INTERVAL OPTIMIZATION PROBLEMS ON HADAMARD MANIFOLDS

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ABSTRACT. In this article, we introduce the interval optimization problems (IOPs) on Hadamard manifolds as well as study the relationship between them and the interval variational inequalities. To achieve the theoretical results, we build up some new concepts about gH-directional derivative and gH-Gâteaux differentiability of interval valued functions and their properties on the Hadamard manifolds. The obtained results pave a way to further study on Riemannian interval optimization problems (RIOPs).

1. INTRODUCTION

1.1. **Background and Motivation.** This paper studies a new problem set, which is called the interval optimization problems on Hadamard manifolds. First, as below, we elaborate the motivation about why we focus on this problem. The variational inequalities have been investigated since the dawn of the sixties [22] and plenty of results are already established, see [11, 14] and references therein. From [32], it is known that the interval optimization problems (IOPs) and interval variational inequalities (IVIs) possess a close relationship under some assumptions. In addition, the interval programming [2, 7, 10, 16, 19, 27, 30–32] is one of the approaches to tackle the uncertain optimization problems, in which an interval is used to characterize the uncertainty of a variable. Because the variation bounds of the uncertain variables can be obtained only through a small amount of uncertainty information, the interval programming can easily handle some optimization problems.

Nowadays, many important concepts and methods of optimization problems have been extended from Euclidean space to Riemannian manifolds, particularly to Hadamard manifolds [1, 5, 6, 8–10, 18]. In general, a manifold has no linear structure, nonetheless, it is locally identified with Euclidean space. In this setting, the Euclidean metric is replaced by Riemannian metric, which is smoothly varying inner product defined on the tangent space of manifold at each point, and the line segments is replaced by minimal geodesics. This means that the generalization of optimization problems from Euclidean spaces to Riemannian manifolds is very important, for example, some nonconvex problems on Euclidean space can be viewed as convex problems on Riemannian manifolds.

1.2. **Contribution and Related works.** Wu [30, 31] studied the duality and optimality conditions for IOPs by using Hukuhara derivative. The extension about optimality conditions were studied by Osuna-Gomez et al. [25], and Gosh et al. [16], using the generalized Hukuhara derivative. Zhang et al. [32] studied about the relationship between IOPs and variational inequalities. And, the concepts of generalized convexity was studied by Jayswal et al. [20]. However, all of them were studied on $\mathbb{R}^n$. To our best knowledge, there is very limited study on Riemannian interval optimization problems (RIOPs) in the literature. In [10], the authors studied the KKT conditions for optimization problems with interval valued objective functions on Hadamard manifolds, which is just a routine extension.

In this paper, we further investigate the interval optimization problems on Hadamard manifolds, and characterize the relationship between them and the interval variational inequalities. Since $\mathbb{R}^n$ is a special case of Hadamard manifold, that will be the extension of previous results to the generalized space. To achieve the theoretical results, we build up some new concepts about gH-directional derivative and gH-Gâteaux differentiability of interval valued functions and their properties on the Hadamard manifolds. The analysis differs from the one used in traditional variational inequalities and nonlinear programming problems. The obtained results pave a way to further study on Riemannian interval optimization problems (RIOPs).

The paper is organized as follows. In Section 2, we formulate the problem set, introduce the notations and recall notions of Rienmannian manifolds, tangent space, geodesically convex and exponential mapping. We also recall some background materials regarding the set of closed, bounded intervals, gH-difference and some properties of interval valued functions as well as interval valued functions on Hadamard manifolds. In Section 3, we study the gH-continuity, the gH-directional derivative and gH-Gâteaux differentiability of interval valued functions on Hadamard manifolds. Then, we characterize the relationship between the gH-directional differentiability and geodesically convex of Riemannian interval valued functions. In Section 4, we introduce the RIOPs and the necessary and sufficient conditions for efficient points of the RIOPs. Besides, we define the Riemannian interval variational inequalities problems (RIVIPs) and establish the relationship between them and RIOPs. Finally, we draw a conclusion in Section 5.

2. Premilinaries

In this section, we review some background materials about Riemannian manifolds with a special case, the Hadamard manifolds. In particular, we study the intervals, interval valued functions, interval valued functions on Hadamard manifolds. We first recall some definitions and properties about Riemannian manifolds, which will be used in subsequent analysis. These materials can be found in textbooks on Riemannian geometry, such as [13, 21, 26].

Let $\mathcal{M}$ be a Riemannian manifold, we denote by $T_x\mathcal{M}$ the tangent space of $\mathcal{M}$ at $x \in \mathcal{M}$, and the tangent bundle of $\mathcal{M}$ is denoted by $T\mathcal{M} = \cup_{x \in \mathcal{M}} T_x\mathcal{M}$. For every $x, y \in \mathcal{M}$, the Riemannian distance $d(x, y)$ on $\mathcal{M}$ is defined by the minimal length over the set of all piecewise smooth curves joining x to y. Let ∇ is the Levi-Civita connection on Riemannian manifold $\mathcal{M}$, $\gamma : I \subset \mathbb{R} \longrightarrow \mathcal{M}$ is a smooth curve on $\mathcal{M}$, a vector field X is called parallel along γ if $\nabla_{\gamma'} X = 0$, where $\gamma' = \frac{\partial \gamma(t)}{\partial t}$. We

say that γ is a geodesic if γ' is parallel along itself, in this cases $\|\gamma'\|$ is a constant. When $\|\gamma'\| = 1, \gamma$ is said to be normalized. A geodesic joining x to y in $\mathcal{M}$ is called minimal if its length equals $d(x, y)$.

For any $x \in \mathcal{M}$, let V be a neighborhood of $0_x \in T_x\mathcal{M}$, the exponential mapping $\exp_x : V \longrightarrow \mathcal{M}$ is defined by $\exp_x(v) = \gamma(1)$ where γ is the geodesic such that $\gamma(0) = x$ and $\gamma'(0) = v$. It is known that the derivative of $\exp_x$ at $0_x \in T_x\mathcal{M}$ is the identity map; furthermore, by the Inverse Theorem, it is a local diffeomorphism. The inverse map of $\exp_x$ is denoted by $\exp_x^{-1}$. A Riemannian manifold is complete if for any $x \in \mathcal{M}$, the exponential map $\exp_x$ is defined on $T_x\mathcal{M}$. A simply connected, complete Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. If $\mathcal{M}$ is a Hadamard manifold, for all $x, y \in \mathcal{M}$, by the Hopf-Rinow Theorem and Cartan-Hadamard Theorem (see [21]), $\exp_x$ is a diffeomorphism and there exists a unique normalized geodesic joining x to y, which is indeed a minimal geodesic.

Example 2.1. Hyperbolic spaces. We equip $\mathbb{R}^{n+1}$ with the Minkowski product defined by

$$\langle x, y \rangle_1 = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i,$$

where $x = (x_0, x_1, \ldots, x_n)$, $y = (y_0, y_1, \ldots, y_n)$; and define

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_1 = -1, x_0 > 0\}.$$

Then, $\langle \cdot, \cdot \rangle_1$ induces a Riemannian metric g on the tangent spaces $T_p\mathbb{H}^n \subset \mathbb{R}^{n+1}$, for all $p \in \mathbb{H}^n$. The section curvature of $(\mathbb{H}^n, g)$ is -1 at every point.

Example 2.2. Manifold of symmetric positive definite matrices (SPD). The space of $n \times n$ symmetric positive definite matrices with real entries, denoted by S^n_{++}, is a Hadamard manifold if it is equipped with the below Riemannian metric:

$$g_A(x, Y) = \text{Tr}(A^{-1}XA^{-1}Y), \forall A \in S^n_{++}, \quad X, Y \in T_A S^n_{++}.$$

For more examples, please refer to [4]. From now on, through the whole paper, when we mention $\mathcal{M}$, it means that $\mathcal{M}$ is a Hadamard manifold.

Definition 2.3 (Totally convex set [28])**.** A subset $\mathcal{D} \subseteq \mathcal{M}$ is said totally convex if $\mathcal{D}$ contains every geodesic γ_{xy} of $\mathcal{M}$, whose end points x, y are in $\mathcal{D}$.

Definition 2.4 (Geodesically convex set [28])**.** A subset $\mathcal{D} \subseteq \mathcal{M}$ is said geodesically convex if $\mathcal{D}$ contains the minimal geodesic γ_{xy} of $\mathcal{M}$, whose end points x, y are in $\mathcal{D}$.

It is easy to see that both total convexity and geodesic convexity are the generalization of convexity in Euclidean space. The total convexity is stronger than geodesic convexity, but when the geodesic between any two points are unique, they coincide.

Example 2.5. Consider S^n_{++} as in Example 2.2. Given $a > 0$ and let

$$D_a = \{X \in S^n_{++} \mid \det X = a\},$$

then D_a is a nonconvex subset of S_{++}^n. In fact, from [29], the minimal geodesic joining $P, Q \in S_{++}^n$ is described by

$$\gamma(t) = P^{1/2}(P^{-1/2}QP^{-1/2})^t P^{1/2}, \quad \forall t \in [0, 1].$$

If $P, Q \in D_a$, then for all $t \in [0, 1]$ we have

$$
\begin{aligned}
\det(\gamma(t)) &= \det(P^{1/2}(P^{-1/2}QP^{-1/2})^t P^{1/2}) \\
&= \det(P)^{1/2}(\det(P)^{-1/2}\det(Q)\det(P)^{-1/2})^t \det(P)^{1/2} \\
&= a^{1-t}a^t \\
&= a.
\end{aligned}
$$

This means that $\gamma(t) \in D_a$, for all $t \in [0, 1]$, that is, D_a is a geodesically convex subset of S_{++}^n.

Following the notations used in [12], let $\mathcal{I}(\mathbb{R})$ be the set of all closed, bounded interval in $\mathbb{R}$, i.e.,

$$\mathcal{I}(\mathbb{R}) = \{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R}, \ \underline{a} \leq \overline{a}\}.$$

The Hausdorff metric d_H on $\mathcal{I}(\mathbb{R})$ is defined by

$$d_H(A, B) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}, \quad \forall A = [\underline{a}, \overline{a}], \ B = [\underline{b}, \overline{b}] \in \mathcal{I}(\mathbb{R}).$$

Then, $(\mathcal{I}(\mathbb{R}), d_H)$ is a complete metric space, see [23]. The Minkowski sum and scalar multiplications is given respectively by

$$
\begin{aligned}
A + B &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
\lambda A &= \begin{cases} [\lambda\underline{a}, \lambda\overline{a}] & \text{if } \lambda \geq 0, \\ [\lambda\overline{a}, \lambda\underline{a}] & \text{if } \lambda < 0. \end{cases}
\end{aligned}
$$

where $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$. Note that, $A - A = A + (-1)A \neq 0$. A crucial concept in achieving a useful working definition of derivative for interval-valued functions is trying to derive a suitable difference between two intervals.

Definition 2.6 (*gH*-difference of intervals [27]). Let $A, B \in \mathcal{I}(\mathbb{R})$. The *gH*-difference between A and B is defined as the interval C such that

$$C = A -_{gH} B \quad \Longleftrightarrow \quad \begin{cases} A = B + C \\ \text{or} \\ B = A - C. \end{cases}$$

Proposition 2.7 ([27]). *For any two intervals* $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, *the gH-difference* $C = A -_{gH} B$ *always exists and*

$$C = \left[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \ \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\right].$$

Proposition 2.8 ([23]). *Suppose that* $A, B, C \in \mathcal{I}(\mathbb{R})$. *Then, the following properties hold.*

   (a) $d_H(A, B) = 0$ *if and only if* $A = B$.
   (b) $d_H(\lambda A, \lambda B) = |\lambda| d_H(A, B)$, *for all* $\lambda \in \mathbb{R}$.
   (c) $d_H(A + C, B + C) = d_H(A, B)$.
   (d) $d_H(A + B, C + D) \leq d_H(A, C) + d_H(B, D)$.
   (e) $d_H(A, B) = d_H(A -_{gH} B, 0)$.

(f) $d_H(A -_{gH} B, A -_{gH} C) = d_H(B -_{gH} A, C -_{gH} A) = d_H(B, C)$.

Notice that, for all $A \in \mathcal{I}(\mathbb{R})$, we define $||A|| := d_H(A, 0)$, then $||A||$ is a norm on $\mathcal{I}(\mathbb{R})$ and $d_H(A, B) = ||A -_{gH} B||$. There is no natural ordering on $\mathcal{I}(\mathbb{R})$, therefore we need to define it.

Definition 2.9 ([31]). Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two elements of $\mathcal{I}(\mathbb{R})$. We write $A \preceq B$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$. We write $A \prec B$ if $A \preceq B$ and $A \neq B$. Equivalently, $A \prec B$ if and only if one of the following cases holds:

- $\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$.
- $\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$.
- $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

We write, $A \nprec B$ if none of the above three cases hold. If neither $A \prec B$ nor $B \prec A$, we say that none of A and B dominates the other.

Lemma 2.10. *For two elements A, B, C and D of $\mathcal{I}(\mathbb{R})$, there hold*

- (a) $A \preceq B \iff A -_{gH} B \preceq \boldsymbol{0}$.
- (b) $A \nprec B \iff A -_{gH} B \nprec \boldsymbol{0}$.
- (c) $A \preceq B \implies A -_{gH} C \preceq B -_{gH} C$,
- (d) $A \preceq B -_{gH} C \implies B \nprec A + C$.
- (e) $0 \preceq (A -_{gH} B) + (C -_{gH} D) \implies 0 \preceq (A + C) -_{gH} (B + D)$.

Proof. (a) The proofs of part(a) can be found in [16].

(b) Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, since $A \nprec B$ then $A = B$, or $\underline{a} > \underline{b}$, or $\overline{a} > \overline{b}$. If $\underline{a} > \underline{b}$, or $\overline{a} > \overline{b}$ then $\max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\} > 0$. Thus, there holds $A -_{gH} B \nprec \boldsymbol{0}$. For the other direction, if $A -_{gH} B \nprec \boldsymbol{0}$ then $A = B$ or $\max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\} > 0$. This says that $\underline{a} > \underline{b}$, or $\overline{a} > \overline{b}$, which implies $A \nprec B$.

(c) Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, and $C = [\underline{c}, \overline{c}]$, it is clear that

$$A -_{gH} C = [\min\{\underline{a} - \underline{c}, \overline{a} - \overline{c}\}, \max\{\underline{a} - \underline{c}, \overline{a} - \overline{c}\}],$$
$$B -_{gH} C = [\min\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\}, \max\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\}].$$

If $A \preceq B$, then $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, which yield

$$\begin{cases} \underline{a} - \underline{c} \leq \underline{b} - \underline{c} \\ \overline{a} - \overline{c} \leq \overline{b} - \overline{c} \end{cases} \implies A -_{gH} C \preceq B -_{gH} C.$$

(d) Assume $B \preceq A + C$, by part(c), we know that

$$B -_{gH} C \preceq (A + C) -_{gH} C = A,$$

which indicates $A = B -_{gH} C$. From the definition of gH-difference, we have $B = A + C$ or $C = B - A$. If $C = B - A$, by the assumption $B \preceq A + C$, there has

$$B \preceq A + B - A \implies A \in \mathbb{R}.$$

Therefore, $B = A + C$. In other words, there holds

$$A \preceq B -_{gH} C \implies B \nprec A + C,$$

which is the desired result.

(e) Let $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, $C = [\underline{c}, \overline{c}]$ and $D = [\underline{d}, \overline{d}]$.

Case 1: If

$$\begin{cases} A -_{gH} B & = [\underline{a} - \underline{b}, \overline{a} - \overline{b}] \\ C -_{gH} D & = [\underline{c} - \underline{d}, \overline{c} - \overline{d}] \end{cases} or \begin{cases} A -_{gH} B & = [\overline{a} - \overline{b}, \underline{a} - \underline{b}] \\ C -_{gH} D & = [\overline{c} - \overline{d}, \underline{c} - \underline{d}] \end{cases},$$

then

$$\begin{cases} \underline{a} - \underline{b} + \underline{c} - \underline{d} & \geq 0 \\ \overline{a} - \overline{b} + \overline{c} - \overline{d} & \geq 0 \end{cases} \Longrightarrow \begin{cases} (\underline{a} + \underline{c}) - (\underline{b} + \underline{d}) & \geq 0 \\ (\overline{a} + \overline{c}) - (\overline{b} + \overline{d}) & \geq 0. \end{cases}$$

Case 2: If

$$\begin{cases} A -_{gH} B & = [\overline{a} - \overline{b}, \underline{a} - \underline{b}] \\ C -_{gH} D & = [\underline{c} - \underline{d}, \overline{c} - \overline{d}] \end{cases} \Longrightarrow \begin{cases} \overline{a} - \overline{b} + \underline{c} - \underline{d} & \geq 0 \\ \underline{a} - \underline{b} + \overline{c} - \overline{d} & \geq 0, \end{cases}$$

together with

$$\begin{cases} \underline{a} - \underline{b} & \geq \overline{a} - \overline{b} \\ \overline{c} - \overline{d} & \geq \underline{c} - \underline{d} \end{cases}$$

we have

$$\begin{cases} (\underline{a} + \underline{c}) - (\underline{b} + \underline{d}) & \geq 0 \\ (\overline{a} + \overline{c}) - (\overline{b} + \overline{d}) & \geq 0. \end{cases}$$

Case 3: If

$$\begin{cases} A -_{gH} B & = [\underline{a} - \underline{b}, \overline{a} - \overline{b}] \\ C -_{gH} D & = [\overline{c} - \overline{d}, \underline{c} - \underline{d}] \end{cases} \Longrightarrow \begin{cases} \overline{a} - \overline{b} + \underline{c} - \underline{d} & \geq 0 \\ \underline{a} - \underline{b} + \overline{c} - \overline{d} & \geq 0, \end{cases}$$

together with

$$\begin{cases} \overline{a} - \overline{b} & \geq \underline{a} - \underline{b} \\ \underline{c} - \underline{d} & \geq \overline{c} - \overline{d} \end{cases}$$

we have

$$\begin{cases} (\underline{a} + \underline{c}) - (\underline{b} + \underline{d}) & \geq 0 \\ (\overline{a} + \overline{c}) - (\overline{b} + \overline{d}) & \geq 0. \end{cases}$$

$\square$

Remark 2.11. The inverse of Lemma 2.10(c)-(d) are not true. To see this, taking $A = [1, 2]$, $B = [0, 5]$ and $C = [-1, 3]$, then

$$A -_{gH} C = [-1, 2], \quad B -_{gH} C = [1, 2].$$

This means that $A -_{gH} C \preceq B -_{gH} C$, but we do not have $A \preceq B$. If taking $A = \mathbf{0}$, $B = [0, 3]$, $C = [1, 2]$ then

$$A + C = [1, 2], \quad B -_{gH} C = [-1, 1],$$

which says $B \not\preceq A + C$, but we do not have $A \preceq B -_{gH} C$.

Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty set, a mapping $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is called a Riemannian interval valued function (RIVF). We write $f(x) = [\underline{f}(x), \overline{f}(x)]$ where $\underline{f}, \overline{f}$ are real valued functions satisfy $\underline{f}(x) \leq \overline{f}(x)$, for all $x \in \mathcal{M}$. Since $\mathbb{R}^n$ is a Hadamard manifold, an interval valued function (IVF for short) $f : U \subseteq \mathbb{R}^n \longrightarrow \mathcal{I}(\mathbb{R})$ is also a RIVF.

Definition 2.12 ([30]). Let $U \subseteq \mathbb{R}^n$ be a convex set. An IVF $f : U \longrightarrow \mathcal{I}(\mathbb{R})$ is said to be convex on U if

$$f(\lambda x_1 + (1 - \lambda)x_2) \preceq \lambda f(x_1) + (1 - \lambda)f(x_2),$$

for all $x_1, x_2 \in U$ and $\lambda \in [0, 1]$.

Definition 2.13 ([16]). Let $U \subseteq \mathbb{R}^n$ be a nonempty set. An IVF $f : U \longrightarrow \mathcal{I}(\mathbb{R})$ is said to be *monotonically increasing* if for all $x, y \in U$ there has

$$x \leq y \implies f(x) \preceq f(y).$$

The function f is said to be *monotonically decreasing* if for all $x, y \in U$ there has

$$x \leq y \implies f(y) \preceq f(x).$$

It is clear to see that if an IVF is monotonically increasing (or monotonically decreasing), if and only if both the real-valued functions $\underline{f}$ and $\overline{f}$ are monotonically increasing (or monotonically decreasing).

Definition 2.14 ([16]). Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty set. An RIVF $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is said to be *bounded below* on $\mathcal{D}$ if there exists an interval $A \in \mathcal{I}(\mathbb{R})$ such that

$$A \preceq f(x), \quad \forall x \in \mathcal{D}.$$

The function f is said to be *bounded above* on $\mathcal{D}$ if there exists an interval $B \in \mathcal{I}(\mathbb{R})$ such that

$$f(x) \preceq B, \quad \forall x \in \mathcal{D}.$$

The function f is said to be *bounded* if it is both bounded below and above.

It is easy to verify that if an RIVF f is bounded below (or bounded above) if and only if both the real-valued functions $\underline{f}$ and $\overline{f}$ are bounded below (or bounded above).

Definition 2.15 ([10]). Let $\mathcal{D} \subseteq \mathcal{M}$ be a geodesically convex set and $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a RIVF. f is called geodesically convex on $\mathcal{D}$ if

$$f(\gamma(t)) \preceq (1 - t)f(x) + tf(y), \quad \forall x, y \in \mathcal{D} \text{ and } \forall t \in [0, 1],$$

where $\gamma : [0, 1] \longrightarrow \mathcal{M}$ is the minimal geodesic joining x and y.

Proposition 2.16 ([10]). *Let $\mathcal{D}$ be a geodesically convex subset of $\mathcal{M}$ and f be a RIVF on $\mathcal{D}$. Then, f is geodesically convex on $\mathcal{D}$ if and only if $\underline{f}$ and $\overline{f}$ are geodesically convex on $\mathcal{D}$.*

Proof. This is a direct consequence from Definition 2.15 and Definition 2.9. □

Example 2.17. Consider the set

$$\mathcal{D} = \{A \in S_{++}^n \mid \det(A) > 1\}.$$

For all $X, Y \in \mathcal{D}$, we have the minimal geodesic joining X, Y defined by

$$\gamma(t) = X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2}, \quad \forall X, Y \in D \text{ and } \forall t \in [0, 1].$$

For any $t \in [0, 1]$, we also obtain

$$\det(X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2}) = (\det(X))^{1-t}(\det(Y))^t > 1,$$

which says that $\mathcal{D}$ is a geodesically convex subset of S_{++}^n. Moreover, on set $\mathcal{D}$, we define a RIVF as below:

$$f : D \longrightarrow \mathcal{I}(\mathbb{R})$$
$$X \longmapsto [0, \ln(\det(X))]$$

Then, for any $X, Y \in \mathcal{D}$ and $t \in [0, 1]$, we have

$$
\begin{aligned}
f(\gamma(t)) &= \left[0, \ln \det(X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2})\right] \\
&= [0, (1-t)\ln(\det(X)) + t\ln(\det(Y))] \\
&= (1-t)[0, \ln(\det(X))] + t[0, \ln(\det(Y))] \\
&= (1-t)f(X) + tf(Y),
\end{aligned}
$$

which shows that f is a geodesically convex RIVF on $\mathcal{D}$.

Proposition 2.18. *The RIVF $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is geodesically convex if and only if for all $x, y \in \mathcal{D}$ and $\gamma : [0, 1] \longrightarrow \mathcal{M}$ is the minimal geodesic joining x and y, the IVF $f \circ \gamma$ is convex on $[0, 1]$.*

Proof. Assume f is geodesically convex, for all $x, y \in \mathcal{D}$ if $\gamma : [0, 1] \longrightarrow \mathcal{M}$ is the minimal geodesic joining x to y, then the restriction of γ to $[t_1, t_2], t_1, t_2 \in [0, 1]$ joins the points $\gamma(t_1)$ to $\gamma(t_2)$. We re-parametrize this restriction

$$\alpha(s) = \gamma(t_1 + u(t_2 - t_1)), s \in [0, 1].$$

Since f is geodesically convex, for all $s \in [0, 1]$, we have

$$
\begin{aligned}
&f(\alpha(s)) \preceq (1-s)f(\alpha(0)) + sf(\alpha(1)) \\
\Rightarrow\quad &f(\gamma((1-s)t_1 + st_2)) \preceq (1-s)f(\gamma(t_1)) + sf(\gamma(t_2)) \\
\Rightarrow\quad &(f \circ \gamma)((1-s)t_1 + st_2) \preceq (1-s)(f \circ \gamma)(t_1) + s(f \circ \gamma)(t_2),
\end{aligned}
$$

which says the IVF $f \circ \gamma$ is convex on $[0, 1]$.
Conversely, for all $x, y \in \mathcal{D}$ and $\gamma : [0, 1] \longrightarrow \mathcal{M}$ is the minimal geodesic joining x and y, suppose that $f \circ \gamma : [0, 1] \longrightarrow \mathcal{I}(\mathbb{R})$ is a convex IVF. In other words, for all $t_1, t_2 \in [0, 1]$, there has

$$(f \circ \gamma)((1-s)t_1 + st_2)) \preceq (1-s)(f \circ \gamma)(t_1) + s(f \circ \gamma)(t_2), \ \forall s \in [0, 1].$$

Letting $t_1 = 0$ and $t_2 = 1$ gives

$$(f \circ \gamma)(s) \preceq (1-s)(f \circ \gamma)(0) + s(f \circ \gamma)(1), \ \forall s \in [0, 1],$$

or

$$f(\gamma(s)) \preceq (1-s)f(x) + sf(y), \ \forall s \in [0, 1].$$

Then, f is a geodesically convex RIVF. $\square$

Lemma 2.19. *If f is a geodesically convex RIVF on $\mathcal{D}$ and A is an interval, then the sublevel set*

$$\mathcal{D}^A = \{x \in \mathcal{D} : f(x) \preceq A\},$$

is a geodesically convex subset of $\mathcal{D}$.

Proof. For all $x, y \in \mathcal{D}^A$, there has $f(x) \preceq A$ and $f(y) \preceq A$. Let $\gamma : [0,1] \longrightarrow \mathcal{M}$ be the minimal geodesic joining x and y. For all $t \in [0,1]$, by the convexity of f, we have

$$f(\gamma(t)) \preceq (1-t)f(x) + tf(y) \preceq (1-t)A + tA = A.$$

Thus, $f(\gamma(t)) \in \mathcal{D}^A$ for all $t \in [0,1]$, which says that $\mathcal{D}^A$ is a geodesically convex subset of $\mathcal{D}$. $\qquad\square$

3. The gH-continuity and gH-differentiability of Riemannian interval valued functions

In this section, we generalize the gH-continuous and gH-differentiable property of interval valued functions to the settings on the Hadamard manifolds. The relationship between gH-differentiability and geodesically convex property of the RIVFs is also established.

The limits of IFV and the related properties were introduced by Aubin and Cellina [3]. The generalization concept for RIVF was introduced by Chen [10].

Definition 3.1 ([10, Definition 3.1]). Let $f : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ be a RIVF, $x_0 \in \mathcal{M}$, $A = [\underline{a}, \overline{a}] \in \mathcal{I}(\mathbb{R})$. We say $\lim_{x \to x_0} f(x) = A$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in \mathcal{M}$ and $d(x, x_0) < \delta$, there holds $d_H(f(x), A) < \epsilon$.

Lemma 3.2 ([10, Lemma 3.1]). *Let $f : M \longrightarrow \mathcal{I}(\mathbb{R})$ be a RIVF, $A = [\underline{a}, \overline{a}] \in \mathcal{I}(\mathbb{R})$. Then,*

$$\lim_{x \to x_0} f(x) = A \quad \Longleftrightarrow \quad \begin{cases} \lim_{x \to x_0} \underline{f}(x) = \underline{a}, \\ \lim_{x \to x_0} \overline{f}(x) = \overline{a}. \end{cases}$$

Proof. If $\lim_{x \to x_0} f(x) = A$, then for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in \mathcal{M}$ and $d(x, x_0) < \delta$, there have

$$d_H(f(x), A) < \epsilon$$
$$\implies \max\{|\underline{f}(x) - \underline{a}|, |\overline{f}(x) - \overline{a}|\} < \epsilon$$
$$\implies \begin{cases} |\underline{f}(x) - \underline{a}| < \epsilon \\ |\overline{f}(x) - \overline{a}| < \epsilon \end{cases}.$$

Consequently, we have

$$\begin{cases} \lim_{x \to x_0} \underline{f}(x) = \underline{a}, \\ \lim_{x \to x_0} \overline{f}(x) = \overline{a}. \end{cases}$$

On other hand, if we have

$$\begin{cases} \lim_{x \to x_0} \underline{f}(x) = \underline{a}, \\ \lim_{x \to x_0} \overline{f}(x) = \overline{a}, \end{cases}$$

for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in \mathcal{M}$ and $d(x, x_0) < \delta$, there has

$$\begin{cases} |\underline{f}(x) - \underline{a}| < \epsilon \\ |\overline{f}(x) - \overline{a}| < \epsilon \end{cases} \implies d_H(f(x), A) < \epsilon.$$

which says $\lim_{x \to x_0} f(x) = A$. Thus, the proof is complete. $\qquad\square$

Remark 3.3. From Proposition 2.8, we know that $d_H(f(x), A) = d_H(f(x) -_{gH} A, \mathbf{0})$, which yields

$$\lim_{x \to x_0} f(x) = A \quad \Longleftrightarrow \quad \lim_{x \to x_0} (f(x) -_{gH} A) = \mathbf{0}.$$

Definition 3.4 (gH-continuity). Let f be a RIVF on a nonempty open subset $\mathcal{D}$ of $\mathcal{M}$, $x_0 \in \mathcal{D}$. The function f is said to be gH-continuous at x_0 if for all $v \in T_{x_0}\mathcal{M}$ with $\exp_{x_0} v \in \mathcal{D}$, there has

$$\lim_{||v|| \to 0} \left(f(\exp_{x_0}(v)) -_{gH} f(x_0) \right) = \mathbf{0}.$$

We call f is gH-continuous on $\mathcal{D}$ if f is gH-continuous at every $x \in \mathcal{D}$.

Remark 3.5. We point out couple remarks regarding gH-continuity.
   (1) When $\mathcal{M} \equiv \mathbb{R}^n$, f become an IVF and $\exp_{x_0}(v) = x_0 + v$. In other words, Definition 3.4 generalizes the concept of the gH-continuity of the IVF setting, see [15].
   (2) By Lemma 3.1 and Remark 3.1, we can see that f is gH-continuous if and only if $\underline{f}$ and $\overline{f}$ are continuous.

Theorem 3.6. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a geodesically convex set with nonempty interior and $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a geodesically convex RIVF. Then, f is gH-continuous on* $\operatorname{int} \mathcal{D}$.

Proof. Let $x_0 \in \operatorname{int} \mathcal{D}$ and $B(x_0, r)$ be an open ball center at x_0 and of sufficient small radius r. Choose $A \in \mathcal{I}(\mathbb{R})$ such that the geodesically convex set $\mathcal{D}^A = \{x \in \mathcal{D} : f(x) \preceq A\}$ contains $\overline{B}(x_0, r)$. Let $\gamma : [-1, 1] \longrightarrow \mathcal{M}$ be a minimal geodesic in $\overline{B}(x_0, r)$ such that $\gamma(-1) = x_1, \gamma(0) = x_0, \gamma(1) = x_2$. For convenience, we denote $\gamma(t) = x$ where $t = \frac{d(x_0, x)}{r} \in [0, 1]$. By the convexity of f, we have

$$f(\gamma(t)) \preceq (1 - t)f(x_0) + tf(x_2) \preceq (1 - t)f(x_0) + tA,$$

which together with Lemma 2.10 implies

$$(3.1) \qquad\qquad f(x) -_{gH} f(x_0) \preceq t(A - f(x_0)).$$

The minimal geodesic joining x_1 and x is the restriction $\gamma(u), u \in [-1, t]$. Setting $u = -1 + s(t + 1), s \in [0, 1]$, we obtain the re-parametrization

$$\alpha(s) = y(-1 + s(t + 1)), \ s \in [0, 1].$$

It is clear to see that

$$\alpha(0) = \gamma(-1) = x_1, \ \alpha\left(\frac{1}{t + 1}\right) = \gamma(0) = x_0, \ \alpha(1) = \gamma(t) = x.$$

Due to the convexity of f, we have

$$f(\alpha(s)) \preceq (1 - s)f(x_1) + sf(x) \preceq (1 - s)A + sf(x), \ \forall s \in [0, 1].$$

Letting $s = \frac{1}{1+t}$ yields

$$f(x_0) \preceq \frac{t}{t + 1}A + \frac{1}{t + 1}f(x),$$

which together with Lemma 2.10 further implies

$$(3.2) \qquad f(x_0) -_{gH} f(x) \preceq [(tA + f(x) -_{gH} tf(x_0)] -_{gH} f(x)).$$

From (3.1) and (3.2), plugging in $t = \frac{d(x_0,x)}{r}$, we obtain $\lim_{x \to x_0} f(x) = f(x_0)$. Then, the proof is complete. □

Definition 3.7 ([24]). Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty open set and consider a function $f : \mathcal{D} \longrightarrow \mathbb{R}$. We say that f has directional derivative at $x \in \mathcal{D}$ in the direction $v \in T_x\mathcal{M}$ if the limit

$$f'(x,v) = \lim_{t \to 0^+} \frac{f(\exp_x(tv)) - f(x)}{t}$$

exists, where $f'(x,v)$ is called the directional derivative of f at x in the direction $v \in T_x\mathcal{M}$. If f has directional derivative at x in every direction $v \in T_x\mathcal{M}$, we say that f is directional differentiable at x.

Definition 3.8 (gH-directional differentiability [10]). Let f be a RIVF on a nonempty open subset $\mathcal{D}$ of $\mathcal{M}$. The function f is said to have gH-directional derivative at $x \in \mathcal{D}$ in direction $v \in T_x\mathcal{M}$, if there exists a closed bounded interval $f'(x,v)$ such that the limits

$$f'(x,v) = \lim_{t \to 0^+} \frac{1}{t}(f(\exp_x(tv)) -_{gH} f(x))$$

exists, where $f'(x,v)$ is called the gH-directional derivative of f at x in the direction of v. If f has gH-directional derivative at x in every direction $v \in T_x\mathcal{M}$, we say that f is gH-directional differentiable at x.

Lemma 3.9 ([10]). *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty open set and consider a RIVF $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$. Then, f has gH-directional derivative at $x \in \mathcal{D}$ in the direction $v \in T_x\mathcal{M}$ if and only if $\underline{f}$ and $\overline{f}$ have directional derivative at x in the direction v. Furthermore, we have*

$$f'(x,v) = \left[\min\{\underline{f}'(x,v), \overline{f}'(x,v)\}, \max\{\underline{f}'(x,v), \overline{f}'(x,v)\}\right],$$

where $\underline{f}'(x,v)$ and $\overline{f}'(x,v)$ are the directional derivatives of $\underline{f}$ and $\overline{f}$ at x in the direction v, respectively.

Theorem 3.10. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty open geodesically convex set. If $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is a geodesically convex RIVF, then at any $x_0 \in \mathcal{D}$, gH-directional derivative $f'(x_0,v)$ exists for every direction $v \in T_{x_0}\mathcal{M}$.*

To prove Theorem 3.10, we need two Lemmas.

Lemma 3.11. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty geodesically convex set and consider a geodesically convex RIVF $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$. Then, $\forall x_0 \in \mathcal{D}, v \in T_{x_0}\mathcal{M}$, the function $\phi : \mathbb{R}^+\backslash\{0\} \longrightarrow \mathcal{I}(\mathbb{R})$, defined by*

$$\phi(t) = \frac{1}{t}(f(\exp_{x_0}(tv)) -_{gH} f(x_0)),$$

for all $t > 0$ such that $\exp_{x_0}(tv) \in \mathcal{D}$, is monotonically increasing.

Proof. For all t, s such that $0 \leq t \leq s$, by the convexity of f, for all $\lambda \in [0,1]$, we have

$$f(\exp_{x_0}(\lambda(sv))) \leq (1 - \lambda)f(x_0) + \lambda f(\exp_{x_0}(sv)).$$

Since $\frac{t}{s} \in [0, 1]$, there holds

$$f(\exp_{x_0}(tv)) \leq \frac{s-t}{s}f(x_0) + \frac{t}{s}f(\exp_{x_0}(sv)),$$

or

$$f(\exp_{x_0}(tv) -_{gH} f(x_0))$$
$$\leq \left[\frac{s-t}{s}f(x_0) + \frac{t}{s}f(\exp_{x_0}(sv))\right] -_{gH} f(x_0)$$
$$= \left[\min\left\{\frac{s-t}{s}\underline{f}(x_0) + \frac{t}{s}\underline{f}(\exp_{x_0}(sv)) - \underline{f}(x_0), \frac{s-t}{s}\overline{f}(x_0) + \frac{t}{s}\overline{f}(\exp_{x_0}(sv)) - \overline{f}(x_0)\right\},\right.$$
$$\left. \max\left\{\frac{s-t}{s}\underline{f}(x_0) + \frac{t}{s}\underline{f}(\exp_{x_0}(sv)) - \underline{f}(x_0), \frac{s-t}{s}\overline{f}(x_0) + \frac{t}{s}\overline{f}(\exp_{x_0}(sv)) - \overline{f}(x_0)\right\}\right]$$
$$= \left[\min\left\{\frac{t}{s}(\underline{f}(\exp_{x_0}(sv)) - \underline{f}(x_0)), \frac{t}{s}(\overline{f}(\exp_{x_0}(sv)) - \overline{f}(x_0))\right\}\right.$$
$$\left. \max\left\{\frac{t}{s}(\underline{f}(\exp_{x_0}(sv)) - \underline{f}(x_0)), \frac{t}{s}(\overline{f}(\exp_{x_0}(sv)) - \overline{f}(x_0))\right\}\right]$$
$$= \frac{t}{s}(f(\exp_{x_0}(sv)) -_{gH} f(x_0)).$$

Then, the proof is complete. $\qquad\square$

Lemma 3.12. *Let $\mathcal{D} \subseteq \mathcal{M}$ be an open geodesically convex set. If $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is a geodesically convex RIVF, then for all $x_0 \in \mathcal{D}$ and $v \in T_{x_0}\mathcal{M}$, there exists $t_0 \in \mathbb{R}$ such that $\phi(t) = \frac{1}{t}\left(f(\exp_{x_0}(tv)) -_{gH} f(x_0)\right)$ is bounded below for all $t \in (0, t_0]$.*

Proof. For all $v \in T_{x_0}\mathcal{M}$, let γ be the geodesic such that $\gamma(0) = x_0$ and $\gamma'(0) = v$. Since $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty open geodesically convex set, there exists $t_1, t_2 \in \mathbb{R}$ such that $0 \in (t_1, t_2)$ and the restriction of γ on $[t_1, t_2]$ is contained in $\mathcal{D}$. Let $\lambda \in (0, t_2]$ and fix the point $\gamma(\lambda)$. The restriction of γ to $[t_1, \lambda]$ joins $\gamma(t_1)$ and $\gamma(\lambda)$. We can re-parametrize this restriction

$$\alpha(s) = \gamma(t_1 + s(\lambda - t_1)), \ s \in [0, 1].$$

Using the convexity of f gives

$$f(\alpha(s)) \preceq (1-s)f(\alpha(0)) + sf(\alpha(1)) \implies f(\alpha(s)) \preceq (1-s)f(\gamma(t_1)) + sf(\gamma(\lambda)).$$

Plugging in $s = \frac{t_1}{t_1 - \lambda}$ leads to

$$f(x_0) \preceq \frac{\lambda}{\lambda - t_1}f(\gamma(t_1)) + \frac{-t_1}{\lambda - t_1}f(\gamma(\lambda)).$$

Then, we have

$$(\lambda - t_1)[\underline{f}(x_0), \overline{f}(x_0)] \preceq \left[-t_1\underline{f}(\gamma(\lambda)) + \lambda\underline{f}(\gamma(t_1)), \ -t_1\overline{f}(\gamma(\lambda)) + \lambda\overline{f}(\gamma(t_1))\right]$$

or

$$\begin{cases} \dfrac{1}{-t_1}(\underline{f}(x_0) - \underline{f}(\gamma(t_1))) \leq \dfrac{1}{\lambda}(\underline{f}(\gamma(\lambda)) - \underline{f}(x_0)), \\ \dfrac{1}{-t_1}(\overline{f}(x_0) - \overline{f}(\gamma(t_1))) \leq \dfrac{1}{\lambda}(\overline{f}(\gamma(\lambda)) - \overline{f}(x_0)). \end{cases}$$

Thus, the proof is complete. $\qquad\square$

As below, we provide the proof of Theorem 3.10:

Proof. Let any $x_0 \in \mathcal{D}, v \in T_{x_0}\mathcal{M}$. Define an IVF $\phi : \mathbb{R}^+\backslash\{0\} \longrightarrow \mathcal{I}(\mathbb{R})$ by

$$\phi(t) = \frac{1}{t}(f(\exp_{x_0}(tv)) -_{gH} f(x_0)).$$

If $\phi(t) = [\underline{\phi}(t), \overline{\phi}(t)]$, by Lemma 3.11 and Lemma 3.12, we have both real-valued functions $\underline{\phi}$ and $\overline{\phi}$ are monotonically increasing and bounded below with t enough small. Therefore, the limits $\lim_{t\to 0^+} \underline{\phi}(t)$ and $\lim_{t\to 0^+} \overline{\phi}(t)$ exist or the limit $\lim_{t\to 0^+} \phi(t)$ exists. Thus, the function $\overline{f}$ has gH-directional derivative at $x_0 \in \mathcal{D}$ in the direction v. □

Theorem 3.13. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty open geodesically convex set of $\mathcal{M}$, and $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a geodesically convex function on $\mathcal{D}$, then*

$$f'(x, \exp_x^{-1} y) \preceq f(y) -_{gH} f(x), \quad \forall x, y \in \mathcal{D}.$$

Proof. For all $x, y \in \mathcal{D}$ and $t \in (0, 1]$, by the convexity of f, we have

$$f(\gamma(t)) \preceq tf(y) + (1 - t)f(x),$$

where $\gamma : [0, 1] \longrightarrow \mathcal{M}$ is the minimal geodesic joining x and y. Applying Lemma 2.10 yields that

$$
\begin{aligned}
&f(\gamma(t)) -_{gH} f(x) \\
\preceq\ & [tf(y) + (1 - t)f(x)] -_{gH} f(x) \\
=\ & [\min\{t\underline{f}(y) + (1 - t)\underline{f}(x) - \underline{f}(x), t\overline{f}(y) + (1 - t)\overline{f}(x) - \overline{f}(x)\}, \\
& \max\{t\underline{f}(y) + (1 - t)\underline{f}(x) - \underline{f}(x), t\overline{f}(y) + (1 - t)\overline{f}(x) - \overline{f}(x)\}] \\
=\ & [\min\{t(\underline{f}(y) - \underline{f}(x)), t(\overline{f}(y) - \overline{f}(x))\}, \max\{t(\underline{f}(y) - \underline{f}(x)), t(\overline{f}(y) - \overline{f}(x))\}] \\
=\ & t[f(y) -_{gH} f(x)].
\end{aligned}
$$

Then, we achieve

$$\frac{1}{t}[f(\gamma(t)) -_{gH} f(x)] \preceq f(y) -_{gH} f(x), \quad \forall x, y \in \mathcal{D}, \text{ and } t \in (0, 1].$$

As a result, when $t \longrightarrow 0^+$, we obtain

$$f'(x, \exp_x^{-1} y) \preceq f(y) -_{gH} f(x), \quad \forall x, y \in \mathcal{D}.$$

Thus, the proof is complete. □

Corollary 3.14. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty open geodesically convex set of $\mathcal{M}$ and suppose that the RIVF $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is geodesically convex on $\mathcal{D}$, then*

$$f(y) \not\prec f'(x, \exp_x^{-1} y) + f(x), \quad \forall x, y \in \mathcal{D}.$$

Proof. The result follows immediately from Theorem 3.13 and Lemma 2.10. □

Definition 3.15 ([16]). Let $\mathcal{V}$ be a linear subspace of $\mathbb{R}^n$. The IVF $F : \mathcal{V} \longrightarrow \mathcal{I}(\mathbb{R})$ is said to be generalized linear (g-linear for short) if
   (a) $F(\lambda v) = \lambda F(v)$, for all $v \in \mathcal{V}$, $\lambda \in \mathbb{R}$; and
   (b) for all $v, w \in \mathcal{V}$, either $F(v) + F(w) = F(v + w)$ or none of $F(v) + F(w)$ and $F(v + w)$ dominates the other.

Definition 3.16 (gH-Gâteaux differentiability). Let f be a RIVF on a nonempty open subset $\mathcal{D}$ of $\mathcal{M}$ and $x_0 \in \mathcal{D}$. The function f is called gH-Gâteaux differentiable at x_0 if f is gH-directional differentiable at x_0 and there exists a gH-continuous, g-linear IVF $f_G(x_0) : T_{x_0}\mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ such that

$$f_G(x_0)(v) = f'(x_0, v), \forall v \in T_{x_0}\mathcal{M}$$

The function f is called gH-Gâteaux differentiable on $\mathcal{D}$ if f is gH-Gâteaux differentiable at every $x \in \mathcal{D}$.

Example 3.17. Let $\mathcal{M} := \mathbb{R}^2$ with the standard metric. Then, $\mathcal{M}$ is a flat Hadamard manifold. We consider the RIVF given as below:

$$f : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$$

$$(x_1, x_2) \longmapsto \begin{cases} \dfrac{x_1 x_2^2}{x_1^4 + x_2^2}[1, 2] & \text{if } (x_1, x_2) \neq (0, 0), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

For all $v = (v_1, v_2) \in T_{(0,0)}\mathcal{M} \equiv \mathbb{R}^2$, we compute

$$\begin{aligned}
f'((0,0), v) &= \lim_{t \to 0^+} \frac{1}{t} \left(f((0,0) + tv) -_{gH} f((0,0)) \right) \\
&= \lim_{t \to 0^+} \frac{1}{t} \frac{t^3 v_1 v_2^2}{t^4 v_1^4 + t^2 v_2^2}[1, 2] \\
&= \lim_{t \to 0^+} \frac{v_1 v_2^2}{t^2 v_1^4 + v_2^2}[1, 2] \\
&= v_1[1, 2].
\end{aligned}$$

On the other hand, for all $h = (h_1, h_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
&f'((0,0), v + h) -_{gH} f'((0,0), v) \\
=\ &[\min\{v_1 + h_1 - v_1, 2(v_1 + h_1) - 2v_1\}, \max\{v_1 + h_1 - v_1, 2(v_1 + h_1) - 2v_1\}] \\
=\ &[\min\{h_1, 2h_1\}, \max\{h_1, 2h_1\}]
\end{aligned}$$

which says $\lim\limits_{||h|| \to 0} (f'((0,0), v + h) -_{gH} f'((0,0), v)) = \mathbf{0}$. In other words, $f'((0,0), \cdot)$ is a gH-continuous IVF. Hence, $f'((0,0), \cdot)$ is a g-linear, gH-continuous IVF or f is gH-Gâteaux differentiable at $(0, 0)$ and $f_G((0,0))(v) = v_1[1, 2]$ for all $v = (v_1, v_2) \in T_{(0,0)}\mathcal{M}$.

Example 3.18. We consider a RIVF defined by

$$f : S_{++}^n \longrightarrow \mathcal{I}(\mathbb{R})$$

$$X \longmapsto \begin{cases} [\ln(\det(X)), \ln(\det(X^2))] & \text{if } \det(X) \geq 1, \\ [\ln(\det(X^2)), \ln(\det(X))] & \text{otherwise.} \end{cases}$$

For all $v \in T_I S_{++}^n \equiv S^n$, where S^n is the space of $n \times n$ symmetric matrices and I is the $n \times n$ identity matrix, by denoting $Y = \exp_I(v)$ for all $t \in (0, 1]$, we have

$$\ln\left(\det(I^{1/2}(I^{-1/2}YI^{-1/2})^t I^{1/2})\right) = t\ln(\det(Y)),$$

which implies

$$f'(I)(v)$$
$$= \lim_{t\to 0+} \frac{1}{t}[f(\exp_I(tv)) -_{gH} f(I)]$$
$$= \lim_{t\to 0+} \frac{1}{t}\left[\min\{t\underline{f}(Y), t\overline{f}(Y)\}, \max\{t\underline{f}(Y), t\overline{f}(Y)\}\right],$$
$$= [\min\{\ln(\det(Y)), 2\ln(\det(Y))\}, \max\{\ln(\det(Y)), 2\ln(\det(Y))\}]$$
$$= \begin{cases} [\ln(\det(Y)), 2\ln(\det(Y))] & \text{if } \det(Y) \geq 1 \\ [2\ln(\det(Y)), \ln(\det(Y))] & \text{otherwise.} \end{cases}$$

This concludes that f is gH-directional differentiable at I.
On the other hand, for all $v \in S^n, \lambda \in \mathbb{R}$, we know that

$$\exp_I(\lambda v) = I^{1/2}\mathrm{Exp}(I^{-1/2}(\lambda v)I^{-1/2})I^{1/2}$$
$$= \mathrm{Exp}(\lambda v),$$

where Exp denotes to the matrix exponential. From [17], we also have

$$\det(\exp_I(\lambda v)) = \det(\mathrm{Exp}(\lambda v)) = e^{\mathrm{Tr}(\lambda v)} = \left(e^{\mathrm{Tr}\,v}\right)^{\lambda}$$
$$\implies \ln(\det(\exp_I(\lambda v))) = \lambda \ln(\det(\exp_I(v))).$$

To sum up, the function $f'(x)(.)$ is a g-linear IVF. Moreover, for all $v, h \in S^n$, it follows from [17] that

$$\exp_I(v + h) = \mathrm{Exp}(v + h) = \mathrm{Exp}(v).\,\mathrm{Exp}(h).$$

Thus, we obtain

$$\det(\exp_I(v + h)) = \det(\mathrm{Exp}(v)).\det(\mathrm{Exp}(h))$$
$$\implies \ln(\det(\exp_I(v + h))) = \ln(\det(\mathrm{Exp}(v))) + \ln(\det(\mathrm{Exp}(h))).$$
$$\implies \lim_{||h||\to 0}(f_G(I)(v + h) -_{gH} f_G(I)(v))$$
$$= \lim_{||h||\to 0}[\min\{\ln(\det(\mathrm{Exp}(h))), 2\ln(\det(\mathrm{Exp}(h)))\},$$
$$\max\{\ln(\det(\mathrm{Exp}(h))), 2\ln(\det(\mathrm{Exp}(h)))\}]$$
$$= 0.$$

which says that $f'(I)(\cdot)$ is a gH-continuous IVF. Thus, f is gH-Gâteaux differentiable at I.

Remark 3.19. We point out that the gH-Gâteaux differentiability does not imply the gH-continuity of RIVF. In fact, in Example 3.17, the function f is gH-Gâteaux differentiable at $(0, 0)$, but

$$\lim_{||h||\to 0}(f(h_1, h_2) -_{gH} f((0, 0))) = \lim_{||h||\to 0} \frac{h_1 h_2^2}{h_1^4 + h_2^2}[1, 2],$$

does not exist, which indicates that f is not gH-continuous at $(0, 0)$.

4. Interval optimization problems on Hadamard manifolds

This section is devoted to building up some theoretical results on the interval optimization problems on Hadamard manifolds. To proceed, we introduce the so-called "efficient point" concept, which is parallel to the role of traditional "minimizer".

Definition 4.1. (Efficient point) Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty set and $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a RIVF. A point $x_0 \in \mathcal{D}$ is said to be an efficient point of the Riemannian interval optimization problem (RIOP):

$$(4.1) \qquad\qquad \min_{x \in \mathcal{D}} f(x)$$

if $f(x) \nprec f(x_0)$, for all $x \in \mathcal{D}$.

Sice the objective function $f(x) = [\underline{f}(x), \overline{f}(x)]$ in RIOP (4.1) is an interval-valued function, we can consider two corresponding scalar problems for (4.1) as follows:

$$(4.2) \qquad\qquad \min_{x \in \mathcal{D}} \underline{f}(x)$$

and

$$(4.3) \qquad\qquad \min_{x \in \mathcal{D}} \overline{f}(x)$$

Proposition 4.2. *Consider problems (4.2) and (4.3).*
  (a) *If $x_0 \in \mathcal{D}$ is an optimal solution of problems (4.2) and (4.3) simultaneously , then x_0 is an efficient point of the RIOP (4.1).*
  (b) *If $x_0 \in \mathcal{D}$ is an unique optimal solution of problems (4.2) or (4.3) , then x_0 is an efficient point of the RIOP (4.1).*

Proof. (a) If $x_0 \in \mathcal{D}$ is an optimal solution of problems (4.2) and (4.3) simultaneously, then

$$\begin{cases} \underline{f}(x_0) & \leq \underline{f}(x) \\ \overline{f}(x_0) & \leq \overline{f}(x) \end{cases}, \forall x \in \mathcal{D} \Rightarrow f(x) \nprec f(x_0), \forall x \in \mathcal{D},$$

or x_0 is an efficient point of RIOP (4.1).
(b) If $x_0 \in \mathcal{D}$ is an unique optimal solution of problems (4.2) or (4.3), then

$$\begin{bmatrix} \underline{f}(x_0) < \underline{f}(x) \\ \overline{f}(x_0) < \overline{f}(x) \end{bmatrix}, \forall x \in \mathcal{D} \backslash \{x_0\},$$

which says $f(x) \nprec f(x_0)$ for all $x \in \mathcal{D}$, or equivalently x_0 is an efficient point of the RIOP (4.1). $\square$

Proposition 4.3. *Consider the RIOP (4.1) with $f(x) = [\underline{f}(x), \overline{f}(x)]$. Given any $\lambda_1, \lambda_2 > 0$, if $x_0 \in \mathcal{D}$ is an optimal solution of the following problem*

$$(4.4) \qquad\qquad \min_{x \in \mathcal{D}} h(x) = \lambda_1 \underline{f}(x) + \lambda_2 \overline{f}(x),$$

then x_0 is an efficient point of the RIOP (4.1).

Proof. Assume that x_0 is not an efficient point of RIOP (4.1), then there exists $x' \in \mathcal{D}$ such that

$$f(x') \prec f(x_0) \implies \lambda_1 \underline{f}(x') + \lambda_2 \overline{f}(x') < \lambda_1 \underline{f}(x_0) + \lambda_2 \overline{f}(x_0).$$

This says that x_0 is not an optimal solution of (4.4), which is a contradiction. Thus, x_0 is an efficient point of the RIOP (4.1).   □

Theorem 4.4 (Characterization I of efficient point). *Let* $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ *be a RIVF on a nonempty open subset* $\mathcal{D}$ *of* $\mathcal{M}$ *and* $x_0 \in \mathcal{D}$ *such that* f *is* gH-*directional differentiable at* x_0.

    (a) *If* x_0 *is an efficient point of RIOP (4.1), then for all* $x \in \mathcal{D}$
$$f'(x_0, \exp_{x_0}^{-1} x) \not\prec \mathbf{0} \text{ or } f'(x_0, \exp_{x_0}^{-1} x) = [a, 0] \text{ for some } a < 0.$$

    (b) *If* $\mathcal{D}$ *is geodesically convex,* f *is geodesically convex on* $\mathcal{D}$ *and*
$$f'(x_0, \exp_{x_0}^{-1} x) \not\prec \mathbf{0} \quad \forall x \in \mathcal{D},$$
    then x_0 *is an efficient point of the RIOP (4.1).*

Proof. For each $x \in \mathcal{D}$, let $v = \exp_{x_0}^{-1} x$. Since f is gH-directional differentiable at x_0, then
$$f'(x_0, \exp_{x_0}^{-1} x) = f'(x_0, v) = \lim_{t \to 0^+} \frac{1}{t}(f(\exp_{x_0}^{-1}(tv)) -_{gH} f(x_0)).$$

(a) If x_0 is an efficient point of RIOP (4.1), then
$$f(\exp_{x_0}(tv)) \not\prec f(x_0), \forall t > 0$$
$$\Rightarrow f(\exp_{x_0}(tv)) -_{gH} f(x_0) \not\prec 0, \forall t > 0 (\text{ by Lemma 2.10})$$
$$\Rightarrow \frac{1}{t}(f(\exp_{x_0}(tv)) -_{gH} f(x_0)) \not\prec 0, \forall t > 0$$
$$\Rightarrow \begin{bmatrix} f'(x_0, v) \not\prec \mathbf{0} \\ f'(x_0, v) = [a, 0], \text{ for some } a < 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} f'(x_0, \exp_{x_0}^{-1} x) \not\prec \mathbf{0} \\ f'(x_0, \exp_{x_0}^{-1} x) = [a, 0], \text{ for some } a < 0 \end{bmatrix}$$

(b) For all $x \in \mathcal{D}$, by the convexity of f and applying Theorem 3.13, we have
$$(4.5) \qquad f'(x_0, \exp_{x_0}^{-1} x) \preceq f(x) -_{gH} f(x_0) \Rightarrow f(x) -_{gH} f(x_0) \not\prec \mathbf{0}.$$

On the other hand, by Lemma 2.10, there has
$$(4.6) \qquad\qquad f(x) \not\prec f(x_0) \Leftrightarrow f(x) -_{gH} f(x_0) \not\prec \mathbf{0}.$$

From (4.5) and (4.6), it is clear to see that
$$f(x) \not\prec f(x_0), \forall x \in \mathcal{D}.$$

Then, x_0 is an efficient point of the RIOP(4.1).   □

Example 4.5. Consider the RIOP $\min_{x \in D} f(x)$ with f and D are defined as in Example 2.4. For all $X, Y \in D$ we have
$$f'(X, \exp_X^{-1} Y) = \begin{cases} [0, \ln(\det(YX^{-1}))] & \text{if } \det(Y) \geq \det(X), \\ [\ln(\det(YX^{-1})), 0] & \text{otherwise.} \end{cases}$$

Note that, for all $X \in D$, we can find $Y \in D$ such that $\det Y < \det X$, which indicates that this RIOP does not have efficient point.

Theorem 4.6 (Characterization II of efficient point). *Let $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a RIVF on a nonempty open subset $\mathcal{D}$ of $\mathcal{M}$ and $x_0 \in \mathcal{D}$ such that f is gH-Gâteaux differentiable at x_0.*

(a) *If x_0 is an efficient point of the RIOP (4.1), then*

$$0 \in f_G(x_0)(\exp_{x_0}^{-1} x), \quad \forall x \in \mathcal{D}.$$

(b) *If $\mathcal{D}$ is geodesically convex, f is a geodesically convex RIVF on $\mathcal{D}$ and*

$$0 \in [\underline{f}_G(x_0)(\exp_{x_0}^{-1} x), \overline{f}_G(x_0)(\exp_{x_0}^{-1} x)), \quad \forall x \in \mathcal{D}.$$

where $f_G(x_0)(\exp_{x_0}^{-1} x) = [\underline{f}_G(x_0)(\exp_{x_0}^{-1} x), \overline{f}_G(x_0)(\exp_{x_0}^{-1} x)]$, then x_0 is an efficient point of the RIOP (4.1).

Proof. For all $x \in \mathcal{D}$, letting $v = \exp_{x_0}^{-1} x$ and due to f being gH-Gâteaux differentiable at x_0, the function f has gH-directional derivative at x_0 in direction v and by Theorem 4.1 we have

$$f_G(x_0)(\exp_{x_0}^{-1} x) = f'(x_0, v) \not\prec \mathbf{0} \text{ or } f_G(x_0)(\exp_{x_0}^{-1} x) = [a, 0] \text{ for some } a < 0.$$

$T_{x_0}\mathcal{M}$ is a linear space then $-v \in T_{x_0}\mathcal{M}$. Because f is gH-Gâteaux differentiable at x_0, hence $f_G(x_0)(\cdot)$ is g-linear. Then, we obtain

$$f_G(x_0)(-v) = -f_G(x_0)(v),$$

Assume $\underline{f}_G(x_0)(v) > 0$, then we have

$$- \underline{f}_G(x_0)(v) < 0$$

$$\Rightarrow \begin{cases} f_G(x_0)(-v) = [-\overline{f}_G(x_0)(v), -\underline{f}_G(x_0)(v)] \prec \mathbf{0} \\ f_G(x_0)(-v) \neq [a, 0] \text{ for some } a < 0 \end{cases},$$

which is a contradiction or $\underline{f}_G(x_0)(v) \leq 0$. Thus, we show that $0 \in f_G(x_0)(\exp_{x_0}^{-1} x)$. For the remaining part, let $x \in \mathcal{D}$, we have

$$0 \in [\underline{f}_G(x_0)(\exp_{x_0}^{-1} x), \overline{f}_G(x_0)(\exp_{x_0}^{-1} x)) \implies f_G(x_0)(\exp_{x_0}^{-1} x) \not\prec \mathbf{0}$$

$$\implies f'(x_0, \exp_{x_0}^{-1} x) \not\prec \mathbf{0},$$

which together with the convexity of f and Theorem 4.4 proves that x_0 is an efficient point of the RIOP (4.1). $\qquad \square$

Example 4.7. Let $\mathcal{M} = \mathbb{R}_{++} := \{x \in \mathbb{R} \,|\, x > 0\}$ be endowed with the Riemannian metric given by

$$\langle u, v \rangle_x = \frac{1}{x^2} uv, \quad \forall u, v \in T_x M \equiv \mathbb{R}.$$

Then, it is known that $\mathcal{M}$ is a Hadamard manifold. For all $x \in \mathcal{M}$, $v \in T_x\mathcal{M}$, the geodesic $\gamma : \mathbb{R} \longrightarrow \mathcal{M}$ such that $\gamma(0) = x, \gamma'(0) = v$ is described by

$$\gamma(t) = \exp_x(tv) = xe^{(v/x)t} \quad \text{and} \quad \exp_x^{-1} y = x \ln \frac{y}{x}, \quad \forall y \in \mathcal{M}.$$

We consider the RIOP $\min_{x \in \mathcal{M}} f(x)$ with $f : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ is defined by

$$f(x) = \left[x, x + \frac{1}{x}\right], \quad \forall x \in \mathcal{M}.$$

For all $x \in \mathcal{M}$, $v \in \mathbb{R}$, we compute

$$
\begin{aligned}
f'(x, v) &= \lim_{t \longrightarrow 0^+} \frac{1}{t}(f(\exp_x(tv)) -_{gH} f(x))) \\
&= \lim_{t \longrightarrow 0^+} \frac{1}{t} \left[\min \left\{ x(e^{(v/x)t} - 1), x(e^{(v/x)t} - 1) + \frac{1}{x}(e^{(v/x)t} - 1) \right\}, \right. \\
&\qquad \left. \max \left\{ x(e^{(v/x)t} - 1), x(e^{(v/x)t} - 1) + \frac{1}{x}(e^{(v/x)t} - 1) \right\} \right] \\
&= \left[\min \left\{ v, v - \frac{1}{x^2}v \right\}, \max \left\{ v, v - \frac{1}{x^2}v \right\} \right] \\
&= v \left[1 - \frac{1}{x^2}, 1 \right],
\end{aligned}
$$

which says that f is gH-directional differentiable on $\mathcal{M}$. We can also easily verify that $f'(x, \cdot)$ is gH-continuous and g-linear, and hence f is gH-Gâteaux differentiable on $\mathcal{M}$.

On the other hand, by the Cauchy-Schwarz inequality, for all $x > 0$, we have

$$
x + \frac{1}{x} \geq 2, \text{ and } x + \frac{1}{x} = 2 \Leftrightarrow x = 1,
$$

then

$$
\left[x, x + \frac{1}{x} \right] \not\prec [1, 2], \forall x > 0,
$$

or $x = 1$ is an efficient point of this RIOP.

Particularly, at $x_0 = 1 \in \mathcal{M}$, we have

$$
\begin{aligned}
f_G(1)(\exp_1^{-1} x) &= [\min \{\ln x, 0\}, \max \{\ln x, 0\}], \\
&= \begin{cases} [\ln x, 0] & \text{if } x < 1 \\ [0, \ln x] & \text{if } x \geq 1. \end{cases}
\end{aligned}
$$

Remark 4.8. Note that there are similar results in [16, Theorem 3.2 and Theorem 4.2], which are not correct.

(1) From Example 4.1 and Example 4.2, we see that, at $x_0 \in \mathcal{D} \subseteq \mathcal{M}$, if there exists $x \in \mathcal{D}$ such that $f'(x_0, \exp_{x_0}^{-1} x) = [a, 0]$ for some $a < 0$, we still do not have enough conditions to answer the question: is x_0 an efficient point?

(2) Theorem 4.4 and Theorem 4.6 are the generalization of the Euclidean concepts in [16, Theorem 3.2 and Theorem 4.2]. We think their statements are not correct as pointed out as above. Hence, we fix their errors and provide correct versions as in Theorem 4.4 and Theorem 4.6.

The interval variational inequality problems (IVIPs) was introduced by Kinderlehrer and Stampacchia [22]. There are some relationships between the IVIPs and the IOPs. Let $\mathcal{D}$ be a nonempty subset of $\mathcal{M}$ and $T : \mathcal{D} \longrightarrow \mathbb{L}_{gH}(T\mathcal{M}, \mathcal{I}(\mathbb{R}))$ be a mapping such that $T(x) \in \mathbb{L}_{gH}(T_x\mathcal{M}, \mathcal{I}(\mathbb{R}))$, where $\mathbb{L}_{gH}(T_x\mathcal{M}, \mathcal{I}(\mathbb{R}))$ denotes the space of gH-continuous, g-linear mapping from $T_x\mathcal{M}$ to $\mathcal{I}(\mathbb{R})$ and $\mathbb{L}_{gH}(T\mathcal{M}, \mathcal{I}(\mathbb{R})) = \bigcup_{x \in \mathcal{M}} \mathbb{L}_{gH}(T_x\mathcal{M}, \mathcal{I}(\mathbb{R}))$. Now, we define the Riemannian interval inequality problems (RIVIPs) as follows:

(a) The Stampacchia Riemannian interval variational inequality problem (RSIVIP) is a problem, which to find $x_0 \in \mathcal{D}$ such that

$$T(x_0)(\exp_{x_0}^{-1} y) \not\prec \mathbf{0}, \quad \forall y \in \mathcal{D}.$$

(b) The Minty Riemannian interval variational inequality problem (RMIVIP) is a problem to find $x_0 \in \mathcal{D}$ such that

$$T(y)(\exp_{x_0}^{-1} y) \not\prec \mathbf{0}, \quad \forall y \in \mathcal{D}.$$

Definition 4.9 (Pseudomonotone). With a mapping T defined as above, we call T is pseudomonotone if for all $x, y \in \mathcal{D}, x \neq y$, there holds

$$T(x)(\exp_x^{-1} y) \not\prec \mathbf{0} \quad \Longrightarrow \quad T(y)(\exp_x^{-1} y) \not\prec \mathbf{0}.$$

Definition 4.10 (Pseudoconvex). Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty geodesically convex set and $f : \mathcal{M} \longrightarrow \mathcal{I}(\mathbb{R})$ be a gH-Gâteaux differentiable RIVF. Then, f is called pseudoconvex if for all $x, y \in \mathcal{D}$, there holds

$$f_G(x)(\exp_x^{-1} y) \not\prec \mathbf{0} \quad \Longrightarrow \quad f(y) \not\prec f(x).$$

Proposition 4.11. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty set and consider a mapping $T : \mathcal{D} \longrightarrow \mathbb{L}_{gH}(T\mathcal{M}, \mathcal{I}(\mathbb{R}))$ such that $T(x) \in \mathbb{L}_{gH}(T_x\mathcal{M}, I(\mathbb{R}))$. If T is pseudomonotone, then every solution of the RSIVIP is a solution of the RMIVIP.*

Proof. Suppose that x_0 is a solution of the RSIVIP. Then, we know that

$$T(x_0)(\exp_{x_0}^{-1} y) \not\prec \mathbf{0}, \quad \forall y \in \mathcal{D},$$

which together with the pseudomonotonicity of T yields

$$T(y)(\exp_{x_0}^{-1} y) \not\prec \mathbf{0}, \quad \forall y \in \mathcal{D}.$$

Then, x_0 is a solution of the RMIVIP. □

It is observed that if a RIVF $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ is gH-Gâteaux differentiable at $x \in \mathcal{D}$, then $f_G(x) \in \mathbb{L}_{gH}(T_x\mathcal{M}, \mathcal{I}(\mathbb{R}))$. It means there are some relationships between the RIOPs and the RIVIPs.

Theorem 4.12. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty set, $x_0 \in \mathcal{D}$ and $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a gH-Gâteaux differentiable RIVF at x_0. If x_0 is a solution of the RIOP (4.1) and $f_G(x_0)(\exp_{x_0}^{-1} y) \neq [a, 0]$ for all $a < 0$, then x_0 is a solution of the RSIVIP with $T(x_0) = f_G(x_0)$.*

Proof. Since f is gH-Gâteaux differentiable on an open set containing $\mathcal{D}$, the function f is gH-directional differentiable at x_0. In light of Theorem 4.4, it follows that

$$f_G(x_0)(\exp_{x_0}^{-1} y) \not\prec \mathbf{0}, \quad \forall y \in \mathcal{D}.$$

Consider $T : \mathcal{D} \longrightarrow \mathbb{L}_{gH}(T\mathcal{M}, \mathcal{I}(\mathbb{R}))$ such that $T(x_0) = f_G(x_0)$ for all $x \in \mathcal{D}$, it is clear that

$$T(x_0)(\exp_{x_0}^{-1} y) \not\prec \mathbf{0}, \quad \forall y \in \mathcal{D},$$

which says that x_0 is a solution of the RSIVIP with $T(x_0) = f_G(x_0)$. □

Theorem 4.13. *Let $\mathcal{D} \subseteq \mathcal{M}$ be a nonempty geodesically convex set, $x_0 \in \mathcal{D}$ and $f : \mathcal{D} \longrightarrow \mathcal{I}(\mathbb{R})$ be a pseudoconvex, gH-Gâteaux differentiable RIVF at x_0. If x_0 is a solution of the RSIVIP with $T(x_0) = f_G(x_0)$, then x_0 is an efficient point of the RIOP (4.1).*

Proof. Let x_0 is a solution of the RSIVIP with $T(x_0) = f_G(x_0)$. Suppose that x_0 is not an efficient point of RIOP (4.1). Then, exists $y \in \mathcal{D}$ such that $f(y) \prec f(x_0)$. From the pseudoconvexity of f, we have

$$f_G(x_0)(\exp_{x_0}^{-1} y) \prec \mathbf{0},$$

which is a contradiction. Thus, x_0 is an efficient point of the RIOP (4.1). $\square$

5. Conclusions

In this paper, we study the Riemannian Interval Optimization problems (RIOPs) on Hadamard manifolds, for which we establish the necessary and sufficient conditions of efficient points. Moreover, we introduce a new concept of gH-Gâteaux differentiability of the Riemannian interval valued functions (RIVFs), which is the generalization of gH-Gâteaux differentiability of the interval valued functions (IVFs). The Riemannian interval variational inequalities problems (RIVIPs) and their relationship, as well as the relationship between the RIVIPs and the RIOPs are also investigated in this article. Some examples are presented to illustrate the main results.

In our opinions, the obtained results are basic bricks towards further investigations on the Riemannian interval optimization, in particular, when the Riemannian manifolds acts as Hadamard manifolds. For future research, we may either study the theory for more general Riemannian manifolds or design suitable algorithms to solve the RIOPs.

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