consider the steepest descent algorithm to solve the RIOP (2) as below.

Algorithm 3.3. Require: Initial iterate x_0, $\beta \in (0,1), \alpha > 0, \delta \in (0,1]$;

1. **for** $k = 0, ...$ **do**
2. Compute v_k, an approximative solution of problem (12) at $x = x_k$ with tolerance δ;
3. If $h_{x_k}(v_k) > -\alpha$, Stop. Otherwise, go to Step 4;
4. Compute steplength $t_k \in [0,1]$ as the maximum of
$$\left\{t = \frac{1}{2^i} | i \in \mathbb{N}, f(\exp_{x_k}(tv_k)) \preceq_{LU} f(x) + \beta t f'(x_k, v_k)\right\};$$
5. $x_{k+1} = \exp_{x_k}(t_k v_k)$;
6. **end for**

4 Inequality constraints interval valued optimization problems on Hadamard manifolds

4.1 Sovability

Now, we consider the Riemannian interval optimization problem with constraints (CRIOP):

$$\begin{array}{ll} \min & f(x) \\ \text{s.t} & G_i(x) \preceq_{LU} \mathbf{0},\ i = 1, ..., r \\ & x \in \mathcal{M} \end{array} \tag{21}$$

where $f, G_i : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$, $i = 1, ..., r$ are RIVFs. Here $\mathcal{X}$ means the feasible set of CRIOP (21), i.e.,

$$\mathcal{X} := \{x \in \mathcal{M} | G_i(x) \preceq_{LU} \mathbf{0},\ i = 1, ..., r\}.$$

We denote by

$$\text{obj}_P(f, \mathcal{X}) := \{f(x) | x \in \mathcal{X}\}$$

the set of all objective value of CRIOP (21); and $\min(f, \mathcal{X})$ the set of all efficient objective values of the CRIOP (21).

Proposition 4.1. *Consider the CRIOP (21) with $f(x) = [\underline{f}(x), \overline{f}(x)]$ and $\mathcal{X}$ being the feasible set. Given any λ_1, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, if $x_0 \in \mathcal{X}$ is an optimal solution of the following problem*

$$\min_{x \in \mathcal{X}} h(x) = \lambda_1 \underline{f}(x) + \lambda_2 \overline{f}(x),$$

then x_0 is an efficient point to the CRIOP (21).

Proof. In [30], the corresponding proposition with $\lambda_1, \lambda_2 > 0$ was established. Here, we improve it since

$$\arg\min_{x \in \mathcal{X}} h(x) \quad \Longleftrightarrow \quad \arg\min_{x \in \mathcal{X}} \lambda h(x), \ \lambda > 0,$$

for all $h(\cdot)$ is real valued function. In other words, we can assume that $\lambda_1 + \lambda_2 = 1$. $\square$

Note that the constraint $G_i(x) = [\underline{G}_i(x), \overline{G}_i(x)] \preceq_{LU} \mathbf{0}$ is equivalent to $\underline{G}_i(x) \leq \overline{G}_i(x) \leq 0$. Since the objective function $f(x) = [\underline{f}(x), \overline{f}(x)]$, we can further consider two corresponding scalar problems for (21) as follows:

$$\begin{array}{ll} \min & \underline{f}(x) \\ \text{s.t} & \underline{G}_i(x) \leq 0, \ i = 1, ..., r \\ & x \in \mathcal{M} \end{array} \tag{22}$$

and

$$\begin{array}{ll} \min & \overline{f}(x) \\ \text{s.t} & \overline{G}_i(x) \leq 0, \ i = 1, ..., r \\ & x \in \mathcal{M} \end{array} \tag{23}$$

Proposition 4.2. *Consider the CRIOP (21) and the corresponding scalar problems (22) and (23). The following hold.*

(i) *If $x_0 \in \mathcal{M}$ is an optimal solution of problems (22) and (23) simultaneously , then x_0 is an efficient point of the CRIOP (21).*

(ii) *If $x_0 \in \mathcal{X}$ is an unique optimal solution of problems (22) or (23), then x_0 is an efficient point of the CRIOP (21).*

Proof. This is immediate consequence of [30, Proposition 4.2]. $\square$

Karush–Kuhn–Tucker (KKT) conditions are important for optimization problems. The next part of this section is devoted to deriving the KKT conditions for the CRIOP (21). At first, we provide the KKT condition for the following real valued optimization on Hadamard manifold (ROP)

$$\begin{array}{ll} \min & F(x) \\ \text{s.t} & g_i(x) \leq 0, \ i = 1, ..., r \end{array} \tag{24}$$

where $F : \mathcal{M} \longrightarrow \mathbb{R}$ and $g_i : \mathcal{M} \longrightarrow \mathbb{R}, i = 1, ..., r$. Let $X = \{x \in \mathcal{M} | g_i(x) \leq 0, i = 1, ..., r\}$ be the set of feasible point to the ROP (24).

Theorem 4.1. *[10, Theorem 5.1] Consider the ROP (24) with $x_0 \in X$. Suppose that $F, g_i, i = 1, ..., r$ are geodesically convex on $\mathcal{M}$. Furthermore, for every feasible point $x \in X$, there exist scalars $\mu_i \geq 0$, $i = 1, ..., r$ such that*

$$\begin{cases} F'(x_0, \exp_{x_0}^{-1}(x)) + \sum\limits_{i=1}^{m} \mu_i g_i'(x_0, \exp_{x_0}^{-1}(x)) \geq 0, \\ \mu_i g_i(x_0) = 0, \forall i = 1, ..., r. \end{cases}$$

Then, x_0 is an optimal solution to the ROP (24).

Theorem 4.2. *Consider the CRIOP (21) with $x_0 \in \mathcal{X}$. Suppose that $f, G_i, i = 1, ..., r$ are geodesically convex on $\mathcal{M}$. Furthermore, for any feasible point $x \in \mathcal{X}$, there exist scalars $\underline{\mu}_i$, $\overline{\mu}_i \geq 0$, $i = 1, ..., r$ such that*

$$\begin{cases} (\underline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum\limits_{i=1}^{r} \underline{\mu}_i (\underline{G}_i)'(x_0, \exp_{x_0}^{-1}(x)) \geq 0 \\ (\overline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum\limits_{i=1}^{r} \overline{\mu}_i (\overline{G}_i)'(x_0, \exp_{x_0}^{-1}(x)) \geq 0 \\ \overline{\mu}_i \overline{G}_i(x_0) = \underline{\mu}_i \underline{G}_i(x_0) = 0, \forall i = 1, ..., r. \end{cases}$$

Then, x_0 is an efficient point to the CRIOP (21).

Proof. Consider problem (22) and problem (23) with $f, G_i, i = 1, ..., r$ being geodesically convex on $\mathcal{M}$. Following Proposition 2.16 and Lemma 3.9 in [30], the geodesically convexity and gH-directional differentiability of f and $G_i, i = 1, ..., r$ are equivalent to the geodesically convex and directional differentiable properties of $\underline{f}, \overline{f}$ and $\underline{G}_i, \overline{G}_i, i = 1, ..., r$, respectively. Then, by Theorem 4.1, x_0 is an optimal solution of problem (22) and problem (23) simultaneously. Therefore, by Proposition 4.2, x_0 is an efficient point of the CRIOP (21). $\square$

Example 4.1. Let $\mathcal{M}$ be the manifold, which is defined as in Example 3.2. We consider the CRIOP as below:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \mathcal{M}, G(x) \leq \mathbf{0} \end{array} \tag{25}$$

where $f, G : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ are respectively defined by

$$f(x) = \left[x, x + \frac{1}{x}\right], \text{ and } G(x) = [\min\{0, \ln x\}, \max\{0, \ln x\}], \quad \forall x \in \mathcal{M}.$$

We know $\ln x \leq 0 \Longleftrightarrow x \in (0, 1]$, hence the feasible set of above problem is $\mathcal{X} = (0, 1]$. Applying the Cauchy-Schwarz inequality, for all $x > 0$, we have

$$x + \frac{1}{x} \geq 2 \quad \text{and} \quad x + \frac{1}{x} = 2 \Leftrightarrow x = 1,$$

which implies

$$\left[x, x+\frac{1}{x}\right] \not\preceq_{LU} [1,2], \quad \forall x>0.$$

This together with $G(1) = \mathbf{0}$ indicates $x_0 = 1$ is an efficient point to the CRIOP(25). On the other hand, we compute

$$(\underline{f})'(1, \exp_1^{-1}(x)) = \ln x, \quad (\overline{f})'(1, \exp_1^{-1}(x)) = 0,$$
$$(\underline{G})'(1, \exp_1^{-1}(x)) = \ln x, \quad (\overline{G})'(1, \exp_1^{-1}(x)) = 0,$$

for all $x \in \mathcal{X}$. Therefore, for all $x \in (0,1)$, we obtain

$$(\underline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum_{i=1}^{r} \underline{\mu}_i (\underline{G})'_i(x_0, \exp_{x_0}^{-1}(x)) < 0$$
$$(\overline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum_{i=1}^{r} \overline{\mu}_i (\overline{G})'_i(x_0, \exp_{x_0}^{-1}(x)) < 0$$

By Example 3.1, we see that, Theorem 3.2 only provides the sufficient condition.

Theorem 4.3. *Under the same assumption of Theorem 4.2. Suppose that for any feasible point $x \in \mathcal{X}$, there exist scalar $\lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1$ and $\mu_i \geq 0, i = 1, ..., r$ such that*

$$\begin{cases} \lambda_1(\underline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \lambda_2(\overline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum\limits_{i=1}^{r} \mu_i (\overline{G})'_i(x_0, \exp_{x_0}^{-1}(x)) \geq 0 \\ \mu_i \overline{G}_i(x_0) = 0, \ \forall i = 1, ..., r. \end{cases} \tag{26}$$

Then, x_0 is an efficient point to the CRIOP (21).

Proof. Consider the Riemannian real valued problem:

$$\begin{array}{rl} \min & F(x) = \lambda_1 \underline{f}(x) + \lambda_2 \overline{f}(x) \\ \text{s.t} & \overline{G}_i(x) \leq 0, \ i = 1, ..., r \\ & x \in \mathcal{M} \end{array}$$

Then, by the condition (26) and Theorem 4.1, x_0 is an efficient point of the above problem. Since $\overline{G}_i(x_0) \leq 0, i = 1, ..., r$, we have $G_i(x_0) \preceq_{LU} \mathbf{0}$, $i = 1, ..., r$. This implies that $x_0 \in \mathcal{X}$. In view of Proposition 4.1, we have x_0 is an efficient point to the CRIOP (21). $\square$

Theorem 4.4. *Under the same assumption of Theorem 4.2. Suppose that for any feasible point $x \in \mathcal{X}$, there exist scalar $\mu_i \geq 0, i = 1, ..., r$, such that*

$$\begin{cases} \mathbf{0} \preceq_{LU} f'(x_0, \exp_{x_0}^{-1}(x)) + \sum\limits_{i=1}^{r} \mu_i G'_i(x_0, \exp_{x_0}^{-1}(x)) \\ \mu_i G_i(x_0) = \mathbf{0}, \ \forall i = 1, ..., r. \end{cases}$$

Then, x_0 is an efficient point to the CRIOP (21).

Proof. By assumption, we have

$$
\begin{aligned}
(\underline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum_{i=1}^{r} \mu_i (\underline{G}_i)'(x_0, \exp_{x_0}^{-1}(x)) &\geq \min\{(\underline{f})'(x_0, \exp_{x_0}^{-1}(x)), (\overline{f})'(x_0, \exp_{x_0}^{-1}(x))\} \\
&+ \sum_{i=1}^{r} \mu_i \min\{(\underline{G}_i)'(x_0, \exp_{x_0}^{-1}(x)), (\overline{G}_i)'(x_0, \exp_{x_0}^{-1}(x))\} \geq 0, \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
(\overline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum_{i=1}^{r} \mu_i (\overline{G}_i)'(x_0, \exp_{x_0}^{-1}(x)) &\geq \min\{(\underline{f})'(x_0, \exp_{x_0}^{-1}(x)), (\overline{f})'(x_0, \exp_{x_0}^{-1}(x))\} \\
&+ \sum_{i=1}^{r} \mu_i \min\{(\underline{G}_i)'(x_0, \exp_{x_0}^{-1}(x)), (\overline{G}_i)'(x_0, \exp_{x_0}^{-1}(x))\} \geq 0. \qquad (28)
\end{aligned}
$$

Combining (27), (28) and $\mu_i G_i(x_0) = \mathbf{0}$ for all $i = 1, ..., r$, we achieve

$$
\begin{cases}
(\underline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum\limits_{i=1}^{r} \mu_i (\underline{G}_i)'(x_0, \exp_{x_0}^{-1}(x)) \geq 0 \\
(\overline{f})'(x_0, \exp_{x_0}^{-1}(x)) + \sum\limits_{i=1}^{r} \mu_i (\overline{G}_i)'(x_0, \exp_{x_0}^{-1}(x)) \geq 0 \\
\overline{\mu}_i \overline{G}_i(x_0) = \mu_i \underline{G}_i(x_0) = 0, \forall i = 1, ..., r.
\end{cases}
$$

Hence, by Theorem 4.2, we show that x_0 is an efficient point to the CRIOP (21). □

Example 4.2. Let $\mathcal{M} = \mathbb{R}^2$ with standard metric. Then, $\mathcal{M}$ is a flat Hadamard manifold. Consider the RIOP as below:

$$
\begin{array}{ll}
\min & (f(x) := [\min\{x_1 + 2x_2, 2x_1 + x_2\}, \max\{x_1 + 2x_2, 2x_1 + x_2\}]) \\
\text{s.t.} & G_1(x) = [\min\{x_1 - x_2, -x_1\}, \max\{x_1 - x_2, -x_1\}] \preceq_{LU} \mathbf{0}, \\
 & G_2(x) = [\min\{x_2 - x_1, -x_2\}, \max\{x_2 - x_1, -x_2\}] \preceq_{LU} \mathbf{0},
\end{array} \qquad (29)
$$

where $x = (x_1, x_2)$. It is easy to see that the feasible point set is $\mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 = x_2\}$. At $x_0 = (0, 0)$, for all $x = (x_1, x_2) \in \mathcal{X}$, we compute

$$
\begin{aligned}
f'(x_0, \exp_{x_0}^{-1}(x)) &= [\min\{2x_1 + x_2, x_1 + 2x_2\}, \max\{2x_1 + x_2, x_1 + 2x_2\}], \\
G_1'(x_0, \exp_{x_0}^{-1}(x)) &= [-x_1, 0], \\
G_2'(x_0, \exp_{x_0}^{-1}(x)) &= [-x_2, 0].
\end{aligned}
$$

Therefore, with $\mu_1 = \mu_2 = 1$, we have

$$
\begin{cases}
\mathbf{0} \preceq_{LU} f'(x_0, \exp_{x_0}^{-1}(x)) + \sum\limits_{i=1}^{r} \mu_i G_i'(\exp_{x_0}^{-1}(x)) \\
\mu_i G_i(x_0) = 0, \ \forall i = 1, ..., r.
\end{cases}
$$

This says that $x = (0, 0)$ is an efficient point to the CRIOP (29).

Denoting by $\min(f, \mathcal{X})$ the set of optimal values of CRIOP (21), we further define the interval valued Langrangian function for the CRIOP (21) as follow:

$$L(x, \mu) = f(x) + \sum_{i=1}^{r} \mu_i G_i(x),$$

for all $x \in \mathcal{M}$ and $\mu_i \geq 0$ for all $i = 1, ..., r$.

Definition 4.1. A vector $\mu^* = (\mu_1^*, ..., \mu_r^*)$ is said to be a *Lagrange multiplier* for the CRIOP (21) if

$$\mu_i \geq 0, i = 1, ..., r,$$

and

$$\min(f, \mathcal{X}) = \min_{x \in \mathcal{M}} L(x, \mu^*).$$

Proposition 4.3. *Let μ^* be a Lagrange multiplier of the CRIOP (21). Then, x^* is an efficient point to the CRIOP (21) if and only if x^* is a feasible point and*

$$x^* \in \arg\min_{x \in \mathcal{M}} L(x, \mu^*), \quad \mu_i^* G_i(x^*) = \boldsymbol{0}, \ \forall i = 1, ..., r. \tag{30}$$

Proof. If x^* is an efficient point to the CRIOP (21), it is clear that x^* is a feasible point. To see the remaining, since μ_i^* is a Lagrange multiplier, we know

$$f(x^*) \in \min_{x \in \mathcal{M}} L(x, \mu^*).$$

It implies that

$$\begin{aligned} & L(x^*, \mu^*) \not\prec_{LU} f(x^*) \\ \Rightarrow & f(x^*) + \sum_{i=1}^{r} \mu_i^* G_i(x^*) \not\prec_{LU} f(x^*). \end{aligned}$$

Consequently,

$$\sum_{i=1}^{r} \mu_i^* G_i(x^*) \not\prec_{LU} \mathbf{0}.$$

For $i = 1, ..., r$, since $G_i(x^*) \preceq_{LU} \mathbf{0}$, and $\mu_i^* \geq 0$, we have $\mu_i^* G_i(x^*) = \mathbf{0}$.

Conversely, if x^* is a feasible point and (30) holds, then

$$f(x^*) = f(x^*) + \sum_{i=1}^{r} \mu_i^* G_i(x^*) = L(x^*, \mu^*) \in \min_{x \in \mathcal{M}} L(x, \mu^*) = \min(f, \mathcal{X}),$$

which says x^* is an efficient to the CRIOP (21). $\square$

5 Exact penalty approach

Consider the CRIOP (21). Based on the optimization problem with absolute valued penalty or exact l_1 penalty function, we propose the following unconstrained interval valued penalizad optimization problem involving exact l_1 penalty function for the given constrains in (21):

$$\begin{aligned} \min \quad & f(x) + l \sum_{i=1}^{r} G_i^+(x) \\ \text{s.t} \quad & x \in \mathcal{M} \end{aligned} \tag{31}$$

where $f, G_i, i = 1, ..., r$ are defined as in (21); $l > 0$ is a penalty parameter and for given constraint $G_i(x)$; and the function $G_i^+(x)$ is defined by

$$G_i^+(x) = \begin{cases} \mathbf{0} & \text{if } G_i(x) \preceq_{LU} \mathbf{0} \\ [0, \overline{G}_i(x)] & \text{if } 0 \in (\underline{G}_i(x), \overline{G}_i(x)) \\ G_i(x) & \text{if } \mathbf{0} \prec_{LU} G_i(x). \end{cases}$$

In [22], the exact l_1 penalty method for interval valued optimization problems on $\mathbb{R}^n$ is investigated, for which the constraints functions are real valued functions. To the contrast, our work not only generalizes from Euclidean space to Hadamard manifolds, but also study with the more general class of constraint functions.

Next, we connect the relationship between the CRIOP (21) and penalized optimization problem (31).

Theorem 5.1. *Let x_0 be a feasible point of the CRIOP (21). Assume that there exist $\mu^* = (\mu_1^*, \mu_2^*, ..., \mu_r^*) > 0$ such that*

(i) *the KKT optimal conditions of Theorem 4.4 hold;*

(ii) *f and $G_i, i = 1, ..., r$ are geodesically convex on $\mathcal{M}$;*

(iii) *the penalty parameter l is sufficiently large.*

Then, x_0 is an efficient point to penalized optimization problem (31).

Proof. Suppose that x_0 is not an efficient point to penalized optimization problem (31), then there exists $x_1 \in \mathcal{M}$ such that

$$f(x_1) + l \sum_{i=1}^{r} G_i^+(x_1) \prec_{LU} f(x_0) + l \sum_{i=1}^{r} G_i^+(x_0). \tag{32}$$

By using the assumption of f and $G_i, i = 1, ..., r$ being geodesically convex on $\mathcal{M}$, we have

$$\begin{cases} f'(x_0, \exp_{x_0}^{-1}(x_1)) & \preceq_{LU} f(x_1) -_{gH} f(x_0) \\ G_i'(x_0, \exp_{x_0}^{-1}(x_1)) & \preceq_{LU} G_i(x_1) -_{gH} G_i(x_0), i = 1, ..., r, \end{cases}$$

Since $\mu^* > 0$, it further implies

$$\begin{cases} f'(x_0, \exp_{x_0}^{-1}(x_1)) & \preceq_{LU} f(x_1) -_{gH} f(x_0) \\ \mu_i^* G_i'(x_0, \exp_{x_0}^{-1}(x_1)) & \preceq_{LU} \mu_i^* \left(G_i(x_1) -_{gH} G_i(x_0)\right), i = 1, ..., r, \end{cases}$$

Adding the above inequalities yields

$$f'(x_0, \exp_{x_0}^{-1}(x_1)) + \sum_{i=1}^{r} \mu_i^* G_i'(x_0, \exp_{x_0}^{-1}(x_1)) \preceq_{LU} (f(x_1) -_{gH} f(x_0)) + \sum_{i=1}^{r} \mu_i^* \left(G_i(x_1) -_{gH} G_i(x_0)\right).$$

Since the assumption (i) holds, we have

$$\mathbf{0} \preceq_{LU} (f(x_1) -_{gH} f(x_0)) + \sum_{i=1}^{r} \mu_i^* \left(G_i(x_1) -_{gH} G_i(x_0)\right).$$

Then, applying Lemma 2.3(e) gives

$$f(x_0) + \sum_{i=1}^{r} \mu_i^* G_i(x_0) \preceq_{LU} f(x_1) + \sum_{i=1}^{r} \mu_i^* G_i(x_1).$$

Since x_0 satisfies the KKT optimal conditions in Theorem 4.4, we know

$$\mu_i^* G_i(x_0) = \mathbf{0},$$

and hence,

$$f(x_0) \preceq_{LU} f(x_1) + \sum_{i=1}^{r} \mu_i^* G_i(x_1).$$

In addition, Using the definition of $G_i^+(\cdot)$ leads to

$$f(x_0) + l \sum_{i=1}^{r} G_i^+(x_0) \preceq_{LU} f(x_1) + \sum_{i=1}^{r} \mu_i^* G_i^+(x_1) \preceq_{LU} f(x_1) + l \sum_{i=1}^{r} G_i^+(x_1),$$

where $l > \max\{\mu_i^*, ..., \mu_i^r\}$, which contradicts (32). Thus, the proof is complete. □

Lemma 5.1. *Let x_0 be an efficient point of penalized optimization problem (31). Then, for all $x \in \mathcal{X}$, we have*

$$f(x) \not\prec_{LU} f(x_0).$$

Proof. Since x_0 is an efficient point of penalized optimization problem (31), it is clear that

$$f(x) + l \sum_{i=1}^{r} G_i^+(x) \not\prec_{LU} f(x_0) + l \sum_{i=1}^{r} G_i^+(x_0) \quad \text{for all } x \in \mathcal{M}.$$

Note that $\mathcal{X} \subseteq \mathcal{M}$, which indicates

$$f(x) + l\sum_{i=1}^{r} G_i^+(x) \not\prec_{LU} f(x_0) + l\sum_{i=1}^{r} G_i^+(x_0), \quad \forall x \in \mathcal{X}.$$

Using definition of $G_i^+(\cdot)$, we have $G_i^+(x) = \mathbf{0}$ for all $x \in \mathcal{X}$ and $\mathbf{0} \preceq_{LU} G_i^+(x)$ for all $x \in \mathcal{M}$. Consequently, we conclude

$$f(x) \not\prec_{LU} f(x_0), \quad \forall x \in \mathcal{X}$$

which is the desired result. $\square$

Theorem 5.2. *Let x_0 be an efficient point to penalized optimization problem (31). Suppose that for all $x \in \mathcal{X}$ and any $\bar{l} > l$, there has*

$$f(x_0) + \bar{l}\sum_{i=1}^{r} G_i^+(x_0) \preceq_{LU} f(x) + \bar{l}\sum_{i=1}^{r} G_i^+(x).$$

Then, x_0 is an efficient point to the CRIOP (21).

Proof. Since x_0 is an efficient point to penalized optimization problem (31), by applying Theorem 5.1, we have

$$f(x) \not\prec_{LU} f(x_0) \quad \text{for all } x \in \mathcal{X}.$$

Hence, in order to show the desired result, we only need to verify that x_0 is a feasible point of the CRIOP (21). Suppose x_0 is not a feasible point of the CRIOP (21), that is,

$$\mathbf{0} \prec_{LU} \sum_{i=1}^{r} G_i^+(x_0).$$

Let $\overline{x} \in \mathcal{X}$ be any feasible point of the CRIOP (21), and set

$$\bar{l} > \max\left\{ \frac{\overline{f}(\overline{x}) - \overline{f}(x_0)}{\sum\limits_{i=1}^{r} \overline{G_i^+}(x_0)}, l \right\}.$$

Then, we have

$$\overline{f}(\overline{x}) = \overline{f}(\overline{x}) + \bar{l}\sum_{i=1}^{r} \overline{G_i^+}(\overline{x}) \geq \overline{f}(x_0) + \bar{l}\sum_{i=1}^{r} \overline{G_i^+}(x_0) > \overline{f}(\overline{x}),$$

which is a contradiction. Thus, x_0 is a feasible point of (21) and the proof is complete. $\square$

6 Conclusions

In this paper, we study the interval valued optimization problems on Hadamard manifolds, including unconstrained and constrained settings. To achieve the theoretical results, we build up new concepts regarding gH-Fréchet differentiability of interval valued functions and their properties on the Hadamard manifolds. For unconstrained case, the existence of efficient point and steepest descent algorithm for solving the RIOPs was studied. The constrained functions employed for constrained case in this paper are interval valued functions, which is more general than those in the literature. We also illustrate some examples to verify the obtained results. The Lagrange multiplier was considered together with some KKT conditions. Like dealing with the traditional optimization, the exact penalty approach was established for solving the CRIOPs. We believe that, our discovery is a small step, yet in the right direction, towards understanding and realizing the power of interval valued optimization in nonlinear spaces.

There are many open questions, which are not yet answered. We summarize a few important ones as below for future directions.

- The convergence of algorithms in Non-Euclidean spaces are more difficult then the usual one. In this paper, it is only the partial convergence. The full/local convergence are not done yet.
- The steepest descent algorithm is just applied with Amijo's rule stepsize. The other case, example for constant stepsize with Lipschitz objective function, may be studied in the future.
- The second order algorithms are very important in the case of real valued objective function. However, in our knowledge, in case of interval valued objective function, the results are very limited.
- For CRIOPs, the duality is one of important approachs. Wu [39] studied the duality for the case of Euclidean space, which use the Hukuhara difference. The corresponding case for nonlinear space are not done yet. Especially, for more general case, which uses the gH-difference.

References

[1] P. A. Absil, R. Mahony, and R. Sepulchre, *Optimization Algorithms on Matrix Manifolds*, Princeton University Press, Princeton, NJ, 2008.

[2] I. Aguirre-Cipe, R. Lopez, E. Mallea, and L. Vásquez, *A study of interval optimization problems*, Optimization Letters, 15(3), 859-877, 2021.

[3] M. Bacak, *Convex Analysis and Optimization in Hadamard Spaces*, De Gruyter, 2014.

[4] G.C. Bento, O.P. Ferreira, and P.R. Oliveira, *Local convergence of the proximal point method for a special class of nonconvex functions on Hadamard manifolds*, Nonlinear Analysis: Theory, Methods and Applications, 74(2), 564-572, 2010.

[5] G.C. Bento, O.P. Ferreira, and P.R. Oliveira, *Unconstrained steepest descent method for multicriteria optimization on Riemmanian manifolds*, Journal of Optimization Theory and Applications, 154(1),88-107, 2012

[6] G.C. Bento and J. Melo, *Subgradient method for convex feasibility on Riemannian manifolds*, Journal of Optimization Theory and Applications, 152(3), 773-785, 2012.

[7] A. Bhurjee and G. Panda, *Efficient solution of interval optimization problem*, Mathematical Method of Operations Research, 76(3), 273-288, 2013.

[8] N. Boumal, *An Introduction to Optimization on Smooth Manifolds*, Cambridge University Press., November, 2020.

[9] S.I. Chen and N.J. Huang, *Generalized invexity and generalized invariant monotone vector fields on Riemannian manifolds with applications*, Journal of Nonlinear and Convex Analysis, 16(7), 1305-1320, 2015.

[10] S.I. Chen, *The KKT optimality conditions for optimization problem with interval-valued objective function on Hadamard manifolds*, Optimization, 71(3), 613-632, 2022.

[11] V. Colao, G. López, G. Marino, and V. Martín-Márquez, *Equilibrium problem in Hadamard manifolds*, Journal of Mathematical Analysis and Applications, 338, 61-77, 2012.

[12] R.W. Cottle, *Complementarity and Variational Problems*, Defense Technical Information Center, 1974.

[13] P. Diamond and P. E. Kloeden, *Metric Spaces of Fuzzy Sets: Theory and Applications*, World Scientific, 1994.

[14] M.P. do Carmo, *Riemannian Geometry*, Birkhäuser Boston, 1992.

[15] J.X. Da Cruz Neto, O.P. Ferreira, L.R. Lucâmbio Pérez, and S.Z. Németh, *Convex-and monotonetransformable mathematical programming problems and a proximal-like point method*, Journal of Global Optimization, 35, 53–69, 2006.

[16] M.C. Ferris and J.S. Pang, *Complementarity and Variational Problems: State of the Art*, Proceedings in Applied Mathematics, Society for Industrial and Applied Mathematics, 1997.

[17] D. Ghosh, *Newton method to obtain efficient solutions of the optimization problems with interval-valued objective functions*, Journal of Applied Mathematics and Computing, 53(1-2), 709-731, 2017.

[18] D. Ghosh, R. Chauhan, R. Mesiar, and A. Debnath, *Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions*, Information Sciences, 510(4), 317-340, 2020.

[19] W.L. Huang, *Optimization algorithms on Riemannian manifolds with applications*, PhD. thesis, Florida State University, 2013.

[20] H. Ishibuchi and H. Tanaka, *Multiobjective programming in optimization of the interval objective function*, Eur J Oper Res 48(2), 219–225, 1990.

[21] M. Jana and G. Panda, *Solution of nonlinear interval vector optimization problem*, Operational Research - An International Journal, 14, 71–85, 2014.

[22] A. Jayswal and J. Banerjee, *An exact l_1 penalty approach for interval valued programming problem*, Journal of the Operations Research Society of China, 4, 461-481, 2016.

[23] J. Jost, *Riemannian Geometry and Geometric Analysis*, Universitext Book Series, Springer, Berlin Heidelberg, 2011.

[24] P. Kumar, G. Panda, and U.C. Gupta, *Generalized quadratic programming problem with interval uncertainty,* FUZZ IEEE 2013. In: 2013 IEEE international conference on Fuzzy systems (FUZZ), 2013.

[25] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Classical in Applied Mathemactics 31, Society for Industrial and Applied Mathemactics, 2000.

[26] V. Lakshmikantham, T.G. Bhaskar, and J.V. Devi, *Theory of Set Differential Equations in Metric Spaces*, Cambridge Scientific Publishers, 2005.

[27] C. Li, G. López, V. M. Márquez, *Monotone vector fields and the proximal point algorithm on Hadamard manifolds*, J. London Math. Soc., 79(2), 663-683, 2009.

[28] C. Li, B. Mordukhovich, J. Wang, and J.-C. Yao, *Weak sharp minima on Riemannian manifolds*, SIAM Journal on Optimization, 21(4), 1523-1260, 2011.

[29] S. Markov Sofia, *Calculus for Interval Functions of a Real Variable*, Computing 22, 325–337 (1979).

[30] L. T. Nguyen, Y.-L. Chang, C.-C. Hu, and J.-S. Chen, *Interval Optimization Problems on Hadamard manifolds*, Journal of Nonlinear and Convex Analysis, 24(11), 2489-2511, 2023.

[31] P. Petersen, *Riemannian Geometry*, Springer International Publishing, 2016.

[32] S. Schaffler, R. Schultz, and K. Weinzierl, *Stochastic method for the solution of unconstrained vector optimization problems*, J. Optim. Theory Appl., 114, 209–222, 2002.

[33] L. Stefanini, *Generalized Hukuhara differentiability of interval-valued functions and interval differential equations*, Nonlinear Analysis, 71(3-4), 1311-1328, 2008.

[34] J. Tao and Z. Zhang, *Properties of interval valued function space under the gH-difference and their application to semi-linear interval diffirential equations*, Advances in Difference Equations, Article no. 45, 2016.

[35] C. Udriste, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Mathematics and Its Applications, volume 297, Springer, 1994.

[36] N.K. Vishnoi, *Geodesic Convex Optimization: Differentiation on manifolds, geodesics, and convexity*, Lecture Notes, arXiv:1806.06373, 2018.

[37] P. Wofle, *A duality theorem for nonlinear programming*, Quarterly of Applied Mathematics, 19(3), 239-244, 1961.

[38] H.-C. Wu, *The Karush–Kuhn–Tucker optimality conditions in an optimization problem with interval-valued objective function*, European Journal of Operational Research, 176(1), 46-59, 2007.

[39] H.-C. Wu, *Wolfe duality for interval-valued optimization*, Journal of Optimization Theory and Applications, 138(3), 497-509, 2008.